Theories to models in full first-order logic

Text: Melvin Fitting, *FOLATP*, Sections 5.6 to 5.9

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- Can write $\neg A$ as follows, skolemizing k as a constant:
  $k \geq 0 \wedge f(k) \approx a \wedge$
  $\forall i \forall j \exists i' \exists j'((0 \leq j < i \supset f(j) \not\approx a) \wedge (f(i) \not\approx a)$
  $\wedge (i' \approx i+1) \wedge 0 \leq j' < i' \wedge (f(j') \approx a))$

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- Can directly skolemize i' as $i + 1$. Skolemize j' as $g(i, j)$.
  $$\begin{array}{l} k \geq 0 \wedge f(k) \approx a \wedge \\ \forall i \forall j((0 \leq j < i \supset f(j) \not\approx a) \wedge (f(i) \not\approx a) \\ \wedge (i + 1 \approx i + 1) \wedge 0 \leq g(i, j) < i + 1 \wedge (f(g(i, j)) \approx a)) \end{array}$$

An exercise (continued)

- Instantiating for one value of i and consequent values for j (this has to be done for all values of i but we skip that):

$$k \geq 0 \wedge f(k) \approx a \wedge$$
$$\bigwedge_{j=0}^{i-1} f(j) \not\approx a \wedge f(i) \not\approx a \wedge \bigvee_{j=0}^{i} (g(i,j) \approx j \wedge f(g(i,j)) \approx a)$$

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- Uninterpreting by propositional abstraction:

  $$P_{kge0} \wedge P_{fka} \wedge \neg P_{f0a} \wedge \cdots \wedge \neg P_{fia} \wedge P_{fja} \wedge (E_{j0} \vee \cdots \vee E_{ji})$$

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- Satisfiable. Now construct Herbrand model.
  $k \geq 0 \wedge f(k) \approx a \wedge$
  $\neg(f(0) \approx a) \wedge \cdots \wedge \neg(f(i) \approx a) \wedge (f(j) \approx a) \wedge (j \approx 0 \vee \cdots \vee j \approx i)$
  When constructing normal model, unsatisfiability is detected.

Theorem (Löwenheim 1915, Skolem 1920)

If a theory in a countable signature is satisfiable, it has a countable model.

1. Follows from soundness and Gödel's procedure, an enumerating program but not an algorithm.
2. Show that family of subsets of theory which have infinitely many parameters new to them is a consistency property.

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Non-definability: No FOL theory has arbitrarily large finite models without also having an infinite model.

A step-by-step proof (Kurt Gödel 1930)

- The proof goes from FOL to universal sentences (A fragment[1]) eliminating existential quantifiers by (Gen) inference rules, expanding the signature finitely many times with new function symbols (depends on FOL syntax).
- Eliminates universal quantifiers to ZOL by (Inst) axioms.
- Then to PL by uninterpreting terms and equality axioms.

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- Then to PL by uninterpreting terms and equality axioms.
- Satisfiability is obtained at PL by extending to Hintikka theory and extracting a model. This uses all the propositional axioms. Could have been a different proof[2].
- Infinite satisfiablity obtained by König's lemma argument.

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- Lifted to existential quantifer by interpreting new functions.

A forcing construction (Leon Henkin 1949)

- Both for PL and FOL (Fitting, Section 3.6, Section 5.8), main idea is forcing: a family of consistency properties (independent of a proof system), reflecting the extension from a consistent theory to a Hintikka theory.
- Every consistency property can be extended to satisfy closure under subsets and under ascending chain unions.

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- Textbook (Fitting, Section 5.8): Going from PL to FOL, set of fresh parameters maintained, ensuring countably many always remain available to be used. Signature updated with parameters which are in use. Sequential technique.

References for presentations

George Boolos, John Burgess and Richard Jeffrey. *Computability and logic, 5th ed.*, CUP (2007).
Section 21.2 has decidability of FOL restricted to unary predicates. Chapter 24 has decidability of FO theory of linear integer arithmetic.
Section 11.1 has undecidability of FOL, which requires understanding of Turing machines.

John Harrison. *Handbook of practical logic and automated reasoning*, CUP (2009).
Article "Real quantifier elimination" on FO theory of reals available on the web. Hans Schoutens has a short note "Muchnik's proof of Tarski-Seidenberg" (2004). Both require understanding of calculus.