Sequent Calculus for First-Order Logic

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Outline of these lectures

Sequents

- A sequent is a pair (Γ, φ), where Γ is a (possibly empty) finite sequence of formulas, and φ is a formula.
- We write (Γ, φ) as simply "$\Gamma\varphi$".
- (Γ, φ) or "$\Gamma\varphi$" must be read as a claim that "φ is a logical consequence of Γ".
- $\Gamma\varphi$ is correct if $\Gamma \vDash \varphi$ (more precisely the set of formulas in Γ entails φ).

Example Sequents

- $[r(x), \neg r(x) \vee r(y)]\ r(y)$ is a correct sequent.
- $[r(x), \neg r(x) \vee r(y)]\ (r(y) \wedge r(z))$ is not a correct sequent.

Sequent Calculus $\mathcal{G}$: Rules I

Assumption Rule (Ass):

$$\frac{}{\Gamma \;\; \varphi}$$

provided φ belongs to Γ.

Antecedant Rule (Ant):

$$\frac{\Gamma \;\; \varphi}{\Gamma' \;\; \varphi}$$

provided Γ is contained in Γ'.

Proof by Cases Rule (PC):

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Contradiction Rule (Ctr):

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Sequent Calculus $\mathcal{G}$: Rules II

Or-Antecedant Rule (a)
(Or-A-(a)):

$$\frac{\begin{array}{lll}\Gamma & \varphi & \theta \\ \Gamma & \psi & \theta\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \theta\end{array}}$$

Or-Succeedent Rule (a)
(Or-S-(a)):

$$\frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)}$$

Or-Succeedent Rule (b)
(Or-S-(b)):

$$\frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

Sequent Calculus $\mathcal{G}$: Rules III

$\exists$-Introduction in Succedent Rule ($\exists$-Succ):

$$\frac{\Gamma \quad \varphi[\frac{t}{x}]}{\Gamma \quad \exists x \varphi}$$

$\exists$-Introduction in Antecedent Rule ($\exists$-Ant):

$$\frac{\Gamma \quad \varphi[\frac{y}{x}] \quad \psi}{\Gamma \quad \exists x \varphi \quad \psi}$$

provided y is not free in Γ, $\exists x \varphi$, and ψ.

Reflexivity Rule ($=$):

$$\frac{}{t = t}$$

Substitution Rule (Sub):

$$\frac{\Gamma \qquad \qquad \varphi[\frac{t}{x}]}{\Gamma \quad t = t' \quad \varphi[\frac{t'}{x}]}$$

Soundness of Rules

∃-Succ:

$$\frac{\Gamma \ \varphi[\frac{t}{x}]}{\Gamma \ \exists x\varphi}$$

Proof: Suppose $M \models \Gamma$.
Then $M \models \varphi[\frac{t}{x}]$.
Hence $M[\frac{M(t)}{x}] \models \varphi$ (by Subs. Lemma)
Hence $M \models \exists x\varphi$ (semantics of $\exists x\varphi$).

∃-Ant:

$$\frac{\Gamma \ \varphi[\frac{y}{x}] \ \psi}{\Gamma \ \exists x\varphi \ \psi}$$

provided y is not free in Γ, $\exists x\varphi$, and ψ.

Proof: Consider cases where (a) $y = x$ and (b) $y \neq x$ (here consider $M[d/y]$).

Derivations using Sequent Calculus

A derivation of a sequent $\Gamma \; \varphi$ (in the Sequent Calculus $\mathcal{G}$) is a sequence of sequents

$$\begin{array}{c} \Gamma_0 \; \varphi_0 \\ \Gamma_1 \; \varphi_1 \\ \dots \\ \Gamma_n \; \varphi_n \end{array}$$

such that

1. $\Gamma_n \; \varphi_n = \Gamma \; \varphi$, and
2. each $\Gamma_i \; \varphi_i$ is obtained from the rules of $\mathcal{G}$, applied to sequents earlier in the sequence.

We write

$$\vdash_{\mathcal{G}} \Gamma \; \varphi,$$

(or simply $\vdash \Gamma \; \varphi$) to mean there is a derivation of $\Gamma \; \varphi$ in $\mathcal{G}$.

Example Derivation

The following derivation shows that $\vdash [\,]\ (r(x) \vee \neg r(x))$:

1.	$[\,]\ r(x)$	$r(x)$	(by (Ass) rule)
2.	$[\,]\ r(x)$	$(r(x) \vee \neg r(x))$	(by (Or-S(a)) applied to 1, $\neg r(x)$)
3.	$[\,]\ \neg r(x)$	$\neg r(x)$	(by (Ass) rule)
4.	$[\,]\ \neg r(x)$	$(r(x) \vee \neg r(x))$	(by (Or-S(b)) applied to 3, $r(x)$)
5.	$[\,]$	$(r(x) \vee \neg r(x))$	(by (PC) applied to 2,4).

Example Derivations

To show: $\vdash \ [e \circ e = e] \ \exists x(x \circ x = e)$

1.	$[e \circ e = e]$	$e \circ e = e$	(by (Ass) rule)
2.	$[e \circ e = e]$	$\exists x(x \circ x = e)$	(by ($\exists$-Succ) rule on (1) with $e \circ e = e$ viewed as $(x \circ x = e)[\frac{e}{x}]$)

Example Derivations

To show: $\vdash \ [e \circ e = e] \ \exists x(x \circ x = e)$

1.	$[e \circ e = e]$	$e \circ e = e$	(by (Ass) rule)
2.	$[e \circ e = e]$	$\exists x(x \circ x = e)$	(by ($\exists$-Succ) rule on (1) with $e \circ e = e$ viewed as $(x \circ x = e)[\frac{e}{x}]$)

To show $\neg :\vdash [\forall x(x \circ e = x)] \ (e \circ e = e)$

1.	$[\neg\exists x \neg(x \circ e = x)] \ \neg(e \circ e = e)$	$\neg\exists x \neg(x \circ e = x)$	by Ass
2.	$[\neg\exists x \neg(x \circ e = x)] \ \neg(e \circ e = e)$	$\neg(e \circ e = e)$	by Ass
3.	$[\neg\exists x \neg(x \circ e = x)] \ \neg(e \circ e = e)$	$\exists x \neg(x \circ e = x)$	by $\exists$-S(2),
	viewing $\neg(e \circ e = e)$ as	$\neg(x \circ e = x)[\frac{e}{x}]$	
4.	$[\neg\exists x \neg(x \circ e = x)]$	$(e \circ e = e)$	by Ctr(1,3)

New Derivation Rules

$\forall$-Rule:

$$\frac{\Gamma \quad \forall x \varphi}{\Gamma \quad \varphi[\frac{t}{x}]}$$

This rule is derivable from the rules in $\mathcal{G}$ in the sense that if $\vdash \Gamma \; \forall x \varphi$, then we also have $\vdash \Gamma \; \varphi[\frac{t}{x}]$ (for any term t, φ, and x).

1.			
	$\vdots$		
7.	Γ	$\neg\exists x \neg\varphi$	(Since $\Gamma \; \neg\exists x \neg\varphi$ is derivable)
8.	$\Gamma \; \neg\varphi[\frac{t}{x}]$	$\neg\exists x \neg\varphi$	by Ant(7)
9.	$\Gamma \; \neg\varphi[\frac{t}{x}]$	$\neg\varphi[\frac{t}{x}]$	by Ass
10.	$\Gamma \; \neg\varphi[\frac{t}{x}]$	$\exists x \neg\varphi$	by $\exists$-S(9)
11.	Γ	$\varphi[\frac{t}{x}]$	by Ctr(8,10)

New Derivation Rules

∃-Rules:

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \exists x \varphi}$$

1.			
	$\vdots$		
7.	Γ	φ	Since $\Gamma \ \varphi$ is derivable by assumption
7′.	Γ	$\varphi[\frac{x}{x}]$	by viewing φ as $\varphi[\frac{x}{x}]$
8.	Γ	$\exists x \varphi$	by ∃-S(7)

Exercise

Exercise

Show that the following rules are derivable:

$$\frac{\Gamma\ \varphi \qquad \psi}{\Gamma\ \exists x\varphi \quad \psi}$$

provided x is not free in Γ, ψ.

$$\frac{\Gamma \qquad\qquad \varphi}{\Gamma\ x = t \quad \varphi[\frac{t}{x}]}$$

$$\frac{\Gamma \quad t = t'}{\Gamma \quad t' = t}$$

$$\frac{\begin{array}{ll}\Gamma & t_1 = t_2 \\ \Gamma & t_2 = t_3\end{array}}{\Gamma \quad t_1 = t_3}$$

Derivability

Let Φ be a set of formulas, and φ a formula. We say φ is *derivable* (or formally provable) from Φ, written

$$\Phi \vdash \varphi,$$

if there exists a finite sequence Γ from Φ, such that $\vdash \Gamma \, \varphi$.

Example

Left Inverse in Groups

Let Φ_{gr} be the set of formulas (group axioms):

$$\forall x \forall y \forall z\ (op(op(x,y),z) = op(x, op(y,z))) \tag{1}$$

$$\forall x\ (op(x,e) = x) \tag{2}$$

$$\forall x \exists y\ (op(x,y) = e) \tag{3}$$

Then $\Phi_{gr} \models \forall x \exists y (op(y,x) = e)$

Informal proof: Let $(D, \circ)$ be a structure satistying Φ_{gr}. Let x be an arbitrary element of D. Let y be the right inverse of x (i.e. $x \circ y = e$). Let z be the right inverse of y. Then

$$\begin{aligned} y \circ x &= (y \circ x) \circ e \\ &= (y \circ x) \circ (y \circ z) \\ &= (y \circ (x \circ y)) \circ z \\ &= (y \circ e) \circ z \\ &= y \circ z \\ &= e. \end{aligned}$$

Formal Proof in $\mathcal{G}$

Let Γ_{gr} denote the list of the 3 group axioms Φ_{gr}.

1.	Γ_{gr}	$\forall x(x \circ e = x)$	(Ass)
2.	Γ_{gr}	$(y \circ x) \circ e = (y \circ x)$	(Derived $\forall$ rule)
3.	Γ_{gr}	$(y \circ x) = (y \circ x) \circ e$	(Derived term rule)
4.	Γ_{gr} $e = y \circ z$	$(y \circ x) = (y \circ x) \circ (y \circ z)$	Sub(3)
	$\vdots$		
34.	Γ_{gr}	$\forall x \exists y (y \circ x) = e$	(Ant).