Consistency

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January 2025

Outline

What have we done so far?

- First defined the language of PL.
- Gave a model theory (meaning) of truth $s \models B$, satisfiability $Sat\ B$ and consequence $Th \models B$ using assignments s.
- Gave an algorithm to check satisfiability of a formula B.
- Gave a proof-theoretic (pattern-matching) generation of derivations $\Gamma\ B$ and derivability $Th \vdash B$ using a calculus G.
- Showed soundness of G, that if $Th \vdash_G B$ (for some finite $\Gamma \subset Th$, there is a derivation of $\Gamma\ B$ in G), then $Th \models B$.

What is the way to completeness, that if $Th \models B$ then $Th \vdash_G B$?

Exercise (Duality)

Show that A is valid if and only if $\neg A$ is not satisfiable, and A is satisfiable if and only if $\neg A$ is not valid.

Idea of consistency (Bernays, Post *c.*1920)

Proof by contradiction (Ctr):

Premises: $\Gamma \ \neg A \ B$

$\Gamma \ \neg A \ \neg B$

Conclusion: $\Gamma \ A$

2nd contradiction rule (Ctr'):

Premises: $\Gamma \ A$

$\Gamma \ \neg A$

Conclusion: $\Gamma \ B$

- In rule (Ctr), antecedent theory $\Gamma \cup \{\neg A\}$ is inconsistent.
- In rule (Ctr'), antecedent theory Γ is inconsistent.

Definition

Theory Th is called inconsistent ($Inc\ Th$) if for some B, $Th \vdash B$ and $Th \vdash \neg B$. Otherwise Th is called consistent ($Con\ Th$).

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Exercise (Chemistry report makes sense?)

When cobalt but no nickel is present, a brown colour appears in the solution.
Nickel and manganese are absent.
Cobalt is present but only a green colour appears.

Independence, consistency (Bernays, Post *c.*1920)

Is *Th* consistent?

$Th = \{(1) Co \wedge \neg Ni \rightarrow \neg green, (2) \neg Ni \wedge \neg Mn, (3) Co \wedge green\}$

Independence, consistency (Bernays, Post *c.*1920)

Is *Th* consistent?

$Th = \{(1) Co \wedge \neg Ni \rightarrow \neg green, (2) \neg Ni \wedge \neg Mn, (3) Co \wedge green\}$

Answer: By (3) and (And-S), $Th \vdash Co$. By (2) and (And-S), $Th \vdash \neg Ni$. By (And),(1) and (MP), $Th \vdash \neg green$. By (3) and (And-S), $Th \vdash green$.

Th is inconsistent.

Independence, consistency (Bernays, Post *c.*1920)

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Th is inconsistent.

Question: Is there an algorithm to check if a theory Th, or (for simplicity) just a singleton formula $\{A\}$, is consistent?

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Answer: By (3) and (And-S), $Th \vdash Co$. By (2) and (And-S), $Th \vdash \neg Ni$. By (And),(1) and (MP), $Th \vdash \neg green$. By (3) and (And-S), $Th \vdash green$.

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How to check $A \nvdash B$: formula B is *not* derivable from A ?

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Is *Th* consistent?

$Th = \{(1)Co \wedge \neg Ni \rightarrow \neg green, (2)\neg Ni \wedge \neg Mn, (3)Co \wedge green\}$

Answer: By (3) and (And-S), $Th \vdash Co$. By (2) and (And-S), $Th \vdash \neg Ni$. By (And),(1) and (MP), $Th \vdash \neg green$. By (3) and (And-S), $Th \vdash green$.

Th is inconsistent.

Question: Is there an algorithm to check if a theory *Th*, or (for simplicity) just a singleton formula $\{A\}$, is consistent?

How to check $A \nvdash B$: formula *B* is *not* derivable from *A* ?

How to get hold of closure of *A*, that is, set of all formulas generable from it using derivations ? And then check that *B* is not in that set ?

Inconsistency explodes, consistency extends

Lemma (Consistency (EFT, Section IV.7))

(Explosion) $Inc\ Th$ iff (if and only if) for every A, $Th \vdash A$.

(Closure) $Th \vdash A$ iff $Inc\ (Th \cup \{\neg A\})$.

(Extension) If $Con\ Th$, then either $Con\ (Th \cup \{A\})$ or $Con\ (Th \cup \{\neg A\})$.

Proof.

For (Explosion), right to left: For any B, $Th \vdash B$ and $Th \vdash \neg B$.
For left to right: suppose for some B, there is a derivation $\Gamma_1\ B$ and a derivation $\Gamma_2\ \neg B$. Paste these together to obtain derivations $(\Gamma_1 \cup \Gamma_2)\ B$ and $(\Gamma_1 \cup \Gamma_2)\ \neg B$.
Then use the 2nd Contradiction rule (Ctr') to derive any A.

Inconsistency explodes, consistency extends

Lemma (Consistency (EFT, Section IV.7))

(Explosion) $\mathit{Inc}\ \mathit{Th}$ *iff (if and only if) for every* A, $\mathit{Th} \vdash A$.

(Closure) $\mathit{Th} \vdash A$ *iff* $\mathit{Inc}\,(\mathit{Th} \cup \{\neg A\})$.

(Extension) *If* $\mathit{Con}\ \mathit{Th}$, *then either* $\mathit{Con}\,(\mathit{Th} \cup \{A\})$ *or* $\mathit{Con}\,(\mathit{Th} \cup \{\neg A\})$.

Proof.

For (Explosion), right to left: For any B, $\mathit{Th} \vdash B$ and $\mathit{Th} \vdash \neg B$.
For left to right: suppose for some B, there is a derivation $\Gamma_1 \;\; B$ and a derivation $\Gamma_2 \;\; \neg B$. Paste these together to obtain derivations $(\Gamma_1 \cup \Gamma_2) \;\; B$ and $(\Gamma_1 \cup \Gamma_2) \;\; \neg B$.
Then use the 2nd Contradiction rule (Ctr') to derive any A.

For (Closure), left to right: $(\mathit{Th} \cup \{\neg A\}) \vdash A \wedge \neg A$.
For right to left: by (Explosion), $(\mathit{Th} \cup \{\neg A\}) \vdash A$ using derivation $\Gamma \;\; \neg A \;\; A$. By (Ass), $\Gamma \;\; A \;\; A$. Proof by cases (PC) gives $\Gamma \;\; A$.

For (Extension), use the contrapositive.