Universal sentences

Kamal Lodaya

March 2021

Axiom system (Bernays, Hilbert-Ackermann 1928)

$$t ::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n$$
$$A ::= \quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B) \mid \exists x A \mid \forall x A$$

Proof system fHB: zHB +

(UInst) $\forall x A(x) \supset A(t)$, (EInst) $A(t) \supset \exists x A(x)$

(UGen) If $\vdash A \supset B(p)$, then $\vdash A \supset \forall x B(x)$, for p parameter not occurring in $A, \forall x B$

(UGen) If $\vdash A \supset \neg B(p)$, then $\vdash A \supset \neg\exists x B(x)$, for p parameter not occurring in $A, \exists x B$

One can get derived rules like:

(EGen) If $\vdash A(p) \supset B$, then $\vdash \exists x A(x) \supset B$, provided p parameter not occurring in $A, \exists x B$

Exercise (Witness)

Show that if $\exists x A(x)$ is consistent, so is $A(p)$ where p is a fresh parameter.

Refutations (Evert Beth 1955, Alan Robinson 1965)

$$t ::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n$$
$$A ::= \quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B) \mid \exists x A \mid \forall x A$$

Refutation system fR: zR + new expansions

$[\ldots, \forall x A(x), \ldots] \rightarrow [\ldots, A(t), \ldots]$,

$[\ldots, \neg\exists x A(x), \ldots] \rightarrow [\ldots, \neg A(t), \ldots]$,

$[\ldots, \exists x A(x), \ldots] \rightarrow [\ldots, A(p), \ldots]$, p fresh parameter,

$[\ldots, \neg\forall x A(x), \ldots] \rightarrow [\ldots, \neg A(p), \ldots]$, p fresh parameter

Refutations (Evert Beth 1955, Alan Robinson 1965)

$$t ::= \quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n$$
$$A ::= \quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False$$
$$\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B) \mid \exists x A \mid \forall x A$$

Refutation system fR: zR + new expansions

$[\ldots, \forall x A(x), \ldots] \to [\ldots, A(t), \ldots]$,
$[\ldots, \neg\exists x A(x), \ldots] \to [\ldots, \neg A(t), \ldots]$,
$[\ldots, \exists x A(x), \ldots] \to [\ldots, A(p), \ldots]$, p fresh parameter,
$[\ldots, \neg\forall x A(x), \ldots] \to [\ldots, \neg A(p), \ldots]$, p fresh parameter

Refutation:

1	$[\neg(\forall x(P(x) \vee Q(x)) \supset \exists x P(x) \vee \forall x Q(x))]$	*Negated*
2	$[\forall x(P(x) \vee Q(x)]; [\neg(\exists x P(x) \vee \forall x Q(x))]$	1, *NotImplies*
3	$[\neg\exists x P(x)]; [\neg\forall x Q(x)]$	$2b$, *NotOr*
4	$[\neg Q(p)]$	$3b$, *NotForall*
5	$[\neg P(p)]$	$3a$, *NotExists*
6	$[P(p), Q(p)]$	$2a$, *Forall*, *Or*
7	$[Q(p)]$	$5, 6$, *UnitCut*
8	$[]$	$4, 7$, *UnitCut*

FOL Hintikka theory (Jaakko Hintikka 1955)

Definition (Fitting, Definition 5.7.1)

An FOL Hintikka theory Th (also called downwards consistent) has the following conditions in addition to propositional ones.

1. $False, \neg True \notin Th$; for P in At, $\{P, \neg P\} \not\subseteq Th$
2. . . . and so on for the propositional conditions seen earlier
3. If $\forall x A$ in Th, then $\{A(t) \mid t \text{ closed term}\} \subseteq Th$
4. If $\neg\exists x A$ in Th, then $\{\neg A(t) \mid t \text{ closed term}\} \subseteq Th$
5. If $\exists x A$ in Th, then $A(p) \in Th$ for fresh parameter p
6. If $\neg\forall x A$ in Th, then $\neg A(p) \in Th$ for fresh parameter p

Items (5) and (6) of the definition are tricky.

Universal sentences (Tarski 1954, Jerzy Łós 1955)

If the prenex normal form of a sentence only has universal quantifiers, it is universal or in the A fragment. The dual is existential or in the E fragment.

Theorem

Consistent A theory is satisfiable in an FOL model. Valid E sentence is provable in fHB proof system using instantiations.

Universal sentences (Tarski 1954, Jerzy Łós 1955)

If the prenex normal form of a sentence only has universal quantifiers, it is universal or in the A fragment. The dual is existential or in the E fragment.

Theorem

Consistent A theory is satisfiable in an FOL model. Valid E sentence is provable in fHB proof system using instantiations.

Proof.

Grounding: Take all closed term instantiations of $\forall y_1 \ldots \forall y_m A$ to satisfy Hintikka condition (3). Consistency by (UInst).

Universal sentences (Tarski 1954, Jerzy Łós 1955)

If the prenex normal form of a sentence only has universal quantifiers, it is universal or in the A fragment. The dual is existential or in the E fragment.

Theorem

Consistent A theory is satisfiable in an FOL model. Valid E sentence is provable in fHB proof system using instantiations.

Proof.

Grounding: Take all closed term instantiations of $\forall y_1 \ldots \forall y_m A$ to satisfy Hintikka condition (3). Consistency by (UInst).

Removing the universal sentence, get consistent ZOL theory. By ZOL satisfiability, there is ZOL Herbrand model, also FOL model. Since all instances to term elements are satisfied, and terms constitute the domain, the FOL sentence is satisfied.

Universal sentences (Tarski 1954, Jerzy Łós 1955)

If the prenex normal form of a sentence only has universal quantifiers, it is universal or in the A fragment. The dual is existential or in the E fragment.

Theorem

Consistent A theory is satisfiable in an FOL model. Valid E sentence is provable in fHB proof system using instantiations.

Proof.

Grounding: Take all closed term instantiations of $\forall y_1 \ldots \forall y_m A$ to satisfy Hintikka condition (3). Consistency by (UInst).

Removing the universal sentence, get consistent ZOL theory. By ZOL satisfiability, there is ZOL Herbrand model, also FOL model. Since all instances to term elements are satisfied, and terms constitute the domain, the FOL sentence is satisfied.

W.l.g., suppose $\exists x_1 \ldots \exists x_n \neg A$ valid. Then $\forall x_1 \ldots \forall x_n A$ unsat. So $\forall x_1 \ldots \forall x_n A$ is inconsistent. That is, $\vdash \neg \forall x_1 \ldots \forall x_n A$. By (UInst) and (EInst) repeatedly, $\vdash \exists x_1 \ldots \exists x_n \neg A$. □