Mathematical Logic and Theorem Proving

Assignment 2 (First-Order Logic)

(Total marks 90. Due on Mon 24 Feb 2025)

1. Consider the FO signature $S = (r^{(2)}, f^{(1)})$. For each of the following formulas say whether they are satisfiable, unsatisfiable, and/or valid. (*10*)
   (a) $\forall x \exists y (r(x, y) \lor f(x) = y)$
   (b) $\forall x \forall y (r(x, y) \lor \neg r(x, y))$
   (c) $(x = y) \rightarrow f(f(x)) = f(f(y))$.
   (d) $\forall x\, r(x, x) \land (y = z) \land \neg r(f(y), f(z))$.

2. Use the formal semantics of FO formulas to show that (*15*)
   (a) $\forall x \varphi$ is logically equivalent to $\neg \exists x \neg \varphi$.
   (b) $\forall x \varphi \lor \psi$ is logically equivalent to $\forall x (\varphi \lor \psi)$, provided $x \notin \mathit{free}(\psi)$.
   (c) If $x \in \mathit{free}(\psi)$ then the above equivalence in part (b) above does *not* hold.

3. Your friend is having trouble deciding whether the following identities are correct. Help them out by saying which is correct and which is not. Justify your answers. (*20*)
   (a) $\exists x (\varphi \lor \psi) \equiv \exists x \varphi \lor \exists x \psi$
   (b) $\exists x (\varphi \land \psi) \equiv \exists x \varphi \land \exists x \psi$
   (c) $\forall x (\varphi \lor \psi) \equiv \forall x \varphi \lor \forall x \psi$
   (d) $\forall x (\varphi \land \psi) \equiv \forall x \varphi \land \forall x \psi$

4. Consider the FO-signature of equivalence relations $S_{eq} = (r^{(2)})$. Give S_{eq} sentences which define each of the following classes of structures: (*10*)
   (a) The relation r is an equivalence relation with at least two equivalence classes.
   (b) The relation r is an equivalence relation with an equivalence class containing at least two elements.

5. Following up on the suggestion given by Adithya in class, consider the FO signature $S = (\{\}, \{f^{(1)}\}, \{\})$ and the S-sentence (*15*)

$$\begin{aligned} \varphi \quad = \quad & \forall x \forall y (f(x) = f(y) \rightarrow x = y) \land \\ & \exists y \forall x \neg (f(x) = y). \end{aligned}$$

   (a) What does the sentence φ state?
   (b) The sentence φ appears to characterize S-structures with infinite domains. Yet we argued in class that infiniteness is not elementarily definable. Where is the catch?

6. A set $X \subseteq \mathbb{N}$ is called a *spectrum* if there is an FO signature S and an S-sentence φ such that

$$X = \{n \in \mathbb{N} \mid \text{there exists a model with exactly } n \text{ elements satisfying } \varphi\}.$$

Show that the following sets are spectrums: *(20)*

(a) The set $\{1, 3, 4\}$.

(b) The set $\{3 \cdot k \mid k \in \mathbb{N}, k \geq 1\}$ (i.e. the set of non-zero multiples of 3).

(c) The set of squares greater than 0 (i.e. $\{1, 4, 9, 16, \ldots\}$.