Assignment 1

E0 205 Mathematical Logic and Theorem Proving

Weightage 5%. Due date: 2pm Monday 15 March 2021.

1. Using the BD proof system (refer to slides if required), which gives basic arithmetic properties of $\mathbb{Z}$, namely that addition and multiplication are commutative and associative with unit, subtraction is defined in terms of addition, and multiplication distributes over addition, give proofs of:
   (a) The Square Equation $(x+y)^2 \approx x^2 + 2xy + y^2$
   (b) The Difference of Squares Equation $x^2 - y^2 \approx (x+y)(x-y)$
   (c) Addition Monotonicity $((x > y) \wedge (a \geq b)) \supset ((x+a) > (y+b))$
2. Reflexivity of equality and the replacement property are axioms of Zeroth-order logic ZOL. Show that the following are theorems of ZOL.
   (a) Transitivity $((x \approx y) \wedge (y \approx z)) \supset (x \approx z)$
   (b) Symmetry $x \approx y \supset y \approx x$
3. (Fitting Exercise 4.1.7) Give a proof in ZOL of the contrapositive
   $(\neg B \supset \neg A) \equiv (A \supset B)$.
4. Fix arithmetic signature $(\mathbb{R}^{\geq 0}, +, \times, \sqrt{}, >)$. That is, all nonnegative real numbers are constants, with functions and binary predicate symbol $>$ as expected. Define a model M over this signature.
   (a) Check $(\sqrt{3} \leq \sqrt{5}) \vee (\sqrt{5} \leq \sqrt{3})$ evaluates to *true*/is satisfied over M.
   (b) How will you deal with $(x \leq y) \vee (y \leq x)$? Is it M-valid?
5. (a) For the arithmetic signature $(0, 1, +, -, \times)$ (remember that equality is always available in ZOL), define the Herbrand model for a theory given by axioms which are valid over the integers modulo two $\mathbb{Z}/2\mathbb{Z}$.
   (b) Suppose you have an axiom system which is sound for a class of models $\mathcal{C}$ and unsound for models outside $\mathcal{C}$. Show that when you add a new axiom to it, the class of models may shrink, but it cannot grow.
6. Show that A is valid if and only if $\neg A$ is not satisfiable, and A is satisfiable if and only if $\neg A$ is not valid.