Consistency and Completeness for First-Order Logic

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Outline of these lectures

Soundness of Sequent Calculus

Theorem (Soundness for sequents)

If $\vdash \Gamma\varphi$ *then* $\Gamma \vDash \varphi$.

Theorem (Soundness for derivations)

If $X \vdash \varphi$ *then* $X \vDash \varphi$.

Completeness of Sequent Calculus

Theorem (Completeness for sequents)

If $\Gamma \vDash \varphi$ *then* $\vdash \Gamma\varphi$.

Theorem (Completeness for derivations)

If $X \vDash \varphi$ *then* $X \vdash \varphi$.

Completeness (of a Bernays-Hilbert style proof system) was shown by Gödel in 1928. The proof we do is due to Henkin (1949).

Consistency

Fix an FO-signature S.

Definition (Consistency)

We say a set of S-formulas is consistent if it is not the case that $X \vdash \psi$ and $X \vdash \neg\psi$, for some S-formula ψ.

Examples:

- $\{r(x), r(y)\}$ is consistent (why?)
- $\{r(x), \neg r(x)\}$ is inconsistent (why?)

Consistency

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Examples:

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Observation: Every satisfiable set of formulas must be consistent.

Some Surprising Facts about Consistency

Lemma (Consistency)

1. *X is inconsistent iff for all φ, $X \vdash \varphi$.*
2. *X is consistent iff there is some φ, such that $X \nvdash \varphi$.*
3. *X is consistent iff all finite subsets of X are consistent.*

For all formulas φ:

4. *$X \vdash \varphi$ iff $X \cup \{\neg\varphi\}$ is inconsistent.*
5. *$X \vdash \neg\varphi$ iff $X \cup \{\varphi\}$ is inconsistent.*
6. *If X is consistent, either $X \cup \{\varphi\}$ is consistent or $X \cup \{\neg\varphi\}$ is consistent.*

Completeness of Sequent Calculus

Theorem (Completeness for derivations)

If $X \vDash \varphi$ then $X \vdash \varphi$.

Sufficient to show:

Theorem

If a set of formulas T is consistent, then it is satisfiable.

(Because $X \nvdash \varphi$
implies $X \cup \{\neg\varphi\}$ is consistent (by Consistency Lemma (4))
implies $X \cup \{\neg\varphi\}$ is satisfiable
implies $X \nvDash \varphi$.)

Key Idea of Proof

For a consistent set X, construct a term model, in which X is satisfied.

Basic plan:

- Show how to construct a term model M^X based on X.
- (Henkin's Theorem) If X is negation complete and contains witnesses, then

$$M^X \vDash \varphi \text{ iff } X \vdash \varphi.$$

- Show that for consistent X with finitely many free vars, we can extend X to X' which is negation-complete and contains witnesses.
- Now follows that X' (and hence X) is satisfiable.
- Reduce case of X with infinitely many free vars to finite case by using new constants.

Term Model

Let X be a consistent set of S-formulas. First attempt:

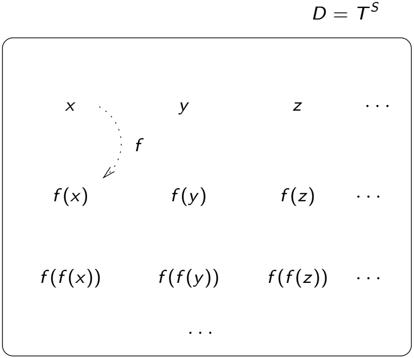
Definition

Define $M^X = (D, I, A)$ where

- $D = T^S$ is the set of all S-terms
- I is given by:
    - $I(c) = c$
    - $I(f)$ is given by: $I(f)(t) = f(t)$
    - $I(r) = \{(t_1, \ldots, t_n) \mid X \vdash r(t_1, \ldots, t_n)\}$.
- $A(x) = x$.

Term Model

Example term model for $S = (f^{(1)})$:



Term Model

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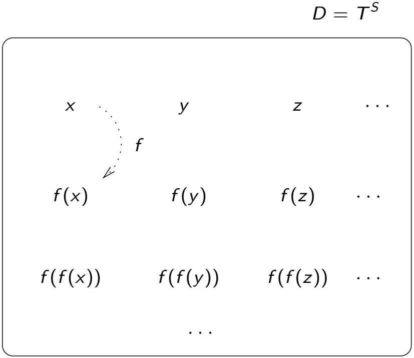


Issue with this: Can never satisfy $f(x) = f(y)$ when x and y are distinct variables.

Solution: Use equivalence classes of terms as domain elements.

Term Model: Better Attempt

Let X be a consistent set of S-formulas. Define equivalence $\sim_X$ (or simply $\sim$) on S-terms:

Definition (Equiv on terms)

$t \sim_X t'$ iff $X \vdash t = t'$.

Define $[t]_\sim$ (or simply $[t]$) to be equivalence class of a term t under $\sim$.

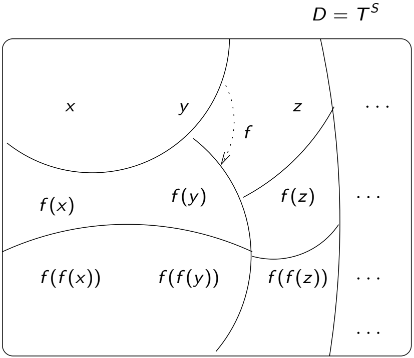
Definition (Term Model)

Define $M^X = (D, I, A)$ where

- D is equiv classes of $\sim$, i.e. $D = \{[t] \mid t \in T^S\}$
- I is given by:
  - $I(c) = [c]$
  - $I(f)$ is given by: $I(f)([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$
  - $I(r) = \{([t_1], \ldots, [t_n]) \mid X \vdash r(t_1, \ldots, t_n)\}$.
- $A(x) = [x]$.

Example Term Model

Example term model for $S = (f^{(1)})$, $X = \{x = y\}$:



Issues still to fix

Exercise

Consider $S = (r^{(1)})$. Describe M^X and tell whether it satisfies the formulas in X:

- $X = \{r(x) \vee r(y)\}$
- $X = \{\exists x\, r(x)\}$.

Issues still to fix

Exercise

Consider $S = (r^{(1)})$. Describe M^X and tell whether it satisfies the formulas in X:

- $X = \{r(x) \vee r(y)\}$
- $X = \{\exists x\, r(x)\}$.

These sets of formulas are not satisfied in their term models.

Negation Complete and Witnessing

Definition (Negation Complete)

A set of formulas X is negation complete if for each formula φ, we have $X \vdash \varphi$ or $X \vdash \neg\varphi$.

Definition (Witnessing)

A set of formulas X is said to contain witnesses if for each formula $\exists x\varphi$, there is a term t such that $X \vdash (\exists x\varphi \to \varphi[\frac{t}{x}])$

Henkin's Theorem

Theorem (Henkin)

Let X be a consistent, negation complete and witnessing set of formulas. Then

$$M^X \models \varphi \text{ iff } X \vdash \varphi.$$

Proof of Henkin's Theorem

By induction on structure of φ:

- For $\varphi = t = t'$
- For $\varphi = r(t_1, \ldots, t_n)$
- For $\varphi = \neg\psi$
- For $\varphi = \psi \vee \chi$
- For $\varphi = \exists x \psi$.

Extending consistent sets to negation complete and witnessing

Claim: Negation Complete

Every consistent set of formulas X can be extended to a consistent negation complete set of formulas X'.

Claim: Witnessing

Every consistent set of formulas X with a finite number of free vars can be extended to a consistent witnessing set of formulas X'.

Hence, as a corollary of Henkin's theorem:

Theorem (Consistent Satisfiability for finitely many free vars)

Every consistent set X with finitely many free vars, is satisfiable (in the term model $M^{X'}$ for the consistent, negation complete and witnessing extension X' of X).

Proof of Negation Complete Claim

Claim: Negation Complete

Every consistent set of formulas X can be extended to a consistent negation complete set of formulas X'.

Proof: Consider enumeration of all S-formulas $\varphi_0, \varphi_1, \ldots$, and define $Y_0 = X$ and

$$Y_{n+1} = \begin{cases} Y_n \cup \{\varphi_n\} & \text{if } Y_n \cup \{\varphi_n\} \text{ is consistent} \\ Y_n & \text{otherwise} \end{cases}$$

with $Y = \bigcup_{i \geq 0} Y_i$.
Argue that Y is consistent.
Y is negation complete: Consider $\varphi = \varphi_n$, and suppose $Y \nvdash \neg\varphi$.
Then $Y_n \cup \{\varphi\}$ must be consistent (by Consistency Lemma).
Hence $Y_{n+1} = Y_n \cup \{\varphi_n\}$. Hence $\varphi \in Y$ and $Y \vdash \varphi$.

Proof of Witnessing Claim

Claim: Witnessing

Every consistent set of formulas X with a finite number of free vars can be extended to a consistent witnessing set of formulas X'.

Proof: Let

$$\exists x_0 \varphi_0, \exists x_1 \varphi_1, \ldots$$

be an enumeration of all formulas beginning with $\exists$. For each $\exists x_n \varphi_n$ define witnessing formula

$$\psi_n = \exists x_n \varphi_n \rightarrow \varphi_n[\frac{y_n}{x_n}]$$

where y_n is smallest index var which does not occur free in X, $\exists x_0 \varphi_0, \ldots, \exists x_n \varphi_n$. Define $Y_n = X \cup \{\psi_o, \ldots, \psi_{n-1}\}$. Argue that $X' = \bigcup_{n \geq 0} Y_n$ is consistent by showing that each Y_n is consistent. (X' is clearly witnessing).

Each Y_n is Consistent

If not, let Y_{n+1} be the first inconsistent Y_i. Consider an arbitrary formula φ. Then $Y_{n+1} \vdash \varphi$, and hence for some $\Gamma \subseteq Y_n$:

$$\vdots$$

7.	$\Gamma\ (\neg\exists x_n\varphi_n \vee \varphi_n[\frac{y_n}{x_n}])$	φ	
8.	$\Gamma\ \neg\exists x_n\varphi_n$	φ	(derived Or-rule on 7)
9.	$\Gamma\ \varphi_n[\frac{y_n}{x_n}]$	φ	(derived Or-rule on 7)
10.	$\Gamma\ \exists x_n\varphi_n$	φ	(by $\exists$-Ant on 9, y_n not free in $\Gamma, \exists x_n\varphi_n$,
11.	Γ	φ	(by (PC) on 8,10)

Hence Γ (and hence Y_n) must be inconsistent, which is a contradiction.

My derived Or rule:

$$\frac{\Gamma\ (\psi \vee \chi)\quad \varphi}{\Gamma\ \psi\quad \varphi} \qquad \frac{\Gamma\ (\psi \vee \chi)\quad \varphi}{\Gamma\ \chi\quad \varphi}$$

Case of infinitely many free vars

Consider consistent X (with possibly infinitely many free vars)

- Consider a new signature $S' = S \cup \{c_0, c_1, \ldots\}$, where c_i's are new constants.
- For each S-formula φ define S'-formula φ' obtained from φ by substituting c_n for each free x_n in φ.
- Let $X' = \{\varphi' \mid \varphi \in X\}$.
- Argue that X' is consistent
- By Henkin's theorem for finite free vars case, X' (which contains no free vars) is satisfiable, say in a model $M = (D, I, A)$.
- Argue that X is satisfied in M.

Consistent Sets are Satisfiable

Theorem (Consistent Satisfiability)

Every consistent set X is satisfiable in a term model.