Sequent calculus

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Outline

Propositional logic (PL) over symbols Pr (propositional variables).

$$A ::= p \in Pr \mid (\neg A) \mid (A \vee B) \mid (A \wedge B) \mid (A \rightarrow B) \mid (A \leftrightarrow B)$$

Let $false \stackrel{\text{def}}{=} p_0 \wedge (\neg p_0)$, $true \stackrel{\text{def}}{=} \neg false$, $A \wedge B \stackrel{\text{def}}{=} \neg((\neg A) \vee (\neg B))$, $A \rightarrow B \stackrel{\text{def}}{=} (\neg A) \vee B$

A propositional assignment s is a function assigning a boolean value $p[s]$ in $\{T, F\}$ to every propositional variable p in Pr.

This is lifted to formulas: every Boolean connective has a truth table (EFT, Section III.2) giving a truth value $A[s]$ to the formula A.

Example: $q \wedge (\neg(p \rightarrow r))[s] = F$ over assignment s given by $p[s] = F, q[s] = T, r[s] = T$.

Model checking (Alfred Tarski 1935)

Given an assignment s and formula A, the truth value $A[s]$ in $\{T, F\}$ can be given recursively, using the notation $s \models A$ (s satisfies A, or s is a model of A) for $A[s] = T$. s is a model of theory Th (set of formulas) if s satisfies every formula in Th.

$$
\begin{array}{lll}
s \models p & \text{iff} & p[s] = T \\
s \models \neg A & \text{iff} & \text{not } (s \models A) \\
s \models A \vee B & \text{iff} & s \models A \text{ or } s \models B \\
s \models A \wedge B & \text{iff} & s \models A \text{ and } s \models B \\
s \models A \to B & \text{iff} & (\text{if } s \models A \text{ then } s \models B) \\
s \models A \leftrightarrow B & \text{iff} & (s \models A \ \text{ iff } \ s \models B)
\end{array}
$$

Definition

A formula A is valid ($\models A$) if for every assignment s, $s \models A$. It is satisfiable ($Sat\ A$) if for some assignment s, $s \models A$.

Valid formulas: Double negation, De Morgan, Distributivity
$(\neg\neg A) \leftrightarrow A$; $\neg(A \vee B) \leftrightarrow ((\neg A) \wedge (\neg B))$;
$\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B))$; $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$;
$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$

Normal forms (Emil Post 1921)

Negation normal form (NNF) formula: $\neg$ only for atomic subformulas. A literal is propositional symbol p or negation $\neg p$.

Disjunctive normal form (DNF): disjunction of ≥ 1 conjunctions of literals. Checking satisfiability of formulas in DNF can be done in time linear in the length of the formula.
Example: $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$

Conjunctive normal form (CNF): conjunction of ≥ 1 disjunctions of literals. Checking satisfiability of formulas in CNF can be done in time exponential in the length of the formula. Any PL formula can be reduced to an equisatisfiable one in CNF in time polynomial in the length of the formula (Richard Karp 1972).

Example:
$$
\begin{aligned}
&(p_{11} \vee p_{12}) \wedge (p_{21} \vee p_{22}) \wedge (p_{31} \vee p_{32}) \\
&\wedge(\neg p_{11} \vee \neg p_{21}) \wedge (\neg p_{21} \vee \neg p_{31}) \wedge (\neg p_{11} \vee \neg p_{31}) \\
&\wedge(\neg p_{12} \vee \neg p_{22}) \wedge (\neg p_{22} \vee p_{32}) \wedge (\neg p_{12} \vee \neg p_{32})
\end{aligned}
$$

Set of sets of literals notation for CNF formulas:
$\langle [p_{11}, p_{12}]; [p_{21}, p_{22}]; [p_{31}, p_{32}]; [\neg p_{11}, \neg p_{21}]; [\neg p_{21}, \neg p_{31}]$
$; [\neg p_{11}, \neg p_{31}]; [\neg p_{12}, \neg p_{22}]; [\neg p_{22}, \neg p_{32}]; [\neg p_{12}, \neg p_{32}] \rangle$

Consequence (Alfred Tarski 1935)

Definition (Consequence)

Formula B is a consequence of Th ($Th \models B$) if for every assignment s which is a model of Th ($s \models Th$), s is also a model of B ($s \models B$).

In other words, every assignment which satisfies all the formulas in Th also satisfies the formula B.

Formula B is valid if $\emptyset \models B$ (it is a consequence of the empty theory), that is, every assignment is a model of B. Thus checking consequence is a generalization of checking validity.

Exercise

1. *Show for formulas A, B that $\{A, \neg A\} \models B$.*
2. *Let Fma be the set of all formulas. When does $Fma \models B$?*
3. *Let $Th = \{A, A \to B\}$. Show that $Th \models B$, but $\neg B$ is not a consequence of Th.*
4. *Let $Cl = \{B \mid Th \models B\}$. Show that $Cl \models A$ iff $Th \models A$.*

Derivations, derivability (Paul Bernays 1918)

- A sequent $\Gamma \ B$ is a pair: a (possibly empty) finite set Γ of antecedent formulas and a succedent formula B. The notation $\Gamma \ A_1 \ \ldots \ A_n \ B$ with more antecedent formulas $A_1, \ldots, A_n$ stands for $(\Gamma \cup \{A_1, \ldots, A_n\}) \ B$.
Example sequents: $[p \ (p \to q)] \ q$, $[p \ (p \to q)] \ r$.
- A derivation of sequent $\Gamma_n \ B_n$ ((EFT, Section IV.1) uses the notation $\vdash \Gamma_n \ B_n$ for derivation of sequents) is a nonempty sequence of steps, which are sequents:
$$\begin{array}{l} \Gamma_1 \ B_1 \\ \Gamma_2 \ B_2 \\ \ldots \\ \Gamma_n \ B_n \end{array}$$
- Each step is an instance of the conclusion of one of seven rules (below), from instances of (0 or more) premisses in the rule which appeared in earlier steps.
- Formula B is derivable from set Th ((EFT) uses $Th \vdash B$) if for some finite $\Gamma \subset Th$ there is a derivation of sequent $\Gamma \ B$.

- A sequent calculus G has rules for derivations. The one we use (in later slides) has two structural rules and five connective rules. For each rule we give a correctness argument, that is, if the premisses of the rule are consequences, then so is the conclusion.
- We also give some derived rules, by means of derivations.
- Here is an example derivation of $[\,](p \vee \neg p)$, which will be justified by the rules in the next slides. For convenience we write $[\,]$ to show the empty set of formulas even though it is not required.

1	$[\,]\ p\ p$	(Ass)
2	$[\,]\ p\ (p \vee \neg p)$	(1,Or-S)
3	$[\,]\ (\neg p)\ (\neg p)$	(Ass)
4	$[\,]\ (\neg p)\ (p \vee \neg p)$	(3,Or-S)
5	$[\,]\ (p \vee \neg p)$	(2,4,PC)

Structural rules (Gerhard Gentzen 1934)

- Assumption rule (Ass):
  Side condition: $A \in \Gamma$. Conclusion: $\Gamma \; A$.

  Correctness: s is a model of every formula in Γ, so $s \models A$.

  Example: You tell me you have studied mathematics all through school. I conclude you have studied IX standard math.
- Antecedent rule (Ant):
  Premiss: $1 \; \Gamma_1 \; B$. Side condition: $\Gamma_1 \subset \Gamma_2$.
  Conclusion: $\Gamma_2 \; B$.

  Correctness: Suppose premiss: if $s \models \Gamma_1$ then $s \models B$. If $s \models \Gamma_2$ then s is a model of every formula in Γ_2. Then it is a model of every formula in Γ_1 as well, so $s \models \Gamma_1$. By correctness of premiss, $s \models B$. So conclusion is correct.

  Example: You tell me you have studied math and science all through school. I conclude you studied IX standard math.

  Derivation:

1	$A_1 \; A_2 \; A_3 \; A_3$	(Ass)
2	$A_2 \; \neg A_2 \; A_3 \; A_1 \; A_4 \; A_3$	(1,Ant)

Connective rules (Gerhard Gentzen 1934)

- (True): Conclusion: Γ *true*.
- Disjunction rules for succedent (Or-S):

Premiss:	1 Γ A	1 Γ A
Conclusion:	Γ $(A \vee B)$	Γ $(B \vee A)$

  Correctness: By premiss, if $s \models \Gamma$ then $s \models A$. Definition of satisfaction says s satisfies the disjunction in any order.
- Disjunction rule for antecedent (Or-A):

  Premisses: 1 Γ A C
  2 Γ B C

  Conclusion: Γ $(A \vee B)$ C

  Correctness: Suppose $\Gamma, A \models C$ and $\Gamma, B \models C$. Let $s \models \Gamma$. Whether $s \models A$ or $s \models B$, both ways it follows that $s \models C$.

- Proof by cases (PC):

Premisses: 1 Γ A B
2 Γ $\neg A$ B

Conclusion: Γ B.

C'ness: Both ways B holds.

Example derivation:

1	[ ] p p	(Ass)
2	[ ] p $(p \vee \neg p)$	(1,Or-S)
3	[ ] $(\neg p)$ $(\neg p)$	(Ass)
4	[ ] $(\neg p)$ $(p \vee \neg p)$	(3,Or-S)
5	[ ] $(p \vee \neg p)$	(2,4,PC)

Connective rules (Gerhard Gentzen 1934)

- Proof by contradiction (Ctr):

  Premisses: $\Gamma \; \neg A \; B$
  $\Gamma \; \neg A \; \neg B$

  Conclusion: $\Gamma \; A$

  Correctness: Suppose $\Gamma, \neg A \models B$ and $\Gamma, \neg A \models \neg B$. Then no model for $\Gamma, \neg A$. So for any $s \models \Gamma$, s has to satisfy A.

Exercise (Wilfrid Hodges 1977)

19th-century mathematician James Smith thought he had proved that the number π is precisely $25/8$. He first assumed that π is $25/8$. Then he did a long derivation in which he did not find any contradictions. He claimed he had a proof. Did he?
Other mathematicians found contradictions from Smith's assumption. Could Smith use (Ctr) to argue that π is precisely $25/8$?

Theorem (Soundness of sequent calculus)

For every derivation of a sequent $\Gamma \; A$, it is the case that it is a consequence $\Gamma \models A$.

Proof.

By induction on the length of the derivation.

For derivations of length 1, the (Ass) and (True) rules are correct.

The other rules preserve correctness: if the premisses are correct, so is the conclusion. □

Corollary (Soundness of derivability)

For a formula A derivable from a set of formulas Th ($Th \vdash A$), it is the case that $Th \models A$.

We will eventually prove completeness of G: the converse of above.

Exercise

Give a derivation for the sequent $[\neg\neg p]\ p$.

Exercise

Give a derivation for the sequent $[p\ \neg p]\ q$.

Exercise (Ian Chiswell and Wilfrid Hodges 2007)

In 1989, the journal Private Eye lost a libel case and was instructed to pay 600,000 British pounds in damages. Coming out of the trial, editor Ian Hislop stood on the courthouse steps and said:

> *"If that is justice, then I am a banana!"*

Formulate this as a rule. Is there a derivation for it?

Solutions and derived rules for negation

- For both exercises we will use Proof by contradiction (Ctr):

Premisses: 1 $\Gamma\ \neg A\ B$ / 2 $\Gamma\ \neg A\ \neg B$ Conclusion: $\Gamma\ A$

- Derivation (and Double Negation (DN) derived rule):

1	$[\neg\neg p]\ \neg\neg p$	(Ass)	$P1$	$\Gamma\ (\neg\neg A)$	
2	$[\neg\neg p\ \neg p]\ \neg\neg p$	(1,Ant)	2	$\Gamma\ \neg A\ \neg\neg A$	(1,Ant)
3	$[\neg\neg p\ \neg p]\ \neg p$	(Ass)	3	$\Gamma\ \neg A\ \neg A$	(Ass)
4	$[\neg\neg p]\ p$	(3,2,Ctr)	$C4$	$\Gamma\ A$	(3,2,Ctr)

- Derivation (and 2nd Contradiction (Ctr') derived rule):

1	$[p\ \neg p]\ p$	(Ass)	$P1$	$\Gamma\ A$	
2	$[p\ \neg p]\ \neg p$	(Ass)	$P2$	$\Gamma\ \neg A$	
3	$[p\ \neg p\ \neg q]\ p$	(1,Ant)	3	$\Gamma\ \neg B\ A$	(1,Ant)
4	$[p\ \neg p\ \neg q]\ \neg p$	(2,Ant)	4	$\Gamma\ \neg B\ \neg A$	(2,Ant)
5	$[p\ \neg p]\ q$	(3,4,Ctr)	$C5$	$\Gamma\ B$	(3,4,Ctr)

- Hint: Think of “I am a banana!” as a representation of *false*.

Exercise

1. *Show the derived rule Contraposition (Cp):*
   Premiss: $1\ \Gamma\ A\ \neg B$. *Conclusion:* $\Gamma\ B\ \neg A$.
2. *Show the derived rule Disjunctive syllogism (DS):*
   Premisses: $\begin{array}{ll} 1 & \Gamma\ A \vee B \\ 2 & \Gamma\ \neg A \end{array}$ *Conclusion:* $\Gamma\ B$.

- Chain rule (Ch): Premisses: $\begin{array}{ll} 1 & \Gamma\ A \\ 2 & \Gamma\ A\ B \end{array}$ Concl: $\Gamma\ B$.

  Derivation: Applying (Ant), $\Gamma\ \neg A\ A$. From (Ass), $\Gamma\ \neg A\ \neg A$. (Ctr') gives $\Gamma\ \neg A\ B$. Use (PC).

- Double negation introduction (DNI):
  Premiss: $1\ \Gamma\ A$. Conclusion: $\Gamma\ \neg\neg A$.

  Derivation: By (Ass), $\Gamma\ \neg A\ \neg A$. By (Cp), $\Gamma\ A\ \neg\neg A$. (Ch) with the premiss gives the result.

Derived rule for conjunction

- And rule (And):

  Premisses: $1 \quad \Gamma \;\; B$

  $2 \quad \Gamma \;\; C$

  Conclusion: $\Gamma \;\; \neg(\neg B \vee \neg C)$ (taken as definition of $B \wedge C$)

  Derivation: Since by (Ass) $\Gamma \;\; (\neg B \vee \neg C) \;\; (\neg B \vee \neg C)$, the aim is to derive $\Gamma \;\; (\neg B \vee \neg C) \;\; \neg(\neg B \vee \neg C)$ and use (Ctr).

  By (1,Ant), $\Gamma \;\; (\neg B \vee \neg C) \;\; B$. By (DNI), $\Gamma \;\; (\neg B \vee \neg C) \;\; \neg\neg B$. By (DS), $\Gamma \;\; (\neg B \vee \neg C) \;\; \neg C$.

  By (2,Ant), $\Gamma \;\; (\neg B \vee \neg C) \;\; C$.
  (Ctr') rule gives the desired $\Gamma \;\; (\neg B \vee \neg C) \;\; \neg(\neg B \vee \neg C)$.

  Now use (Ctr) to get $\Gamma \;\; \neg(\neg B \vee \neg C)$.

Derived rules for implication

- Modus ponens (MP):

  Premisses: 1 $\Gamma\ (A \to B)$
  2 $\Gamma\ A$

  Conclusion: $\Gamma\ B$.

  Derivation: From (Ant) $\Gamma\ \neg A\ A$ and (Ass) $\Gamma\ \neg A\ \neg A$, by (Ctr') $\Gamma\ \neg A\ B$. From (Ass) $\Gamma\ B\ B$, so (Or-A) gives $\Gamma\ (A \to B)\ B$. (Ch) gives the result combining this with (1).

- Implies introduction (Imp):

  Premiss: 1 $\Gamma\ A\ B$. Conclusion: $\Gamma\ (A \to B)$.

  Derivation: From (Or-S) $\Gamma\ A\ (A \to B)$. Using (Ass) and (Or-S) $\Gamma\ \neg A\ (A \to B)$. (PC) gives the result.

What have we done so far?

- First defined the language of PL.
- Gave a semantics (meaning) of truth $s \models B$, satisfiability $Sat\ B$ and consequence $Th \models B$ using models (assignments) s.
- Gave an algorithm to check satisfiability of a formula B.
- Gave a syntactic (pattern-matching) generation of derivations $\Gamma\ B$ using a calculus G.
- Showed soundness of G, that if $Th \vdash_G B$ (for some finite $\Gamma \subset Th$, there is a derivation of $\Gamma\ B$ in G), then $Th \models B$.

How to go about for completeness, that if $Th \models B$ then $Th \vdash_G B$?

What is a strategy to find a proof of a hypothesized theorem?

Go back to 19th-century mathematician James Smith (Wilfrid Hodges 1977) who wanted to show that π is precisely $25/8$: look for a contradiction.