Constrained Horn Clauses

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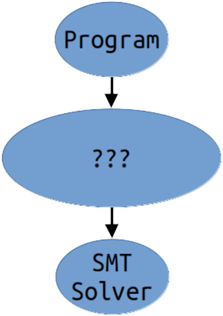
Motivation: Program Safety Verification

Program and an assert: all the states that reach assert satisfy the condition in assert?



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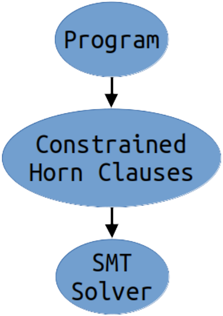


A bit of history

- How different areas of computation are related?

  AI vs Databases vs Programming ??

- Logic was well understood

  Propositional logic $<$ FOL $<$ Higher-order logic etc.

- Can logic unify different areas of computation?

Horn Clauses

In Propositional Logic: A clause with at most one positive literal

$$A_1 \wedge \cdots \wedge A_n \implies B \tag{1}$$

$$A_1 \cdots A_n \tag{2}$$

Exercise 1

$P, Q \ldots$ are propositional variables

$$P \implies Q$$

$$L \wedge M \implies P$$

$$B \wedge L \implies M$$

$$A \wedge P \implies L$$

$$A \wedge B \implies L$$

$$A$$

$$B$$

Can we conclude Q?

Reference: Figure 7.16, Artificial Intelligence A Modern Approach

Exercise 1

$$P, Q \ldots \text{ are propositional variables}$$
$$P \implies Q$$
$$L \wedge M \implies P$$
$$B \wedge L \implies M$$
$$A \wedge P \implies L$$
$$A \wedge B \implies L$$
$$A$$
$$B$$

Forward: $A, B \to L$; $L, B \to M$; $L, M \to P$; $P \to Q$

Horn Clauses

$$\begin{aligned}
\forall \vec{x}_0 \,.\, \mathit{true} &\implies \mathrm{r}_0(\vec{x}_0) && (3)\\
\forall \vec{x}_1 \ldots \vec{x}_{n+1} \,.\, \bigwedge_{1 \le i \le n} \mathrm{r}_i(\vec{x}_i) &\implies \mathrm{r}_{n+1}(\vec{x}_{n+1}) && (4)\\
\forall \vec{x}_1 \ldots \vec{x}_{n+1} \,.\, \bigwedge_{1 \le i \le n} \mathrm{r}_i(\vec{x}_i) &\implies \mathit{false} && (5)
\end{aligned}$$

Applications

- Database: Specify dependencies

  Employees of same department should have same manager: $\forall e_1, d, m_1, e_2, d, m_2 \,.\, Row(e_1, d, m_1) \wedge Row(e_2, d, m_e) \implies Equal(m_1, m_2)$

- Specification of Data-structures

  $append(X, Y, Z)$
  $append(X, Y, Z) \implies$
  $append(cons(U, X), Y, cons(U, Z))$

- Aritifical Intelligence
  - Given a knowledge base as Horn Clauses, can we a sentence?

More details refer: LOGIC PROGRAMMING, Robert Kowalski

Exercise 2

```
int x = 0, y = 0;
while (*) {
  x = x + 1;
  y = y + x;
}
assert (y ≥ 0);
```

Listing 1: A Program from Understanding IC3

'*' denotes non-deterministic loop (i.e. the loop can run any number of iterations)
'int' is mathematical integer (i.e. no overflow)

Is the assert in program safe?

Inductive Invariants

```
int x = y = 0
while (*) {
  x = x + 1
  y = y + x
}
assert(y >= 0)
```

Program Reference: Understanding IC3

Inductive Invariants

```
int x = y = 0
while (*) {
  x = x + 1
  y = y + x
}
assert(y >= 0)
```

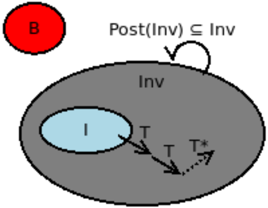
- Safe Inductive Invariants:

  $(x \geq 0 \wedge y \geq 0)$,

  $(x \geq 0 \wedge y - x \geq 0)$

Program Reference: Understanding IC3

Inductive Invariants



Inductive Invariants

Given: $\langle V \cup V', \mathit{Init}, \mathit{Tr} \rangle$ and Prop

$\{x, y, x', y'\}, x = 0 \land y = 0, x' = x + 1 \land y' = y + x'$
and $(y \geq 0)$

Find a relation inv such that:

- Initiation: $\forall V \,.\, \mathit{Init}(V) \Rightarrow \mathrm{inv}(V)$
- Consecution: $\forall V, V' \,.\, \mathrm{inv}(V) \land \mathit{Tr}(V, V') \Rightarrow \mathrm{inv}(V')$
- Safety: $\forall V \,.\, \mathrm{inv}(V) \Rightarrow \mathit{Prop}(V)$

How to specify this problem and how to solve it?
<u>Constrained</u> Horn Clauses

Syntax of Constrained Horn Clauses

A *CHC* over a set of uninterpreted relation symbols $\mathcal{R}$ has the form of one of the following three implications:

$$\forall \vec{x}_1 \,.\, \varphi(\vec{x}_1) \implies \mathrm{r}_1(\vec{x}_1) \tag{6}$$

$$\forall \vec{x}_0 \ldots \vec{x}_{n+1} \,.\, \bigwedge_{0 \le i \le n} \mathrm{r}_i(\vec{x}_i) \wedge \varphi(\vec{x}_0, \ldots, \vec{x}_{n+1}) \implies \mathrm{r}_{n+1}(\vec{x}_{n+1}) \tag{7}$$

$$\forall \vec{x}_0 \ldots \vec{x}_n \,.\, \bigwedge_{0 \le i \le n} \mathrm{r}_i(\vec{x}_i) \wedge \varphi(\vec{x}_0, \ldots, \vec{x}_n) \implies \mathit{false} \tag{8}$$

where:

- $\mathrm{r}_i \in \mathcal{R}$, $\vec{x}_i$ is a vector of variables of length $arity(r_i)$;
- for some i and j, such that $i \neq j$, it could be (though not necessary) that $\mathrm{r}_i = \mathrm{r}_j$;
- φ is a satisfiable quantifier-free formula in a theory T that does not contain any uninterpreted symbols.

Syntax of Constrained Horn Clauses

A *CHC* over a set of uninterpreted relation symbols $\mathcal{R}$ has the form of one of the following three implications:

$$\forall \vec{x}_1 \,.\, \varphi(\vec{x}_1) \implies \mathrm{r}_1(\vec{x}_1) \tag{6}$$

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$$\forall \vec{x}_0 \ldots \vec{x}_n \,.\, \bigwedge_{0 \le i \le n} \mathrm{r}_i(\vec{x}_i) \wedge \varphi(\vec{x}_0, \ldots, \vec{x}_n) \implies \mathit{false} \tag{8}$$

where:

- *body*(C) and *head*(C) denotes the left and right side of the implication in C, resp.;
- A CHC of type (6) is called *fact*, of type (7) *inductive*, and type (8) *query*
- C is *linear* if $|\mathit{rel}(\mathit{body}(C))| \le 1$; otherwise it is *non-linear*.

Semantics

- *interpretation* for $r \in \mathcal{R}$ is a map $\lambda x_1 \ldots \lambda x_{a_r}.\varphi(x_1, \ldots, x_{a_r})$, where φ is a quantifier-free formula without any symbols from $\mathcal{R}$.
- Extended to $\mathcal{R}$ by interpreting each symbol $r \in \mathcal{R}$
- $\varphi[M/\mathcal{R}]$ is the formula obtained by replacing each occurrence of a term of the form $r(x_1, \ldots, x_{a_r})$ by $M(r)(x_1, \ldots, x_{a_r})$
- A system S of CHCs over $\mathcal{R}$ is said to be *satisfiable* if there exists an interpretation M for $\mathcal{R}$ which makes all implications in S valid, i.e., for all $C \in S$, it holds that $\mathit{body}(C)[M/\mathcal{R}] \implies \mathit{head}(C)[M/\mathcal{R}]$.

CHCs - An Example

$$\forall x, y \,.\, x = 0 \land y = 0 \implies \mathrm{inv}(x, y)$$

$$\forall x_0, y_0, x_1, y_1 \,.\, \mathrm{inv}(x_0, y_0) \land x_1 = x_0 + 1 \land y_1 = y_0 + x_1 \implies \mathrm{inv}(x_1, y_1)$$

$$\forall x, y \,.\, \mathrm{inv}(x, y) \land \neg(y \geq 0) \implies \mathit{false}$$

CHCs - Inductive Invariants

- Given $\langle V \cup V', \mathit{Init}, \mathit{Tr} \rangle$ and Prop
  - $\{x, y, x', y'\}, x = 0 \wedge y = 0, x' = x + 1 \wedge y' = y + x'$ and $(y \geq 0)$
- Given $M := \{\text{inv} \mapsto \lambda x, y \,.\, x \geq 0 \wedge y \geq 0\}$
- Initiation: $\forall V \,.\, \mathit{Init}(V) \Rightarrow \text{inv}(V)$
  - $\forall x, y \,.\, (x = 0 \wedge y = 0) \Rightarrow (x \geq 0 \wedge y \geq 0)$
- Consecution: $\forall V, V' \,.\, \text{inv}(V) \wedge \mathit{Tr}(V, V') \Rightarrow \text{inv}(V')$
  - $\forall x, y, x', y' \,.\, (x \geq 0 \wedge y \geq 0) \wedge x' = x + 1 \wedge y' = y + x' \Rightarrow (x' \geq 0 \wedge y' \geq 0)$
- Safety: $\forall V \,.\, \text{inv}(V) \Rightarrow \mathit{Prop}(V)$
  - $\forall x, y \,.\, (x \geq 0 \wedge y \geq 0) \Rightarrow (y \geq 0)$

CHCs - Inductive Invariants

Universally Quantified! How to check satisfiability?
Negate the formulas and check satisfiability
Is any of these is SAT?

$$(x = 0 \land y = 0) \land \neg(x \geq 0 \land y \geq 0)$$

$$(x \geq 0 \land y \geq 0) \land (x' = x + 1 \land y' = y + x') \land (x' \geq 0 \land y' \geq 0)$$

$$(x \geq 0 \land y \geq 0) \land \neg(y \geq 0)$$

CHCs - Refutation

- Given $\langle V \cup V', Init, Tr\rangle$ and $Prop$
  $\{x, y, x', y'\}, x = 0 \wedge y = 0, x' = x + 1 \wedge y' = y + x'$ and $(y = x)$
- Initiation: $\forall V \,.\, Init(V) \Rightarrow \text{inv}(V)$
- Consecution: $\forall V, V' \,.\, \text{inv}(V) \wedge Tr(V, V') \Rightarrow \text{inv}(V')$
- Safety: $\forall V \,.\, \text{inv}(V) \Rightarrow Prop(V)$

$\rightarrow$ $x = 0 \wedge y = 0$

$\rightarrow$ $\text{inv}(0, 0)$, $\text{inv}(0, 0) \wedge x' = x + 1 \wedge y' = y + x'$

$\rightarrow$ $\text{inv}(1, 1)$

$\rightarrow$ $\text{inv}(1, 1)$, $\text{inv}(0, 0) \wedge x' = x + 1 \wedge y' = y + x'$

$\rightarrow$ $\text{inv}(2, 3)$

Exercise 3

Encode the following program as CHCs

```
int x = 0, y = 0
int m = n = *
assume(m ⩾ 0)
while (n≠0) {
   n--;
   if(*) x++;
   else y++;
}
while (x≠0){m--;x--;}
while(y≠0) {m--;y--;}
assert(m==0);
```

Exercise 3 - Solution

$$x = 0 \wedge y = 0 \wedge m \geq 0 \implies \mathrm{inv}_1(x, y, m, n)$$

$$\mathrm{inv}_1(x, y, m, n) \wedge \neg(n = 0) \wedge n' = n - 1 \wedge (x' = x + 1 \vee y' = y + 1)$$
$$\implies \mathrm{inv}_1(x', y', m, n)$$

$$\mathrm{inv}_1(x, y, m, n) \wedge n = 0 \implies \mathrm{inv}_2(x, y, m, n)$$

$$\mathrm{inv}_2(x, y, m, n) \wedge \neg(x = 0) \wedge m' = m - 1 \wedge x' = x - 1$$
$$\implies \mathrm{inv}_2(x', y, m', n)$$

Complexity

When T is LIA, LRA finding a solution is *undecidable*
Reference: The Universal Fragment of Presburger Arithmetic with Unary Uninterpreted Predicates is Undecidable, Horbach et al.

References for CHCs in Program Verification

Horn Clause Solvers for Program Verification

Nikolaj Bjørner, Arie Gurfinkel, Ken McMillan and Andrey Rybalchenko

Microsoft Research, Software Engineering Institute

Abstract. Automatic program verification and symbolic model checking tools interface with theorem proving technologies that check satisfiability of formulas. A theme pursued in the past years by the authors of this paper has been to encode symbolic model problems directly as Horn clauses and develop dedicated solvers for Horn clauses. Our solvers are called Duality, HSF, SeaHorn, and μZ and we have devoted considerable attention in recent papers to algorithms for solving Horn clauses. This paper complements these strides as we summarize main useful properties of Horn clauses, illustrate encodings of procedural program verification into Horn clauses and then highlight a number of useful simplification strategies at the level of Horn clauses. Solving Horn clauses amounts to establishing Existential positive Fixed-point Logic formulas, a perspective that was promoted by Blass and Gurevich.

1 Introduction

We make the overall claim that *Constrained Horn Clauses* provide a suitable basis for automatic program verification, that is, symbolic model checking. To sub-

Position paper

Synthesizing Software Verifiers from Proof Rules

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Encoding rules for various program constructs

So far:

- Horn Clauses
- Constrained Horn Clauses
- Invariant Inference as CHCs

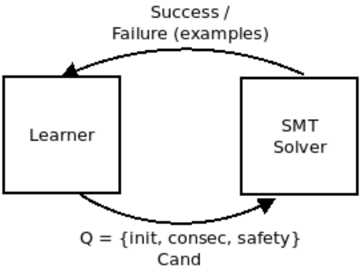
Next:

- How to solve CHCs?

Solving Constrained Horn Clauses using Syntax and Semantics

Reference: Fedyukovich, Prabhu, Madhukar, and Gupta, FMCAD 2018

Guess and Check Framework



Iterative learning: inv $\Leftrightarrow l_0 \wedge l_1 \wedge \cdots \wedge l_n$

A learner using Syntax Guided Synthesis

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

Invariants needed:

for first loop:
$(x + y + n = m)$

for second loop:
$(x + y + n = m) \wedge (n = 0)$

for third loop:
$(x + y + n = m) \wedge (n = 0) \wedge (x = 0)$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$$x = 0 \rightarrow x \geq 0, -x \geq 0$$

$$y = 0 \rightarrow y \geq 0, -y \geq 0$$

$$m \geq 0 \rightarrow m \geq 0$$

$$m = n \rightarrow m \geq n, -m \geq n$$

$$n \neq 0 \rightarrow -n > 0 \vee n > 0$$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$$\{x \geq 0, -x \geq 0, y \geq 0, -y \geq 0,$$
$$m \geq 0, m \geq n, -m \geq n,$$
$$-n > 0 \vee n > 0\}$$

$$c ::= 0$$
$$k ::= 1 \mid -1$$
$$v ::= x \mid y \mid m \mid n$$
$$e ::= k \cdot v \mid k \cdot v + k \cdot v$$
$$cand ::= e \geq c \mid e > c \vee e > c$$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$\{n \geq 0, -n \geq 0, -x > 0 \lor x > 0\}$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$$\{n \geq 0, -n \geq 0, -x > 0 \vee x > 0\}$$

$$c ::= 0$$
$$k ::= 1 \mid -1$$
$$v ::= x \mid n$$
$$e ::= k \cdot v$$
$$cand ::= e \geq c \mid e > c \vee e > c$$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$\{x \geq 0, -x \geq 0, -y > 0 \vee y > 0,$
$y \geq 0, -y \geq 0, m \geq 0, -m \geq 0\}$

Syntax Guided Synthesis

```
int x = y = 0
int m = n = *;
assume(m >= 0);

while (n != 0) {
  n--;
  if (*) then x++;
  else y++;
}

while (x != 0) {
  m--; x--;
}
while (y != 0) {
  m--; y--;
}
assert(m == 0);
```

$$\{x \geq 0, -x \geq 0, -y > 0 \vee y > 0,$$
$$y \geq 0, -y \geq 0, m \geq 0, -m \geq 0\}$$

$c ::= 0$
$k ::= 1 \mid -1$
$v ::= x \mid y \mid m$
$e ::= k \cdot v$
$cand ::= e \geq c \mid e > c \vee e > c$

Insufficiency of the grammars

$c ::= 0$
$k ::= 1 \mid -1$
$v ::= x \mid y \mid m \mid n$
$e ::= k \cdot v \mid k \cdot v + k \cdot v$
$cand ::= e \geq c \mid e >$
$c \vee e > c$

$(x + y + n = m)$

$c ::= 0$
$k ::= 1 \mid -1$
$v ::= x \mid n$
$e ::= k \cdot v$
$cand ::= e \geq c \mid e >$
$c \vee e > c$

$(x + y + n = m) \wedge (n = 0)$

$c ::= 0$
$k ::= 1 \mid -1$
$v ::= x \mid y \mid m$
$e ::= k \cdot v$
$cand ::= e \geq c \mid e >$
$c \vee e > c$

$(x + y + n = m) \wedge (n = 0)$
$\wedge (x = 0)$

Data Candidates

$$a*x + b*y + c*m + d*n + e = 0$$

$$\begin{pmatrix} x & y & m & n \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 5 & 4 \\ 2 & 0 & 5 & 3 \\ 2 & 1 & 5 & 2 \\ 2 & 2 & 5 & 1 \end{pmatrix}$$

$$x + y + n - m = 0$$

Propagation

$$\mathrm{inv}_1(x,y,m,n) \wedge n=0 \wedge$$
$$x_1=x \wedge y_1=y \wedge m_1=m \wedge n_1=n \Rightarrow$$
$$\mathrm{inv}_2(x_1,y_1,m_1,n_1)$$

No change in variables so candidates of inv_1 are likely to be candidates of inv_2

Propagation

$$inv_i(X) \wedge \phi(X,X') \Rightarrow inv_j(X')$$

Forward:

$$Cand_j = \exists X\ Cand_i(X) \wedge \phi(X,X')$$

Backward:

$$Cand_i = \exists X'\ Cand_j(X') \wedge \phi(X,X')$$

CHC References

- The Science, Art and Magic of Constrained Horn Clauses Arie, Gurfinkel and Nikolaj Bjørner
- Horn Clause Solvers for Program Verification, Bjørner, et al.
- Synthesizing Software Verifier from Proof Rules, Grebenshchikov, et al.