Definability in First-Order Logic

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03 February 2025

Outline

Logical Consequence

- For a set T of S-formulas, we say an S-model M satisfies T, written "$M \models T$", iff $M \models \varphi$ for each $\varphi \in T$.
- For a set T of S-formulas, and an S-formula φ, we say φ is a logical consequence of T, written

$$T \models \varphi,$$

iff for every S-model M, whenever $M \models T$ we have $M \models \varphi$.

Examples

- $\{\exists y \forall x (op(x,y) = e)\} \models \forall x \exists y (op(x,y) = e)$
- $\{\forall x \exists y (op(x,y) = e)\} \not\models \exists y \forall x (op(x,y) = e)$
- $\{\forall x \exists y (op(x,y) = e),\ \forall x (x = e)\} \models \exists y \forall x (op(x,y) = e)$

Free Variables in a Formula

A variable occurs "free" in a formula if it is not "in the scope" of any quantifier in the formula. Var x is free in φ if there is an occurence of x with no $\exists x$ or $\forall x$ ancestor in the formula tree of φ.

Example (free occurrences of vars are underlined)

$$\exists x(r(x, \underline{y}) \wedge \forall y(\neg(y = x) \vee r(y, \underline{z}))).$$

A variable can occur both free and bound (like y).

Let $var(t)$ denote the variables that occur in a term t. The set of vars that occur free in φ can be defined inductively by:

$$\begin{aligned}
free(t = t') &= var(t) \cup var(t') \\
free(r(t_1, \ldots, t_n)) &= var(t_1) \cup \cdots \cup var(t_n) \\
free(\neg\varphi) &= free(\varphi) \\
free(\varphi \vee \psi) &= free(\varphi) \cup free(\psi) \\
free(\exists x \varphi) &= free(\varphi) - \{x\} \\
free(\forall x \varphi) &= free(\varphi) - \{x\}.
\end{aligned}$$

Exercise

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What are the variables that occur free in

$$r(y, x) \rightarrow \forall y(\neg(y = z))?$$

Sentences and L_n^S

Let S be an FO signature and $n \in \mathbb{N}$. Then L_n^S denotes the set of S-formulas whose free variables are among $\{v_0, \ldots, v_{n-1}\}$. That is:

$$L_n^S = \{\varphi \in L^S \mid free(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}\}.$$

A sentence is a formula without any free variables (equivalently formulas in L_0^S).

Example $(r^{(2)}, f^{(2)}, c)$-sentence

$$\forall x \exists y (r(x, y) \wedge f(x, y) = c)$$

Sentences don't need the assignment component (A) of a model $M = (D, I, A)$, to determine their truth in the model.

Coincidence Lemma

Lemma

Let φ be an S-formula, and let $M_1 = (D, I_1, A_1)$ and $M_2 = (D, I_2, A_2)$ be two S-structures with a common domain, such that M_1 and M_2 agree on all the free variables and symbols in φ. Then

$$M_1 \models \varphi \text{ iff } M_2 \models \varphi.$$

Proof: Argue that

- For all S-terms t, if M_1 and M_2 agree on symbols and vars in t, then $M_1(t) = M_2(t)$. (By induction on structure of t.)
- For all S-formulas φ, if M_1 and M_2 have the same domain and agree on symbols and free vars in φ, then $M_1 \models \varphi$ iff $M_2 \models \varphi$. (By induction on structure of φ.)

Isomorphic Structures

Two S-structures $M = (D, I, A)$ and $M' = (D', I', A')$ are said to be isomorphic if there exists a bijection $\pi : D \to D'$ such that

- $(d_1, \ldots, d_n) \in I(r)$ iff $(\pi(d_1), \ldots, \pi(d_n)) \in I'(r)$.
- $\pi(I(f)(d_1, \ldots, d_n)) = I'(f)(\pi(d_1), \ldots, \pi(d_n))$
- $I'(c) = \pi(I(c))$.

In this case we write $M \cong M'$.

Natural numbers

The model $(\mathbb{N}, +, 0)$ is isomporphic to $(2 \cdot \mathbb{N}, +, 0)$.

FO cannot Distinguish Isomoporphic Structures

Theorem (Isomorphic Structures)

If M and M' are S-structures such that $M \cong M'$, then

$$M \models \varphi \text{ iff } M' \models \varphi$$

for all S-sentences φ.

Proof: Let $\pi : D \to D'$ be an isomorphism. For a D-assignment A consider the D'-assignment $A' = \pi \circ A$ (first A then π). Argue that

- for all S-terms t: $\pi(((D, I), A)(t)) = ((D', I'), A')(t)$.
- for all S-formulas φ: $((D, I), A) \models \varphi$ iff $((D', I'), A') \models \varphi$.

Theories

An *S-theory* is a set of S-sentences T which is closed under logical consequence.

The theory of a set of S-formulas T, written "$Th(T)$", is the set of S-sentences that are logical consequences of T. That is:

$$Th(T) = \{\varphi \in L_0^S \mid T \models \varphi\}.$$

Theory of Groups $Th(\Phi_{gr})$

Let Φ_{gr} be the set of formulas (group axioms):

$$\forall x \forall y \forall z \; (op(op(x, y), z) = op(x, op(y, z)) \quad (1)$$

$$\forall x \; (op(x, e) = x) \quad (2)$$

$$\forall x \exists y \; (op(x, y) = e) \quad (3)$$

Then $Th(\Phi_{gr})$

- Contains $\forall x \exists y (op(y, x) = e)$, but
- Does not contain $\forall x \forall y (op(x, y) = op(y, x))$.

Exercise

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Consider the axioms of equivalence relations, Φ_{eq}, over the signature $S_{eq} = (r^{(2)})$:

$$\forall x \, r(x, x)$$
$$\forall x \forall y \, (r(x, y) \rightarrow r(y, x))$$
$$\forall x \forall y \forall z \, ((r(x, y) \wedge r(y, z)) \rightarrow r(x, z))$$

Which of the following sentences are in $Th(\Phi_{eq})$?

- $\forall x \exists y \, r(x, y)$
- $\forall x \forall y (\exists z \, (r(x, z) \wedge r(y, z)) \rightarrow \forall w (r(x, w) \rightarrow r(y, w)))$.

Theory of a Structure

The theory of an S-structure M, written "$Th(M)$", is the set of S-sentences that are true in M:

$$Th(M) = \{\varphi \in L_0^S \mid M \models \varphi\}.$$

Theory of Arithmetic $Th(\mathbb{N}, +, \cdot, 0, 1)$

- Contains $\forall x(x \cdot 0 = 0)$, but
- Does not contain $\exists y \forall x(x < y)$ (here "$< (x, y)$" is shorthand for $\exists z((z \neq 0) \wedge (x + z = y)))$

Logical Definability (EFT Secs. III.6, VI.3-4)

Are there sets of FO-sentences that characterize

- A class of structures (like groups, equivalence relations, torsion groups, etc)
- A particular structure like $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$
- Relations like "$<$" in reals, via an FO-formula $\varphi(x, y)$?

Elementary Definability

"Elementary" = "FO-definable"

Let S be an FO-signature.

Definition (Elementary)

A class of S-structures $\mathcal{C}$ is called elementary if there is an S-sentence φ such that $\mathcal{C} = \{M \mid M \models \varphi\}$.

Definition (Δ-Elementary)

A class of S-structures $\mathcal{C}$ is called Δ-elementary if there is a set of S-sentences Φ such that $\mathcal{C} = \{M \mid M \models \Phi\}$.

Definition (Elementary Equivalence)

Two S-structures M and M' are called elementarily equivalent if $Th(M) = Th(M')$.

Some Elementarily Definable Classes of Structures

Cardinality Properties:

- Class of models with 1-element domains is elementary: $\exists x \forall y (y = x)$
- $\varphi_{\geq 2} = \exists v_0 \exists v_1 (\neg v_0 = v_1)$ says that "there are at least two elements in the domain".
- "are at least n elements in the domain"?
- Class of models with infinite domains is Δ-elementary: Take $\Phi_\infty = \{\varphi_{\geq 2}, \varphi_{\geq 3}, \ldots\}$
- "finitely many elements in the domain"?

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- "are at least n elements in the domain"?
- Class of models with infinite domains is Δ-elementary: Take $\Phi_\infty = \{\varphi_{\geq 2}, \varphi_{\geq 3}, \ldots\}$
- "finitely many elements in the domain"?
  Not Δ-elementary! (Proof 2 slides ahead)
- As a consequence, "infiniteness" cannot be elementary.

Exercise

Exercise

Characterize using FO sentences:

- r is an equivalence relation with at least two equivalence classes
- r is an equivalence relation with an equivalence class containing at least two elements.

Compactness

We will later show as an easy consequence of the proof of Gödel's Completeness Theorem:

Theorem (Compactness)

If a set of formulas X is unsatisfiable, then there must be a finite subset of X that is unsatisfiable.

Non-Δ-Elementariness of finiteness

Suppose the class $\mathcal{C}$ of all finite S-structures was Δ-elementary via a set of sentences Φ.

- Consider $\Psi = \Phi \cup \Phi_\infty$.
- Ψ must be unsatisfiable.
- By Compactness Theorem, a finite subset X_0 of Ψ must also be unsat.
- But we can easily construct a model for X_0 (if $\varphi_{\geq 17}$ is largest sentence from Φ_∞ in X_0, then a model M with a domain of 17 elements will satisfy X_0).
- This is a contradiction. Hence $\mathcal{C}$ could not have been Δ-elementary.

Non Δ-elementariness of $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$

The class of models that are isomorphic to $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$ is not Δ-elementary.

A non-standard model for arithmetic is a model M which not isomorphic to $\mathcal{N}$, but is elementarily equivalent to $\mathcal{N}$ (i.e. $Th(M) = Th(\mathcal{N})$).

Theorem (Skolem)

There is a non-standard countable model for arithmetic.

Proof: Consider the set of formulas

$$\Psi = Th(\mathcal{N}) \cup \{\neg(x = 0), \neg(x = 1), \neg(x = \underline{2}), \ldots\}.$$

Here "$\underline{2}$" denotes the term $(1 + 1)$, etc.

Non Δ-elementariness of $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$ (Proof ctd)

- Since every finite subset of Ψ is sat (in $\mathcal{N}$ itself) Ψ must be sat in a model $M = (D, I, A)$.
- Follows that $\mathcal{N}$ and M are elementarily equivalent.
- But M cannot be isomorphic to $\mathcal{N}$ (as any isomporphism π from $\mathbb{N}$ to D must map $\underline{2}$ to $\underline{2}$, etc; and hence $A(x)$ would not be the image of any element under π).

Non Δ-elementariness of classes of groups

- The class of finite groups is not Δ-elementary.
- The class of torsion groups (where every element x is such that $x^n = x \circ x \cdots \circ x = e$ for some n) is not Δ-elementary.