First-Order Logic

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25 January 2023

Outline of these lectures

Why study FO Logic in Computer Science?

- Some of the most exciting results, arguably greatest intellectual achievements, in Mathematics in the last century were to do with FO Logic (Gödel's Completeness and Incompleteness Theorems).
  - Far-reaching consequences in logic and computation
- Spawned the study of Computability (notions of computability, undecidable problems (Entscheidugsproblem or deciding logical consequence), many natural "complete" problems (like SAT) were from logic).
- FO arises naturally in Program Verification and Synthesis (Floyd-Hoare logic, array logics, symbolic techniques for verification and synthesis)
- Normal forms are useful in decision procedures for logical problems.
- Helps to clarify basic notions ("theory of arrays", "theory of linear integer arithmetic") in decision procedures for logic.

Outline of Topics in FO Logic

1. Syntax and Semantics of FO Logic
2. Normal Forms, Substitution lemma etc.
3. Sequent Calculus
4. Completeness
5. Compactness and Lowenheim-Skolem Theorem

Example FO Logic Formula

$$\forall x \exists y (x < y \land op(y, x) = e)$$

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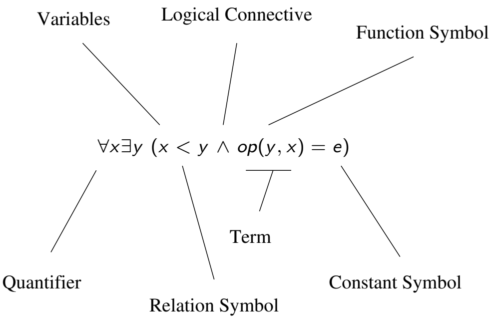


FO Signature

A First-Order signature is a tuple

$$S = (R, F, C)$$

where

- R is a countable set of relation symbols
- F is a countable set of function symbols
- C is a countable set of constant symbols

Each relational/functional symbol comes with an associated "arity".

Example FO signatures

- $S_{gr} = (\{\}, \{op^{(2)}\}, \{e\})$ (Groups)
- $S_{ogr} = (\{<^{(2)}\}, \{op^{(2)}\}, \{e\})$ (Ordered Groups)
- $S_{ar} = (\{\}, \{+^{(2)}, \cdot^{(2)}\}, \{0, 1\})$ (Arithmetic)
- $S_{eq} = (\{r^{(2)}\}, \{\}, \{\})$ (Equivalence Relations)

FO Alphabet

The FO alphabet induced by an FO signature $S = (R, F, C)$ is the set of symbols A_S which is the union of

- $R \cup F \cup C$ (symbols from the signature)
- $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$ (logical connectives)
- $\{(, \text{“,”},)\}$ (parenthesis and comma)
- $\mathbb{V} = \{v_0, v_1, \ldots\}$ (variables)

FO Terms

The set of *S*-terms T^S (of an FO signature S) is given by

$$t ::= x \mid c \mid f(t_1, \ldots, t_n)$$

where $x \in \{v_0, v_1, \ldots\}$ is a variable, $c \in C$ is a constant symbol, and $f \in F$ is a function symbol of arity n.

Example terms

- e, x, $op(e, e)$, $op(x, op(e, e))$ (S_{gr}-terms)
- 0, 1, x, $+(0, 1)$, $\cdot(0, +(0, 1))$ (S_{ar}-terms)

FO Formulas

The set of S-formulas L^S (of an FO signature S) is given by

$$\begin{aligned}\varphi \quad ::= \quad & t_1 = t_2 \mid r(t_1, \ldots, t_n) \quad \text{(Atomic Formulas)} \\ & \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \\ & \mid \exists x \varphi \mid \forall x \varphi\end{aligned}$$

where r is a relation symbol of arity n and $t_1, \ldots, t_n$ are S-terms.

Example formulas

- $\forall x(x = x)$ (S_{gr}-formula)
- $\forall x \exists y(<(x, y) \wedge op(y, x) = e)$ (S_{gr}-formula)
- $\exists y(x = +(y, y))$ (S_{ar}-formula)

Countability of Terms and Formulas

Theorem (Countability)

For any FO signature S, the set of S-terms and S-formulas are countable.

Recall that a set X is countable if there is an onto map from $\mathbb{N}$ to X (or equivalently, an injection from X to $\mathbb{N}$).

Argue that the set A_S^* is countable, hence L^S which is a subset of A_S^* is also countable.

Principle of Structural Induction for Formulas

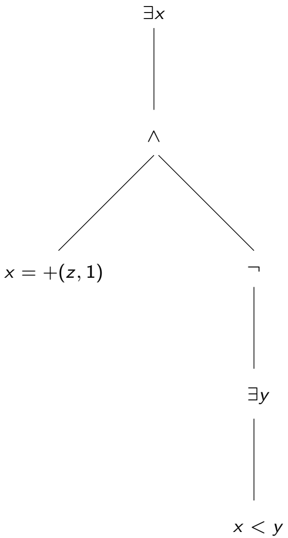
For any FO signature S, if a property P of S-formulas

- holds for all atomic S-formulas, and
- whenever it it holds for S-formulas φ and ψ, it also holds for $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, $\exists x\varphi$, and $\forall x\varphi$;

then P holds for all S-formulas.

Formula structure

$$\exists x(x = +(z,1) \land \neg\exists y(x < y))$$



Exercise

Exercise

Show that in any S_{gr}-formula, the number of opening and closing parenthesis must be equal.

Giving Semantics to FO Formulas

- In logic in general, formulas are interpreted in models or structures.
  Examples:
  - In Propositional Logic models are valuations:

$$\langle p_0 \mapsto \mathit{true}, p_1 \mapsto \mathit{false}, \ldots \rangle \models (p_0 \vee \neg p_1)$$

  - In Temporal Logic models are sequences of valuations:

$$\langle p \mapsto \mathit{true}, q \mapsto \mathit{false} \rangle \; \langle p \mapsto \mathit{false}, q \mapsto \mathit{false} \rangle \cdots \models G(p \rightarrow Fq)$$

- What kind of a model do we need to interpret the S_{gr}-formula

$$\forall x \exists y (op(y, x) = e)?$$

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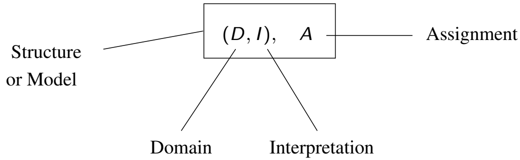
- What kind of a model do we need to interpret the S_{gr}-formula
  $$\forall x \exists y (op(y, x) = e)?$$
- In FO, a model looks like:



Example Structures

Example structures for $S_{gr} = (op, e)$.

- $(\mathbb{Z}, +, 0)$:
  - Domain D: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
  - Interpretation I:
    - $op \mapsto +$ (i.e. the binary function $+ : \mathbb{Z} \to \mathbb{Z}$ given by: $+(i, j) = i + j$).
    - $e \mapsto 0$
- $(\mathbb{Z}_3, (+ \bmod 3), 0)$:
  - Domain D: $\mathbb{Z}_3 = \{0, 1, 2\}$.
  - Interpretation I:
    - $op \mapsto (+ \mod 3)$ (i.e. the binary function "$(+ \mod 3)$" $: \mathbb{Z}_3 \to \mathbb{Z}_3$ given by: $(+ \mod 3)(i, j) = (i + j) \mod 3$).
    - $e \mapsto 0$

Truth of a formula in a model: intuitively

Exercise

Consider the S_{gr}-structure $(\mathbb{Z}, +, 0)$. Are the following formulas true in this model?

- $\forall x \exists y (op(x, y) = e)$
- $\exists x (op(x, x) = e)$
- $\forall x (op(x, x) = e)$

Domain and Interpretation

Let $S = (R, F, C)$ be an FO signature.
A Domain D is a non-empty set.
An S-interpretation I is a map that assigns

- to each relation symbol $r^{(n)} \in R$, a relation $I(r) \subseteq (D \times \cdots \times D)$
- to each function symbol $f^{(n)} \in R$, a function $I(f) : (D \times \cdots \times D) \to D$
- to each constant symbol $c \in C$, an element $I(c) \in D$.

Assignments for Variables

- How do we give meaning to the formula

$$\exists y(op(x, y) = e)?$$

  (The variable x is said to be "free" in this formula).

- Even if we were interested in "sentences" (formulas without free variables), still convenient to have assignments.

Let D be a domain.

- An assignment (in D) is a map $A : \mathbb{V} \to D$.
- For an assignment A, variable $x \in \mathbb{V}$, and $d \in D$, we use $A[d/x]$ to denote the assignment A' given by:

$$A'(y) = \begin{cases} A(y) & \text{if } y \neq x \\ d & \text{otherwise.} \end{cases}$$

Semantics of FO

Let S be an FO signature. An S-structure (or S-model) is a tuple $M = (D, I, A)$, where D is a domain, I an interpretation for symbols in S, and A is an assignment in D.

For an S-term t we define $M(t)$ to be the interpretation of t (the domain element that t maps to). Formally

$$
\begin{aligned}
M(c) &= I(c) \\
M(v) &= A(v) \\
M(f(t_1, \ldots, t_n)) &= I(f)(M(t_1), \ldots, M(t_n))
\end{aligned}
$$

We define the relation "$M \vDash \varphi$" (φ is satisfied in model M) by:

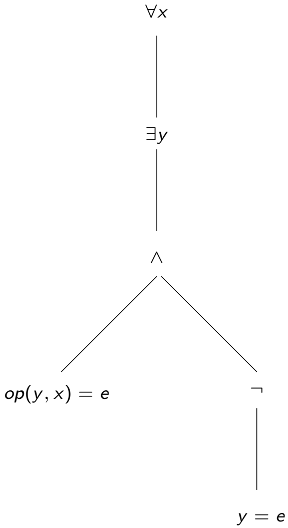
$M \vDash (t = t')$	iff	$M(t) = M(t')$
$M \vDash (r(t_1, \ldots, t_n))$	iff	$(M(t_1), \ldots, M(t_n)) \in I(r)$
$M \vDash \neg\varphi$	iff	$M \nvDash \varphi$
$M \vDash (\varphi \lor \psi)$	iff	$M \vDash \varphi$ or $M \vDash \psi$
$M \vDash \exists x \varphi$	iff	there is $d \in D$ s.t. $(D, I, A[d/x]) \vDash \varphi$
$M \vDash \forall x \varphi$	iff	for each $d \in D$ we have $(D, I, A[d/x]) \vDash \varphi$.

Semantics: Example

Find the truth of the S_{gr}-formula

$$\forall x \exists y ((op(y, x) = e) \wedge \neg(y = e))$$

in the structure $(\mathbb{Z}, +, 0)$.



Exercise

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Consider the FO signature $S = (r^{(1)}, f^{(2)})$. Give models that (a) satisfy and (b) don't satisfy the following formulas:

- $\forall y(f(x, y) = x)$
- $\exists x \forall y(f(x, y) = y)$
- $\exists x(r(x) \wedge \forall y\ r(f(x, y))$.

Satisfiability and Validity

- An S-formula φ is satisfiable if there is an S-model M such that $M \vDash \varphi$.
- An S-formula φ is valid if for every S-model M, we have $M \vDash \varphi$.

Proposition

φ is valid iff $\neg\varphi$ is not satisfiable.

Logical Implication and Equivalence

- φ implies ψ iff every model of φ is also a model of ψ.
- φ is logically equivalent to ψ, written $\varphi \equiv \psi$, iff the set of models of φ coincides with that of ψ.

Examples

- $r(x) \rightarrow r(y)$ is logically equivalent to $\neg r(x) \vee r(y)$.
- $\forall x \varphi$ is logically equivalent to $\neg \exists \neg \varphi$

Logical Consequence

- For a set T of S-formulas, we say a model M satisfies T, written "$M \vDash T$", iff $M \vDash \varphi$ for each $\varphi \in T$.
- For a set T of S-formulas, and an S-formula φ, we say φ is a logical consequence of T, written

$$T \vDash \varphi,$$

iff for every S-model M, whenever $M \vDash T$ we have $M \vDash \varphi$.

Examples

- $\{\exists y \forall x (op(x, y) = e)\} \vDash \forall x \exists y (op(x, y) = e)$
- $\{\forall x \exists y (op(x, y) = e)\} \nvDash \exists y \forall x (op(x, y) = e)$
- $\{\forall x \exists y (op(x, y) = e), \forall x (x = e)\} \vDash \exists y \forall x (op(x, y) = e)$

Theories

The theory of a set of S-formulas T, written "$Th(T)$", is the set of S-formulas that are logical consequences of T. That is:

$$Th(T) = \{\varphi \mid T \models \varphi\}.$$

Theory of Groups $Th(\Phi_{gr})$

Let Φ_{gr} be the set of formulas (group axioms) (using infix $\circ$ instead of op):

$$\forall x \forall y \forall z \; ((x \circ y) \circ z = x \circ (y \circ z)) \tag{1}$$

$$\forall x \; (x \circ e = x) \tag{2}$$

$$\forall x \exists y \; (x \circ y = e) \tag{3}$$

Then $Th(\Phi_{gr})$

- Contains $\forall x \exists y (op(y, x) = e)$, but
- Does not contain $\forall x \forall y (op(x, y) = op(y, x))$.

Theories

The theory of an S-structure M, written "Th(M)", is the set of S-formulas that are true in M:

$$Th(M) = \{\varphi \mid M \vDash \varphi\}.$$

Theory of Arithmetic $Th(\mathbb{N}, +, \cdot, 0, 1)$

- Contains $\forall x(x \cdot 0 = 0)$, but
- Does not contain $\exists y \forall x(x < y)$ (here $<(x, y)$ is shorthand for $\exists z((z \neq 0) \wedge (x + z = y)))$