Propositional Logic

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Outline of these lectures

Propositional Logic

Fix a countable set of propositional variables $Pr = \{p_0, p_1, \ldots\}$
Formulas of Propositional Logic are given by

Propositional Logic Formulas Syntax

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi)$$

Derived or shorthand operators

- *true*: $(p_0 \vee \neg p_0)$
- *false*: $\neg(p_0 \vee \neg p_0)$
- $\varphi \wedge \psi$: $\neg(\neg\varphi \vee \neg\psi)$
- $\varphi \rightarrow \psi$: $(\neg\varphi \vee \psi)$.

PL Semantics

- Valuation (or Assignment) is a map $s : Pr \to \{T, F\}$.
- "s satisfies φ" defined in the expected way (inductively).
  Examples: Let s be a valuation in which $p \mapsto T, q \mapsto F$.
  Then s satisfies $p \vee q$, but s does not satisfy $p \wedge q$,
- φ is satisfiable if there is a valuation which satisfies φ.
- φ is valid (or a tautology) if every valuation satisfes φ.
  Examples: $(p \vee \neg p)$, $(p \to q) \to (\neg q \to \neg p)$.
- A set of formulas Φ is satisfiable if there is a valuation which satisfies all the formulas in Φ.

Logical Consequence

- φ is a logical consequence of a set of formulas Φ (we also say "Φ entails φ"), written

$$\Phi \models \varphi,$$

  if every valuation s that satisfies all the formulas in Φ also satisfies φ.
- Example: $\{p, \neg p \vee q\} \models q$, but
- $\{p_0 \to p_1, p_1 \to p_2, \ldots\} \not\models p_5 \vee p_7$.

Sequent Calculus

- A sequent is a pair (Γ, φ), where Γ is a (possibly empty) finite sequence of formulas, and φ is a formula.
- We write (Γ, φ) as simply "$\Gamma\varphi$".
- (Γ, φ) or "$\Gamma\varphi$" must be read as a claim that "φ is a logical consequence of Γ".
- $\Gamma\varphi$ is correct if $\Gamma \models \varphi$ (more precisely the set of formulas in Γ entails φ).
- Example: $[p, \neg p \vee q]\ q$ is a correct claim.
- Example: $[p, \neg p \vee q]\ q \wedge r$ is not a correct claim.

Sequent Calculus $\mathcal{G}$: Rules I

Antecedant Rule (Ant):

$$\frac{\Gamma \ \varphi}{\Gamma' \ \varphi}$$

provided Γ is contained in Γ′.

Assumption Rule (Ass):

$$\frac{}{\Gamma \ \varphi}$$

provided φ belongs to Γ.

Proof by Cases Rule (PC):

$$\frac{\begin{array}{lcc}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lcc}\Gamma & & \varphi\end{array}}$$

Contradiction Rule (Ctr):

$$\frac{\begin{array}{lcc}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lcc}\Gamma & & \varphi\end{array}}$$

Sequent Calculus $\mathcal{G}$: Rules II

Or-Antecedant Rule (a) (Or-A-(a)):

$$\frac{\begin{array}{lll}\Gamma & \varphi & \theta \\ \Gamma & \psi & \theta\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \theta\end{array}}$$

Or-Antecedant Rule (b) (Or-A-(b)):

$$\frac{\begin{array}{lll}\Gamma & \varphi & \theta \\ \Gamma & \psi & \theta\end{array}}{\begin{array}{lll}\Gamma & (\psi \vee \varphi) & \theta\end{array}}$$

Or-Succeedent Rule (a) (Or-S-(a)):

$$\frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)}$$

Or-Succeedent Rule (b) (Or-S-(b)):

$$\frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

Derivations using Sequent Calculus

A derivation of a sequent $\Gamma \ \varphi$ (in the Sequent Calculus $\mathcal{G}$) is a sequence of sequents

$$\begin{array}{ll} \Gamma_0 & \varphi_0 \\ \Gamma_1 & \varphi_1 \\ \dots & \\ \Gamma_n & \varphi_n \end{array}$$

such that

1. $\Gamma_n \ \varphi_n = \Gamma \ \varphi$, and
2. each $\Gamma_i \ \varphi_i$ is obtained from the rules of $\mathcal{G}$, applied to sequents earlier in the sequence.

We write

$$\vdash_{\mathcal{G}} \Gamma \ \varphi,$$

(or simply $\vdash \Gamma \ \varphi$) to mean there is a derivation of $\Gamma \ \varphi$ in $\mathcal{G}$.

Example Derivation

The following derivation shows that $\vdash [\,]\ (p \vee \neg p)$:

1.	$[\,]\ p$	p	(by (Ass) rule)
2.	$[\,]\ p$	$(p \vee \neg p)$	(by (Or-S(a)) applied to 2, $\neg p$
3.	$[\,]\ \neg p$	$\neg p$	(by (Ass) rule)
4.	$[\,]\ \neg p$	$(p \vee \neg p)$	(by (Or-S(b)) applied to 3, p)
5.	$[\,]$	$(p \vee \neg p)$	(by (PC) applied to 2,4).

Exercise

Exercise

Show that

$$\vdash [p,\ \neg p \vee q]\ q.$$

New Derivation Rules

Second Contradiction Rule (Ctr'):

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

This rule is derivable from the rules in $\mathcal{G}$ in the sense that if $\vdash \Gamma\ \psi$ and $\vdash \Gamma\ \neg\psi$, then we also have $\vdash \Gamma\ \varphi$ (for any ψ and φ).

1.			
	$\vdots$		
7.	Γ	ψ	(Since $\Gamma\ \psi$ is derivable)
8.			
	$\vdots$		
12.	Γ	$\neg\psi$	(Since $\Gamma\ \neg\psi$ is derivable)
13.	$\Gamma\ \neg\varphi$	ψ	(by (Ant)(7, $\neg\varphi$))
14.	$\Gamma\ \neg\varphi$	$\neg\psi$	(by (Ant)(12, $\neg\varphi$))
15.	Γ	φ	(by (Ctr)(13,14))

Exercise

Exercise

Show that the following rule is derivable from the rules in $\mathcal{G}$:
(Modus-Ponens)

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi \vee \varphi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

Soundness and Completeness of Sequent Calculus

Theorem (Soundness and Completeness of $\mathcal{G}$)

$\vdash_{\mathcal{G}} \Gamma\ \varphi$ *if and only if* $\Gamma \vDash \varphi$.

We say

$$\Phi \vdash_{\mathcal{G}} \varphi$$

if for some $\Gamma \subseteq \Phi$ (more precisely the elements in Γ belong to Φ), we have $\vdash_{\mathcal{G}} \Gamma\ \varphi$.
Then:

Theorem (Strong Soundness and Completeness of $\mathcal{G}$)

$\Phi \vdash_{\mathcal{G}} \varphi$ *if and only if* $\Phi \vDash \varphi$.