Mathematical Logic and Theorem Proving

Assignment 1 (Propositional Logic)

(Total marks 70. Due on Wed 05th Feb 2025)

1. Let the signature $Pr(A) \subseteq Pr$ of formula A be the propositional symbols which appear in A. Show that if s is an assignment for A, evaluating its truth value depends only on $s\lceil Pr(A)$, the restriction of its domain to the formula signature. *(5)*

2. Show that the formula $\neg A$ cannot be equivalently defined using any formula made up of connectives $\wedge$ and $\vee$.

   Then show that the formula $\neg A$ cannot be equivalently defined using any formula made up of connectives $\wedge, \vee, \rightarrow$. *(10)*

3. Show that formula B is valid iff (if and only if) its negation $\neg B$ is not satisfiable. What is the name of the proof idea to show from this statement that formula B is satisfiable iff its negation $\neg B$ is not valid? *(5)*

4. Show that the following sequents are derivable using the base rules of the Sequent Calculus: *(10)*

   (a) $[p]\ \neg\neg p$

   (b) $[p \wedge q]\ p$.

5. Consider the following inference rules: *(10)*

$$\begin{array}{ll} \Gamma\ A & C \\ \Gamma\ B & D \\ \hline \Gamma\ A \vee B & C \vee D \end{array}$$

   and

$$\begin{array}{ll} \Gamma\ A & C \\ \Gamma\ B & D \\ \hline \Gamma\ A \vee B & C \wedge D \end{array}.$$

   One of them is correct while the other is not. Give a counter example for the incorrect rule, and show that the other can be derived using the rules of the Sequent Calculus (i.e. given derivations of the premisses, we can derive the conclusion using only the rules of the Sequent Calculus). You may make use of the derived rules like (Ctr'), (Ch), and the contraposition rules (Cp).

6. Show that the following rule is derivable from the rules in the Sequent Calculus (here "$A \wedge B$" is shorthand for $\neg(\neg A \vee \neg B)$): *(5)*

$$\begin{array}{ll} \Gamma & A \\ \Gamma & B \\ \hline \Gamma & (A \wedge B) \end{array}$$

7. Is there an algorithm to check for a finite theory (set of formulas) Γ that $\Gamma \vdash B$? *(5)*

8. Prove the Consistency Extension Lemma. That is, if theory Th is consistent then either $Th \cup \{A\}$ or $Th \cup \{\neg A\}$ is consistent. Using this, show that a maximal consistent theory must either contain A or $\neg A$. *(10)*

9. Show that a theory Th is consistent iff all its finite subsets are consistent. Using this statement and some theorems (to be) proved in this course, show that a theory Th is satisfiable iff all finite subsets of Th are satisfiable. *(10)*