Existential quantifiers

Text: Melvin Fitting, *FOLATP (2nd ed.)*, Sections 6.5,8.3;
Daniel Kroening and Ofer Strichman, *DP*, Section 9.5

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Axiom system (Bernays, Hilbert-Ackermann 1928)

$$
\begin{aligned}
t ::= &\quad x \in V \mid c \in C \mid f(t_1, \ldots, t_n),\ f \in F_n \\
A ::= &\quad P(t_1, \ldots, t_n),\ P \in R_n \mid t_1 \approx t_2 \mid True \mid False \\
&\mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B) \mid (A \equiv B) \mid \exists x A \mid \forall x A
\end{aligned}
$$

Proof system fHB: zHB +

(UInst) $\forall x A(x) \supset A(t)$, (EInst) $A(t) \supset \exists x A(x)$

(UGen) If $\vdash A \supset B(p)$, then $\vdash A \supset \forall x B(x)$, for p parameter not occurring in $A, \forall x B$

(UGen) If $\vdash A \supset \neg B(p)$, then $\vdash A \supset \neg \exists x B(x)$, for p parameter not occurring in $A, \exists x B$

One can get derived rules like:

(EGen) If $\vdash A(p) \supset B$, then $\vdash \exists x A(x) \supset B$, provided p parameter not occurring in $B, \exists x A$

Exercise (Witness)

Show that if $\exists x A(x)$ is consistent, so is $A(p)$ where p is a fresh parameter[1].

[1] The definition of a parameter is extended on the next slide

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- For example $\forall y \forall z(f(y, z) \wedge \exists x(f(x, z) \wedge x < 0))$ is replaced by $\forall y \forall z(f(y, z) \wedge f(g(y, z), z) \wedge g(y, z) < 0)$.

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- For example $\forall y \forall z (f(y, z) \wedge \exists x (f(x, z) \wedge x < 0))$ is replaced by $\forall y \forall z (f(y, z) \wedge f(g(y, z), z) \wedge g(y, z) < 0)$.
- Choose outer $$\begin{aligned} \forall x \exists y \exists z R(x, y, z) &\to \forall x \exists z R(x, g(x), z) \\ &\to \forall x R(x, g(x), h(x)) \end{aligned}$$ or inner $$\begin{aligned} \forall x \exists y \exists z R(x, y, z) &\to \forall x \exists y R(x, y, k(x, y)) \\ &\to \forall x R(x, l(x), k(x, l(x))) \end{aligned}$$?
- Work outside in.
- Idea of parameter as constant extended to parameter as closed term.

Lemma (Skolem function, Fitting, Lemma 8.3.1)

Let formula $B(x, y_1, \ldots, y_n)$ with free variables among shown and g function symbol not occurring in B. If $M = (D, I)$ is a model, there are models $N_1 = (D, J_1),\ N_2 = (D, J_2)$, with I, J_1, J_2 differing only in interpreting g and:

1. *In N_1, $\exists x B \supset B[g(y_1, \ldots, y_n) \text{ for } x]$ is true.*
2. *In N_2, $B[g(y_1, \ldots, y_n) \text{ for } x] \supset \forall x B$ is true.*

Skolem functions (Skolem 1920)

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- J_1 is same as I, except choose x-variant r making $\exists x B^{I,s} = true$, set $g(y_1^s, \ldots, y_n^s) = x^r$.
- Otherwise set it to an arbitrary element of D.
- Similarly for J_2.

Replacing existential quantifiers

Theorem (Skolemization, Fitting, Theorem 8.3.2)

Given $B(x)$ with free variables $y_1, \ldots, y_n$, formula $C(P)$ with atomic P such that $C(\exists x B(x))$ sentence, and $g \in F_n$ not occurring in $C(\exists x B(x))$.

1. *If all occurrences of P are positive in $C(P)$, then $C(\exists x B(x))$ is satisfiable if and only if $C(B(g(y_1, \ldots, y_n)))$ is satisfiable.*
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Proof.

$(\Longleftarrow)$ The formula $B(g(y_1, \ldots, y_n)) \supset \exists x B(x)$ is valid. By replacement, $C(B(g(y_1, \ldots, y_n))) \supset C(\exists x B(x))$ is valid.

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$(\Longrightarrow)$ Let $M = (D, I) \models C(\exists x B(x))$. By Lemma, there is $N = (D, J)$ where $\exists x B(x) \supset B(g(y_1, \ldots, y_n))$. By replacement, $N \models C(\exists x B(x)) \supset C(B(g(y_1, \ldots, y_n)))$.

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Given $B(x)$ with free variables $y_1, \ldots, y_n$, formula $C(P)$ with atomic P such that $C(\exists x B(x))$ sentence, and $g \in F_n$ not occurring in $C(\exists x B(x))$.

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($\Longrightarrow$) Let $M = (D, I) \models C(\exists x B(x))$. By Lemma, there is $N = (D, J)$ where $\exists x B(x) \supset B(g(y_1, \ldots, y_n))$. By replacement, $N \models C(\exists x B(x)) \supset C(B(g(y_1, \ldots, y_n)))$. $N \models C(\exists x B(x))$ as g not occurring in $C(\exists x B(x))$. So $N \models C(B(g(y_1, \ldots, y_n)))$. □

Refutation examples

1	$[\neg(\forall x \exists y R(x, y) \supset \exists y \forall x R(x, y))]$	*Negated*
2	$[\forall x \exists y R(x, y)]; [\neg \exists y \forall x R(x, y)]$	1, *NotImplies*
3	$[\forall x R(x, f(x))]$	$2a$, *Exists Skolem*
4	$[\neg \forall x R(x, t)]$	$2b$, *NotExists*
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Disagreement pair $(x, g(t))$ unifiable,
disagreement pair $(f(x), t)[g(t) \text{ for } x] = (f(g(t)), t)$ not unifiable.
Countermodel: $f(x) = x, \; g(y) = y + 1, \; R(x, y) = x \approx y$.

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5	$[R(q, p)]; [\neg R(q, p)]$	$3, 4, \{(x, q), (p, y)\}$ *unifiable*
6	$[]$	$5a, 5b$, *UnitCut*

(Alan Robinson 1965) gave a unification algorithm for terms, see (Fitting, Section 7.2).

Putting it together (Kurt Gödel 1930)

Theorem (FOL weak completeness)

Finite consistent FOL sentences are satisfiable.

- Given consistent sentence B, replace existential quantifiers by skolemizing to an equisatisfiable universal sentence C.
- By (Witness), which used the (UGen,EGen) inference rules, C is consistent.
  (Instead of fresh parameter constant p, a fresh function parameter term was used in skolemization.)
- By completeness of universal sentences, which used the rest of the proof system but not the (UGen,EGen) inference rules, C is satisfiable.
- By equisatisfiability, so is B.

- Suppose a countable PL theory $Th = \{A_1, A_2, \ldots\}$ is consistent.

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Theorem (FOL strong completeness)

A consistent countable FOL theory Th is satisfiable in a countable FOL model.