Logic of Equality and Uninterpreted Functions (EUF)

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Outline

Logic of Equality and Uninterpreted Functions (EUF) (KS Ch 4)

- Boolean combinations of equality predicates.

EUF syntax

(Formula) $\varphi ::= \mathit{Atom} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$
(Atom) $\mathit{Atom} ::= \mathit{Term} = \mathit{Term}$
(Term) $\mathit{Term} ::= \mathit{Var} \mid \mathit{Const} \mid F(\mathit{Term})$

Example formula φ_1

$$(x_1 = x_2) \wedge (x_2 = x_3) \wedge (F(x_1) \neq F(x_3)).$$

Example formula φ_2

$$F(x) = F(G(y)) \vee x = y$$

Questions we want to answer

Given an EUF formula φ:

- Satisfiability: Does there exist (M, v), with $M = (D, I)$ and $D = \mathbb{Z}$ (or any infinite domain), such that $M, v \vDash \varphi$.
- Validity: Does $M, v \vDash \varphi$, for *every* (M, v), with $M = (D, I)$ and $D = \mathbb{Z}$ (or any infinite domain).

Exercise:

- Give examples of satisfiable, unsatisfiable, valid EUF formulas.
- What is the relation between satisfiabililty and validity?

Importance of EUF logic

Many practical applications. Arguing correctness of:

- Program transformation, compilation.
- Pipelining in a hardware circuit.

Example: Are these programs equivalent?

```
S1: z := (x1 + y1) * (x2 + y2);
```

```
T1: u1 := (x1 + y1);
T2: u2 := (x2 + y2);
T3: z := u1 * u2;
```

We want to check whether:

$$[u_1 = x_1 + y_1 \wedge u_2 = x_2 + y_2 \wedge z = u_1 * u_2] \implies z = (x_1 + y_1) * (x_2 + y_2).$$

We could check whether the EUF formula is valid:

$$[u_1 = F(x_1, y_1) \wedge u_2 = F(x_2, y_2) \wedge z = G(u_1, u_2)] \implies z = G(F(x_1, y_1), F(x_2, y_2)).$$

How do we decide satisfiability of EUF formulas?

How?

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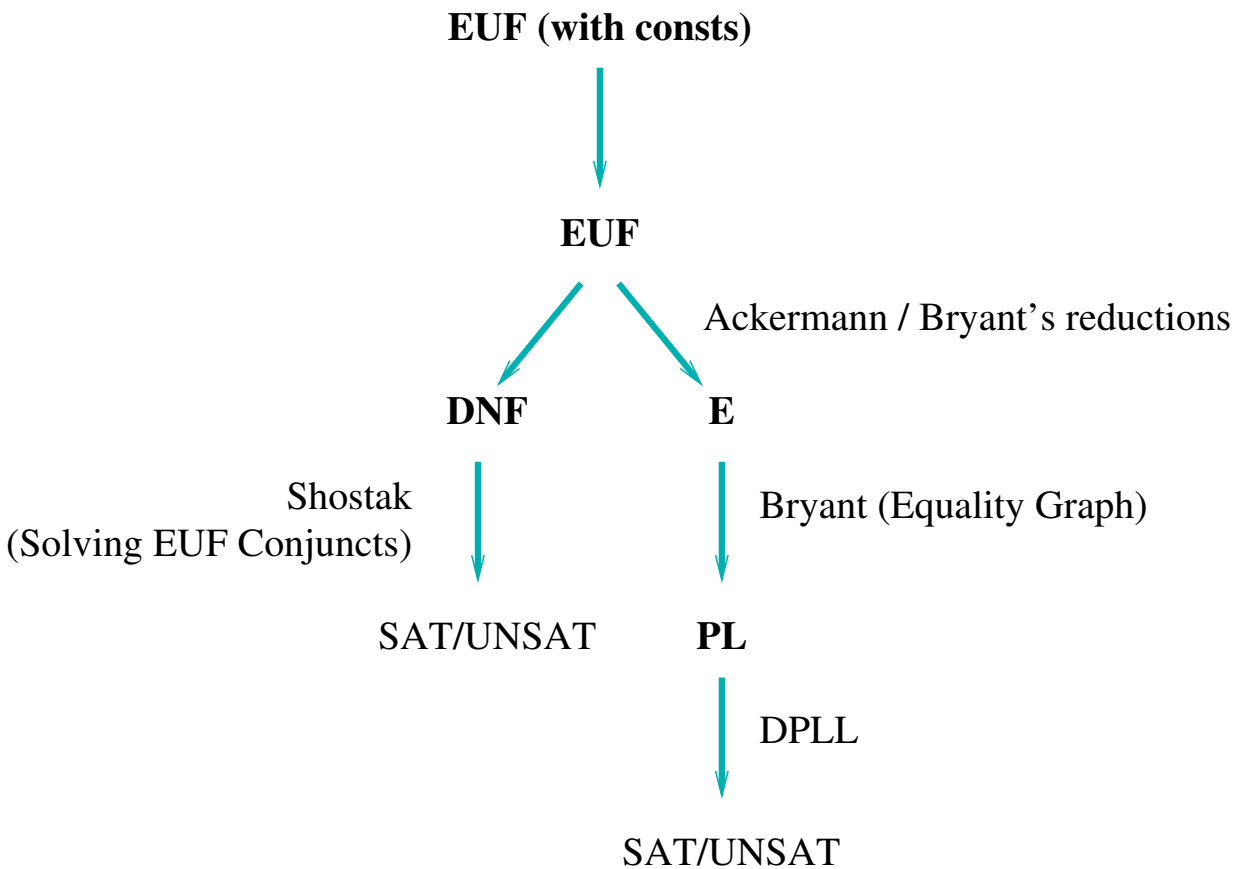
Strategies we will look at:



A brute-force algorithm

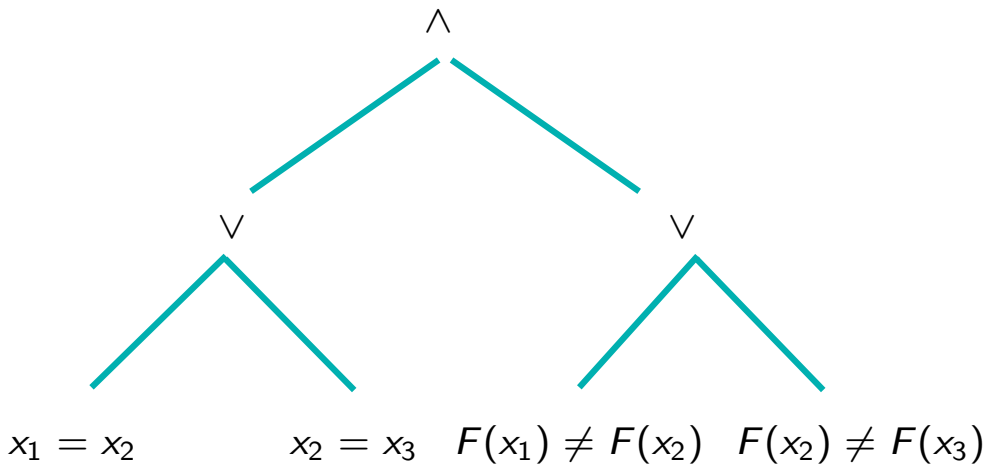
Given EUF formula φ:

1. Let V be the variables in φ, and let k be the number of distinct terms in φ.
2. Let $U = \{1, \ldots, k\}$
3. For each possible valuation v of V over U, check if $u \models \varphi$.
4. If some v satisfies φ, output SAT; else output UNSAT.

Example

Example

$$(x_1 = x_2 \vee x_2 = x_3) \wedge (F(x_1) \neq F(x_2) \vee F(x_2) \neq F(x_3))$$

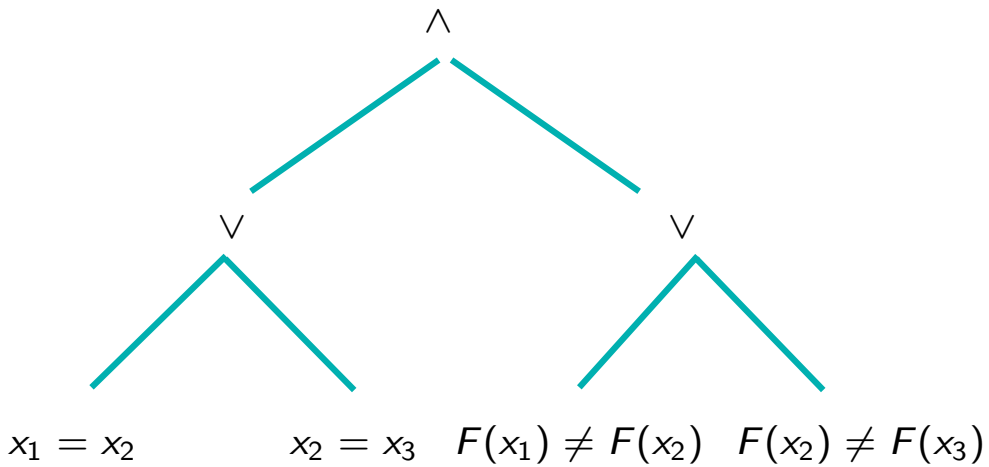


Correctness?

Example

Example

$$(x_1 = x_2 \vee x_2 = x_3) \wedge (F(x_1) \neq F(x_2) \vee F(x_2) \neq F(x_3))$$



Correctness? Claim: If M, v satisfies φ, then we can construct M', v' such that $M', v' \models \varphi$, and v' is over U.

Getting rid of constants (KS Sec 4.1.3)

Given φ in EUF (with consts), replace each constant k in φ by a new variable c_k and add conjuncts saying that $c_k \neq c_{k'}$ for each distinct constants k, k'. Example: Replace φ

$$(y = z \wedge z \neq 1) \vee ((x \neq z) \wedge x = 2)$$

by equisatisfiable φ':

$$[(y = z \wedge z \neq c_1) \vee ((x \neq z) \wedge x = c_2)] \wedge (c_1 \neq c_2).$$

Claim: φ is satisfiable iff φ' is.

Congruence Closure Algorithm (Shostak 1978) (KS Sec 4.3)

Given EUF formula φ as conjunction of literals:

1. Consider all subterms t of φ.
2. If $(t_1 = t_2)$ is a predicate in φ, put t_1, t_2 in the same equivalence class. All other terms in their own singleton equivalence classes.
3. If two classes share a term, merge them.
4. Apply congruence closure: If t_1 and t_2 are in the same equivalence class, then merge the equivalence classes of $F(t_1)$ and $F(t_2)$.
5. If there is a disequality $t_1 \neq t_2$ in φ, with t_1 and t_2 in the same equivalence class, return UNSAT. Else return SAT.

Example:

$$(x_1 = x_2) \wedge (x_2 = x_3) \wedge (x_4 = x_5) \wedge (x_5 \neq x_1) \wedge (F(x_1) \neq F(x_3)).$$

Correctness?

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Example:

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Correctness? Claim: Algo is correct if it outputs SAT/UNSAT.

Ackermann's reduction (KS Sec 11.2.1)

Given EUF formula φ output E formula:

$$\varphi_{flat} \wedge FC_{\varphi},$$

where φ_{flat} replaces function applications like $F(G(x))$ by $f(g_1)$ etc, and FC encodes functional consistency:

$$x = y \implies F(x) \implies F(y).$$

Claim: φ is sat/valid iff $\varphi_{flat} \wedge FC_{\varphi}$ is sat/valid.

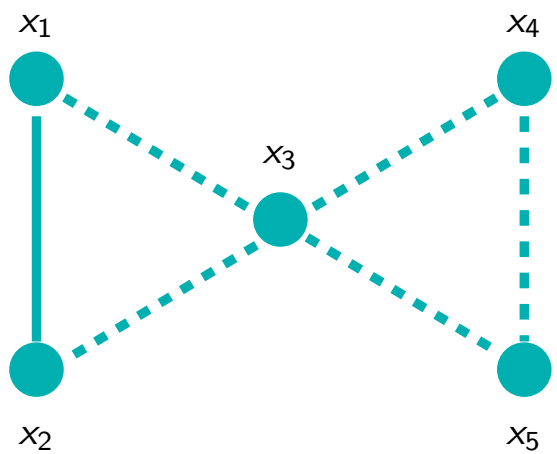
Exercise

Exercise: Give Ackermann's reduction for this formula

$$(x_1 = x_2) \implies F(F(G(x_1))) = F(F(G(x_2))).$$

Equality graph induced by an E formula (KS Sec 11.3)

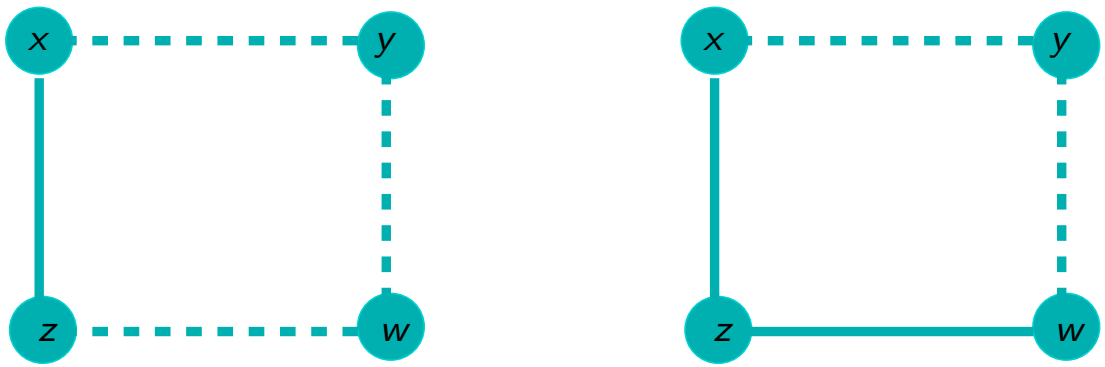
Let φ be a E formula. Then the equality graph G_φ induced by φ is an undirected graph with nodes as variables and "- - -"edge (x_i, x_j) iff the literal $x_i = x_j$ occurs in φ, and "—"edge (x_i, x_j) iff the literal $x_i \neq x_j$ occurs in φ.



- Contradictory cycles (cycle with exactly one disequality edge).
- Is an abstraction of the original formula.
- Can be used to simplify an E formula.

Contradictory Cycles

A conjunction φ of equality constraints is satisfiable iff the equality graph G_φ induced by φ has no contradictory cycles.



Proof (Exercise).

Solving E-formulas: DNF approach

Example φ

$$x = y \wedge [(y = z \wedge \neg(x = z)) \vee (y = z \wedge \neg(x = w))].$$

$e(\varphi)$: $\quad e_{xy} \wedge [(e_{yz} \wedge \neg e_{xz}) \vee (e_{yz} \wedge \neg e_{xw})]$.

Convert $e(\varphi)$ to DNF and check each disjunct using equality graph.

Solving E-formulas: DPLL(T) approach (KS Sec 3.4)

Example φ

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$e(\varphi)$: $\quad e_{xy} \wedge [(e_{yz} \wedge \neg e_{xz}) \vee (e_{yz} \wedge \neg e_{xw})]$.

Check satisfiabilty of $e(\varphi)$ using DPLL:

1. If $e(\varphi)$ is SAT, check if satisfying assignment is T-valid;
   - If T-valid, then return SAT;
   - If not T-valid, add negation of the assignment as conflicting clause to $e(\varphi)$, and go back Step 1.
2. If UNSAT, report UNSAT.

Bryant's Graph-Based reduction of E to PL (KS Sec 11.5)

Given E formula φ output PL formula:

$$e(\varphi) \wedge \mathcal{B}_{trans},$$

where $e(\varphi)$ replaces each literal $x_i = x_j$ by a propositional symbol p_{ij}, and $\mathcal{B}_{trans}$ encodes transitivity constraints based on the non-polar graph induced by φ.

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where $e(\varphi)$ replaces each literal $x_i = x_j$ by a propositional symbol p_{ij}, and $\mathcal{B}_{trans}$ encodes transitivity constraints based on the non-polar graph induced by φ.
Claim: φ is equisatisfiable with $e(\varphi) \wedge \mathcal{B}_{trans}$.

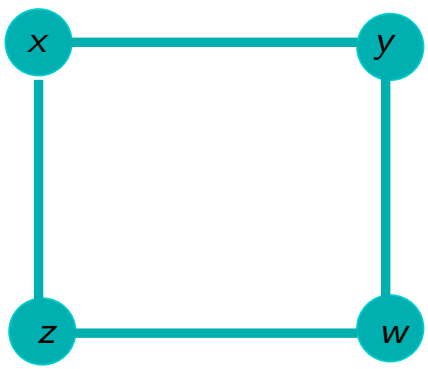
Bryant's Graph-Based reduction: Example

Example φ

$$x = y \wedge [(y = w \wedge w = z \wedge \neg(x = z)) \vee (y = w \wedge \neg(w = z))].$$

$e(\varphi)$: $e_{xy} \wedge [(e_{wy} \wedge e_{wz} \wedge \neg e_{xz}) \vee (e_{wy} \wedge \neg e_{wz})]$.

G_{φ}^{NP}:

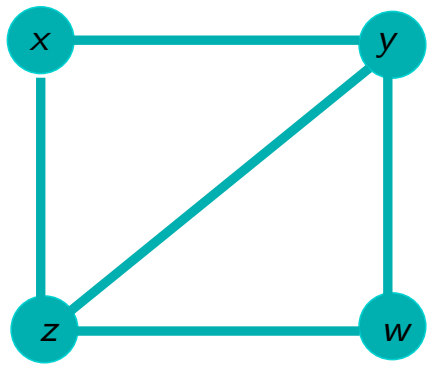


$\mathcal{B}_{trans}$:

$$\begin{aligned}
(e_{xy} \wedge e_{wy} \wedge e_{wz}) &\implies e_{xz} \wedge \\
(e_{wy} \wedge e_{wz} \wedge e_{xz}) &\implies e_{xy} \wedge \\
(e_{wz} \wedge e_{xz} \wedge e_{xy}) &\implies e_{wy} \wedge \\
(e_{xz} \wedge e_{xy} \wedge e_{wy}) &\implies e_{wz}
\end{aligned}$$

Bryant's Graph-Based reduction: Using chordal graph

Make G_φ^{NP} chordal (every simple cycle of ≥ 4 vertices has a chord).



Sufficient to check no contradictary triangles [Bryant-Velev CAV 2000):

$\mathcal{B}_{trans}$:

$$
\begin{aligned}
(e_{xy} \wedge e_{yz}) &\Longrightarrow e_{xz} \wedge \\
(e_{yz} \wedge e_{xz}) &\Longrightarrow e_{xy} \wedge \\
(e_{xz} \wedge e_{xy}) &\Longrightarrow e_{yz} \wedge \\
(e_{wy} \wedge e_{wz}) &\Longrightarrow e_{yz} \wedge \\
(e_{wz} \wedge e_{yz}) &\Longrightarrow e_{wy} \wedge \\
(e_{yz} \wedge e_{wy}) &\Longrightarrow e_{wz}.
\end{aligned}
$$

Using equality graph to simplify E formulas

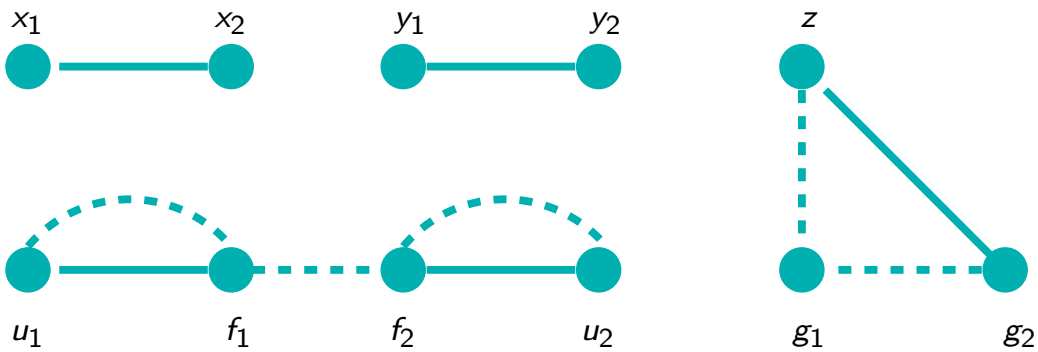
Given E formula φ:

1. Construct equality graph G_φ for φ.
2. If a literal does not occur as part of a contradictory cycle in G_φ, set it to *true*. Obtain φ' in this way.
3. Simplify φ' and go back to Step 2.
4. Ouput φ' as equisatisfiable to φ.

Equality graph: example

Example

$$(u_1 \neq f_1 \vee y_1 \neq y_2 \vee f_1 = f_2) \wedge$$
$$(x_1 \neq x_2 \vee u_2 \neq f_2 \vee g_1 = g_2) \wedge$$
$$(u_1 = f_1 \wedge u_2 = f_2 \wedge z = g_1 \wedge z \neq g_2).$$



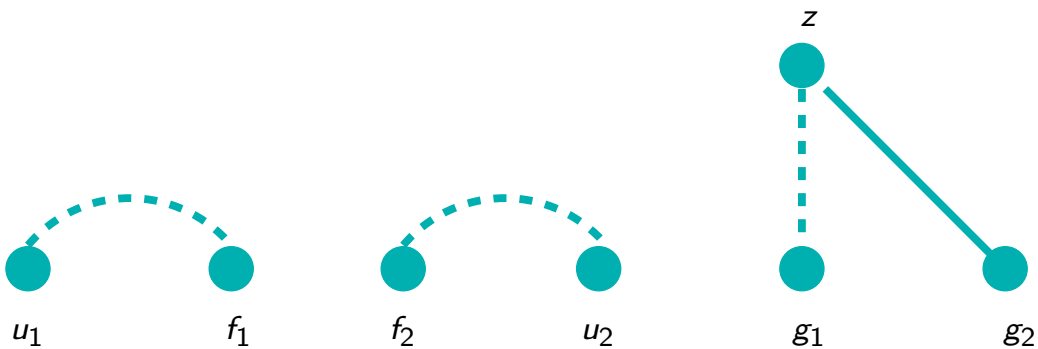
$$(u_1 \neq f_1 \vee \mathit{true} \vee \mathit{true}) \wedge$$
$$(\mathit{true} \vee u_2 \neq f_2 \vee g_1 = g_2) \wedge$$
$$(u_1 = f_1 \wedge u_2 = f_2 \wedge z = g_1 \wedge z \neq g_2).$$

Simplifies to:

$$u_1 = f_1 \wedge u_2 = f_2 \wedge z = g_1 \wedge z \neq g_2.$$

Equality graph: example contd.

$$u_1 = f_1 \wedge u_2 = f_2 \wedge z = g_1 \wedge z \neq g_2.$$



$$\mathit{true} \wedge \mathit{true} \wedge \mathit{true} \wedge \mathit{true}.$$

Simplifies to:

$$\mathit{true}.$$