Substitution and Prenex Normal Form

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Outline

Substitution in Formulas (EFT III.8)

Let φ be an S-formula with a free var x. Let t be an S-term. Then the substitution of t for x in φ, written $\varphi[t/x]$, is the S-formula obtained by replacing each free occurrence of x in φ by t.

If $M = (D, I, A)$ is a model, the formula $\varphi[t/x]$ should say the same thing about the domain element $M(t)$ in M that φ says about $A(x)$ in M.

Example

Let φ be the formula

$$\exists z(z + z = x).$$

- Then $\varphi[y/x]$ is the formula

$$\exists z(z + z = y).$$

- What should $\varphi[z/x]$ be?

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$$\exists z(z + z = x).$$

- Then $\varphi[y/x]$ is the formula

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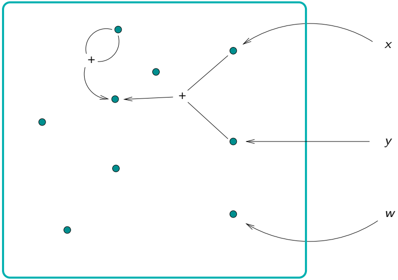
- What should $\varphi[z/x]$ be? Not the formula $\exists z(z + z = z)$, but something like

$$\exists u(u + u = z).$$

Illustrating Substitution Lemma

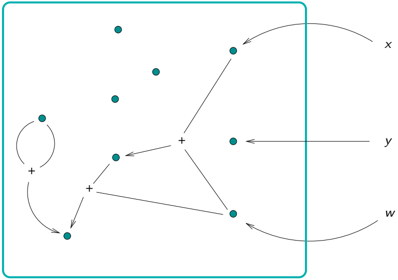
φ:

$$\exists z(x + y = z + z)$$



$\varphi[(x+w)/x,\ w/y]$:

$$\exists z((x + w) + w = z + z)$$



Substitution Definition

Let φ be an S-formula, $x_1, \ldots, x_n$ be distinct variables, and $t_1, \ldots, t_n$ be S-terms. Then $\varphi[t_1/x_1, \ldots, t_n/x_n]$ denotes the formula obtained by simultaneously substituting t_i for x_i in φ.

If $\varphi = \exists z(z + z = x \wedge r(x, y))$, what should $\varphi[y/x, w/y]$ be?

First define substitution on terms: $t[t_1, \ldots, t_n/x_1, \ldots, x_n]$.

Definition (Substitution for terms)

$$
\begin{aligned}
c[t_1/x_1, \ldots, t_n/x_n] &= c \\
x[t_1/x_1, \ldots, t_n/x_n] &= t_i \text{ if } x_i = x, \text{ else } x \\
f(t'_1, \ldots, t'_k)[t_1/x_1, \ldots, t_n/x_n] &= f(t'_1[t_1/x_1, \ldots, t_n/x_n], \ldots, \\
&\qquad t'_k[t_1/x_1, \ldots, t_n/x_n])
\end{aligned}
$$

Substitution Definition ctd.

Definition (Substitution for formulas)

$$
\begin{array}{lcl}
(t = t')[t_1/x_1, \ldots, t_n/x_n] & = & t[t_1/x_1, \ldots, t_n/x_n] = t'[t_1/x_1, \ldots, t_n/x_n] \\
(r(t'_1, \ldots, t'_k))[t_1/x_1, \ldots, t_n/x_n] & = & r(t'_1[t_1/x_1, \ldots, t_n/x_n], \ldots, t'_k[t_1/x_1, \ldots, t_n \\
(\neg\varphi)[t_1/x_1, \ldots, t_n/x_n] & = & \neg(\varphi[t_1/x_1, \ldots, t_n/x_n]) \\
(\varphi \vee \psi)[t_1/x_1, \ldots, t_n/x_n] & = & (\varphi[t_1/x_1, \ldots, t_n/x_n] \vee \psi[t_1/x_1, \ldots, t_n/x_n] \\
(\exists x\varphi)[t_1/x_1, \ldots, t_n/x_n] & = & \exists u(\varphi[u/x, t_{i_1}/x_{i_1}, \ldots, t_{i_m}/x_{i_m}])
\end{array}
$$

where $x_{i_1}, \ldots, x_{i_m}$ are those x_is which occur free in $\exists x\varphi$ and $t_i \neq x_i$, *and* u is x if x does not occur in $t_{i_1}, \ldots, t_{i_m}$; otherwise u is a var which does not occur in $\varphi, t_{i_1}, \ldots, t_{i_m}$.

Exercise

Exercise

What is

1. $r(v_0, f(v_1, v_2))\,[v_2/v_1, v_0/v_2, v_1/v_3]$
2. $\exists v_0 r(v_0, f(v_1, v_2))\,[v_4/v_0, f(v_1, v_2)/v_2]$
3. $\exists v_0 r(v_0, f(v_1, v_2))\,[v_0/v_1, v_2/v_2, v_4/v_0]$

Subsitution Lemma

Lemma (Substitution)

$$M \models \varphi[t_1/x_1, \ldots, t_n/x_n]$$

iff

$$M[M(t_1)/x_1]\cdots[M(t_n)/x_n] \models \varphi.$$

In particular,

$$M \models \varphi[t/x] \text{ iff } M[M(t)/x] \models \varphi.$$

Prenex Normal Form (EFT VIII.4)

Theorem (Prenex Normal Form)

For every FO formula φ we can construct a logically equivalent *formula ψ such that:*

- *ψ is of the form $Q_1 x_1 \cdots Q_n x_n \, \chi$ (with $n \geq 0$), where each Q_i is "$\exists$" or "$\forall$", and χ is quantifier-free;*
- *$free(\varphi) = free(\psi)$;*
- *and the number of quantifiers in φ and ψ are the same.*

Prenex Example

Prenex normal form example

$$\neg\exists x\, p(x) \;\lor\; \forall x\, r(x) \quad \equiv$$

Prenex Example

Prenex normal form example

$$\neg\exists x\, p(x) \;\lor\; \forall x\, r(x) \quad\equiv\quad \forall x \forall y\, (\neg p(x) \lor r(y))$$

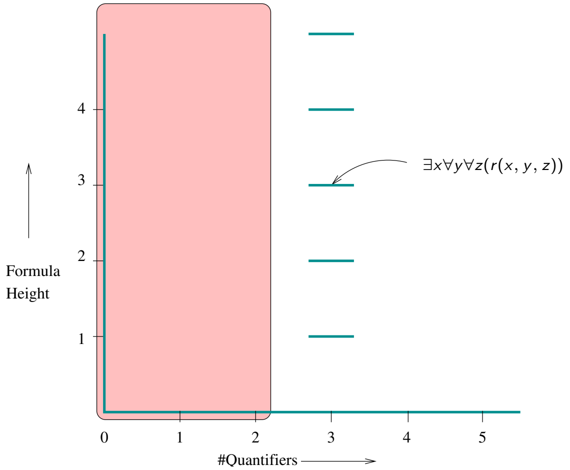
Some Useful Identities

$\varphi \equiv \psi$ means "φ is logically equivalent to ψ".

1. If $\varphi \equiv \psi$ then $\neg\varphi \equiv \neg\psi$.
2. If $\varphi \equiv \psi$ and $\varphi' \equiv \psi'$, then $\varphi \vee \varphi' \equiv \psi \vee \psi'$.
3. If $\varphi \equiv \psi$ then $\exists x\varphi \equiv \exists x\psi$ and $\forall x\varphi \equiv \forall x\psi$.
4. $\neg\exists x\varphi \equiv \forall x\neg\varphi$ and $\neg\forall x\varphi \equiv \exists x\neg\varphi$.
5. If $x \notin \mathit{free}(\psi)$ then:
   - $\exists x\varphi \vee \psi \;\equiv\; \exists x(\varphi \vee \psi)$.
   - $\forall x\varphi \vee \psi \;\equiv\; \forall x(\varphi \vee \psi)$.

Proof of Prenex Theorem

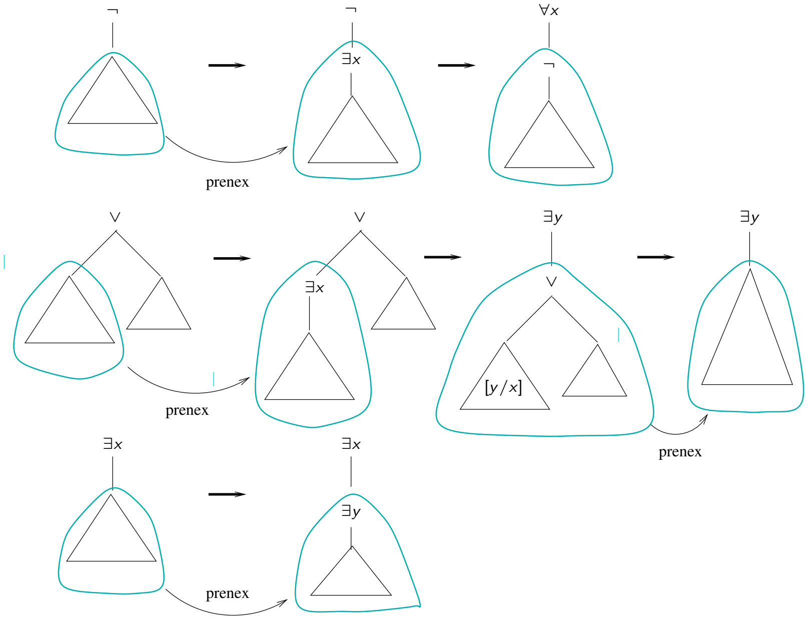
Argue by induction on the number of quantifiers in φ (and in the inductive step, by induction on the height of φ).



Proof of Prenex Theorem

- Base case: 0 quantifiers, hence φ is quantifier-free, and we can take $\psi = \varphi$.
- Induction step: Consider φ with $n + 1$ quantifiers. Use further induction on height of φ ($P(k)$: If φ has $n + 1$ quantifiers and has height k then φ has a prenex equivalent with same number of quantifiers and free vars):
  - Base case: Atomic formula, vacously true since no quantifiers.
  - Induction step: Consider cases
    - $\neg\psi$
    - $\psi \vee \chi$
    - $\exists x \psi$

Prenex Proof



Illustrating Prenex Procedure