Maximality and completeness

Heinz-Dieter Ebbinghaus, Jörg Flum, Wolfgang Thomas, *Mathematical logic*, Sections V.1,V.2,V.4,XI.5

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Two structural rules and five connective rules.

- Antecedent rule (Ant):
  Premiss: $1\ \Gamma_1\ A$. Side condition: $\Gamma_1 \subset \Gamma_2$.
  Conclusion: $\Gamma_2\ A$.
- Assumption rule (Ass):
  Side condition $A \in \Gamma$. Conclusion: $\Gamma\ A$.
- Disjunction rule for antecedent (Or-A):
  Two premisses: $1\ \Gamma\ A\ C; 2\ \Gamma\ B\ C$. Concl: $\Gamma\ (A \vee B)\ C$.
- Disjunction rules for succedent (Or-S):
  Premiss: $1\ \Gamma\ A$. Two conclusions: $\Gamma\ (A \vee B); \Gamma\ (B \vee A)$.
- Proof by cases (PC):
  Two premisses: $1\ \Gamma\ A\ B; 2\ \Gamma\ \neg A\ B$. Conclusion: $\Gamma\ B$.
- Proof by contradiction (Ctr):
  Two premisses: $1\ \Gamma\ \neg A\ B; 2\ \Gamma\ \neg A\ \neg B$. Conclusion: $\Gamma\ A$.

Derivations, derivability (Gerhard Gentzen 1934)

A derivability $Th \vdash_G A$ of calculus G has antecedent theory and succedent formula. A derivation is a finite sequence of steps, EFT IV.1 restricts antecedents to finite lists Γ. A step has a sequent $\Gamma\ B$, following from ≥ 0 previous sequents by applying a rule of G.

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Theorem (Soundness)

If $Th \vdash_G A$, then $Th \models A$.

If for some B, $Th \vdash_G B$ as well as $Th \vdash_G \neg B$, then Th is called inconsistent ($Inc_G\ Th$). Otherwise consistent ($Con_G\ Th$).

Lemma (Consistency (EFT, Section IV.7))

(Explosion) *$Inc\ Th$ iff (if and only if) for every A, $Th \vdash A$.*

(Closure) *$Th \vdash A$ iff $Inc\ (Th \cup \{\neg A\})$.*

(Extension) *If $Con\ Th$ then $Con\ (Th \cup \{A\})$ or $Con\ (Th \cup \{\neg A\})$.*

State (Adolf Lindenbaum 1927, Marshall Stone 1934)

Definition

Theory H is maximal consistent if for every A, $Con\,(H \cup \{A\})$ iff $A \in H$.

Lemma (Maximality)

For maximal consistent theory H:

1. *Either $A \in H$ or $\neg A \in H$. By (Extension), either $H \cup \{A\}$ or $H \cup \{\neg A\}$ is consistent.*
2. *If $H \vdash A$, then $A \in H$. By (Closure), $Inc\,(H \cup \{\neg A\})$. By (1), $A \in H$.*
3. *$A \vee B$ in H iff A in H or B in H. Right to left by Or-S. Left to right: if $A \vee B, \neg A \in H$ then $B \in H$ by (DS).*
4. *If $A \rightarrow B, A \in H$, then B in H (MP)*

Lindenbaum construction (EFT, Theorem V.2.2)

Lemma (Adolf Lindenbaum 1927, Alfred Tarski 1935)

Let Pr be countable. Every consistent theory Th can be extended to a maximal consistent theory H.

Proof.

Enumerate all Pr-formulas $A_1, A_2, \ldots$. Start with $H_0 = Th$.

$$H_i = \begin{cases} H_{i-1} \cup \{A_i\}, \text{ if } Con\,(H_{i-1} \cup \{A_i\}) \\ H_{i-1}, \text{ otherwise} \end{cases}$$

Note that every H_i is consistent by construction.

Claim: $H = \bigcup_{i \in \mathbb{N}} H_i$ is maximal consistent.

If H were inconsistent with $H \vdash B, H \vdash \neg B$, from finite $\Gamma_1, \Gamma_2 \subset H$ and derivations of sequents $\Gamma_1 \; B$ and $\Gamma_2 \; \neg B$, then all formulas of $\Gamma_1 \cup \Gamma_2$ appear by some stage N of the enumeration and get added to H_{N+1} which would make it inconsistent.

If H were not maximal, say $H \cup \{A\}$ was consistent and $A = A_i$ was left out of H_i, for some stage i. Then by the construction, $H_{i-1} \cup \{A_i\}$ was inconsistent, a contradiction. □

Way to completeness (Kurt Gödel 1930)

Converse to Soundness theorem is the Completeness/Adequacy theorem:
if $Th \models A$, then $Th \vdash A$.

We prove the contrapositive. Suppose not $Th \vdash A$. For contrapositive we have to show not $Th \models A$.

By Consistency Lemma (Closure), $Th \cup \{\neg A\}$ is consistent.

By Duality Exercise, sufficient to show that $Th \cup \{\neg A\}$ is satisfiable.

That is, it is sufficient to show:

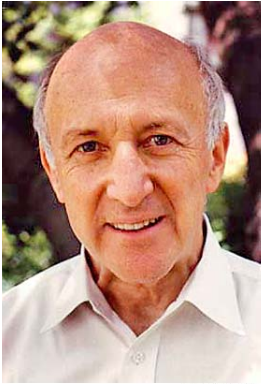
Theorem (Model construction)

For all theories Th, if $Con\ Th$ then $Sat\ Th$.

A similar argument shows the Soundness Theorem implies:
If $Sat\ Th$ then $Con\ Th$. Thus $Con\ Th$ iff $Sat\ Th$ is the proof-theoretic counterpart of satisfiability. Algorithm to check consistency.

Truth lemma (Leon Henkin 1949)

Using Lindenbaum Lemma, model construction reduces to:



George Bergman

Lemma (Truth, EFT, Thm V.1.10)

Maximal consistent theories H are satisfiable.

Proof.

Define model s: $p[s] \stackrel{\text{def}}{=} T$ iff $p \in H$.
By induction on formulas we prove $s \models A$ iff $A \in H$ (truth iff membership).

Base (propositional variables): by definition of s. □

For induction step, assume smaller formulas satisfy hypothesis.

- $s \models \neg A$ iff not ($s \models A$) iff $A \notin H$ (induction hypothesis) iff $\neg A \in H$ (Maximality 1).
- $s \models (A \vee B)$ iff $s \models A$ or $s \models B$ iff $A \in H$ or $B \in H$ (induction hypothesis) iff $(A \vee B) \in H$ (Maximality 3).

- Thus Model Construction Theorem proved using Duality Exercise, Lindenbaum Lemma, Truth Lemma.
- The lemmas used the combination of consistency and maximality to build a state (an assignment).

Consider consistent theory Th:
$\{(\neg(p \wedge \neg q)) \vee ((\neg p) \wedge r), \neg\neg q, r \wedge q, \neg t\}$.

Can extend it to a downwards consistent theory H_1:
$\{(\neg(p \wedge \neg q)) \vee ((\neg p) \wedge r), \neg(p \wedge \neg q), \neg p, \neg\neg q, q, r \wedge q, r, \neg t\}$.

Satisfying model s for H_1 from one maximal consistent set:
$p[s] = F, q[s] = T, r[s] = T, t[s] = F$.

Can extend it to another downwards consistent theory H_2:
$\{(\neg(p \wedge \neg q)) \vee ((\neg p) \wedge r), ((\neg p) \wedge r), \neg p, r, r \wedge q, q, \neg t\}$.

Satisfying model s for H_2 is the same:
$p[s] = F, q[s] = T, r[s] = T, t[s] = F$.

Definition

- *A literal ℓ is an atomic formula (positive) or its negation. Its complement is $\bar{\ell}$: $\bar{p} = \neg p$, $\overline{\neg p} = p$.*
- *A clause is a disjunction $K = [\ell_1, \ldots, \ell_n]$ where every ℓ_i is a literal. Empty clause $[]$ is False.*
- *A PL formula in conjunctive normal form (CNF) is a clause set $Th = \langle K_1; \ldots; K_m \rangle$ (sometimes we write $K_1 \ldots K_m$) where each K_i is a clause. Empty clause set $\langle\rangle$ is True.*
- *The following rewrite rules convert a formula to CNF.*

(DoubleNeg) $[\ldots, \neg\neg A, \ldots] \implies [\ldots, A, \ldots]$,
(Or) $[\ldots, A \vee B, \ldots] \implies [\ldots, A, B, \ldots]$,
(Implies) $[\ldots, A \rightarrow B, \ldots] \implies [\ldots, \neg A, B, \ldots]$,
(NotAnd) $[\ldots, \neg(A \wedge B), \ldots] \implies [\ldots, \neg A, \neg B, \ldots]$,
(And) $[\ldots, A \wedge B, \ldots] \implies \langle [\ldots, A, \ldots]; [\ldots, B, \ldots] \rangle$,
(NotOr) $[\ldots, \neg(A \vee B), \ldots] \implies \langle [\ldots, \neg A, \ldots]; [\ldots, \neg B, \ldots] \rangle$,
(NotImpl) $[\ldots, \neg(A \rightarrow B), \ldots] \implies \langle [\ldots, A, \ldots]; [\ldots, \neg B, \ldots] \rangle$.

Resolution (Gentzen 1934, Alan Robinson 1968)

(Resolution) on pivot A: $\langle [B, A]; [\neg A, C] \rangle \implies [B, C]$ resolvent.
(UnitRes/One-Literal) If $[\ell]$ in clause set Th, delete clauses of Th containing ℓ (including unit clause) and occurrences in Th of complement $\overline{\ell}$. (No choice for satisfying assignment!)

1	$[\neg(((p \wedge q) \vee (r \to s)) \to ((p \vee (r \to s)) \wedge (q \vee (r \to s))))]$	
2*a*	$[p \wedge q, r \to s]$	
2*b*	$[\neg((p \vee (r \to s)) \wedge (q \vee (r \to s)))]$	$1, \mathit{NotImpl} + \mathit{Or}$
3	$[p, r \to s]$ $[q, r \to s]$	$2a, \mathit{And}$
4	$[\neg(p \vee (r \to s)), \neg(q \vee (r \to s))]$	$2b, \mathit{NotAnd}$
5*a*	$[\neg p, \neg(q \vee (r \to s))]$	
5*b*	$[\neg(r \to s), \neg(q \vee (r \to s))]$	$4, \mathit{NotOr}$
6	$[\neg p, \neg q]$ $[\neg p, \neg(r \to s))]$	$5a, \mathit{NotOr}$
7	$[\neg(r \to s), \neg q]$ $[\neg(r \to s)]$	$5b, \mathit{NotOr}$
8	$[p, \neg q]$	$3a, 7a, \mathit{Res}\ r \to s$
9	$[\neg q]$	$6a, 8, \mathit{Res}\ p$
10	$[r \to s]$	$3b, 9, \mathit{UnitRes}\ q$
11	$[]$	$7b, 10, \mathit{Res}\ r \to s$

Resolution is refutation-complete (Robinson 1968)

SAT ALGORITHM (Martin Davis, Hilary Putnam 1960)
Preliminary steps: Put in block form, remove repetitions from clauses, order literals, delete clauses containing literal and its complement. Apart from (UnitRes) and (Resolution), also:

(Affirmative-negative) If some literal ℓ occurs only positively/ only negatively in clause set *Th*, delete clauses in *Th* with ℓ.

Theorem (Completeness)

If a CNF formula B is unsatisfiable, there is a refutation for B.

Proof by induction on number of variables in *B*.

- Base, no variables: must be [].
- Fix ℓ in *B*. If (AN),(UR) are not applicable, for all clauses $[C, \ell]; [D, \overline{\ell}]$, take the resolvent. Drop tautologies $[C, \ell, \overline{\ell}]$. Then the variable in ℓ does not occur in the result.

Either end with $\langle[]\rangle = false$ (a refutation), or with $\langle\rangle = true$, a contradiction as each rule preserved satisfiability.