Mathematical Logic and Theorem Proving

Assignment 3 (FO, DPLL and EUF)

(Total marks 80. Due on Wed 02 Apr 2025)

1. In the Sequent Calculus for First-Order Logic, show that: (*10*)

$$\vdash [x = y,\ y = z]\ f(x) = f(z).$$

2. Show that the following rule is derivable in the FO Sequent Calculus: (*10*)

$$\frac{\Gamma \quad \forall x \varphi}{\Gamma \quad \varphi[\frac{t}{x}]}$$

3. The following can be defined for a run of a DPLL algorithm on a CNF formula, given as a set of clauses.

   - Variables are guessed at decision levels, updating partial assignments.
   - A trail is a sequence of assignments to (distinct) variables, which does not falsify any clause. These assignments may be guesses, or they may be forced by earlier assignments (reasons) using Unit Resolution.
   - The clause from which reasons come is called an antecedent.
   - A trail may lead to conflict (contradiction), forcing the DPLL run to backtrack on an earlier guessed assignment.

   The following definitions relate to an implication graph, which is directed, acyclic and edge-labelled. It may not have a single source.

   - Every node is an assignment to a variable (the last one set on a run), with a path from a source node making up a partial assignment. In addition there may be a special conflict node.
   - The edges are from the reasons of an assigned variable to that assignment, labelled by the antecedent clause.
   - If Unit Resolution leads to an unsatisfied clause K, then there is an edge to the conflict node labelled by K.

   (a) Consider the set of clauses

   $$[\overline{p_1}, \overline{p_4}][\overline{p_2}, p_4][\overline{p_1}, p_2, \overline{p_3}][p_2, p_3].$$

   Suppose a DPLL run starts by guessing $p_1 \mapsto$ *true*. Continue this execution to a logical conclusion, providing decision levels, trail, reasons and antecedents. (*10*)

   (b) On

   $$[p_6, \overline{p_5}][p_7, \overline{p_3}, p_2][p_2, p_3][p_1, p_5, \overline{p_2}][\overline{p_3}, \overline{p_4}][\overline{p_2}, p_4][\overline{p_1}, p_3, p_5][\overline{p_1}, p_2],$$

   suppose a DPLL run starts by guessing $p_6 \mapsto$ *false* and continues, processing clauses left to right, deciding to set variables to *false*, and if required to backtrack, then to set them to *true*. Continue this DPLL run until it reaches a conflict. (*10*)

(c) For the DPLL run in the above question, construct an implication graph for the CDCL algorithm. Identify the dominators (unit implication points, UIP) which represent alternative assignments at the current decision level, and the immediate dominator (first UIP). Which new clause will the algorithm learn from the conflict? *(10)*

4. Prove that an Equality Logic formula φ which is a conjunction of literals, is satisfiable iff its induced equality graph G_φ has no contradictory cycles. *(10)*

5. Consider the following approaches to checking satisfiability of EUF formulas:

   - DNF + Congruence Closure: Convert the given EUF-formula φ into DNF, and check the satisfiability of each disjunct using Shostak's Congruence Closure algorithm. Return SAT if any one disjunct is SAT, else return UNSAT.
   - Ackermann's Reduction: Convert the given EUF formula φ to an Equality Logic formula φ' using Ackermann's reduction. Apply any technique for Equality Logic to check satisfiability of φ'.

   Use each of the above approaches to check satisfiability of the following EUF formulas: *(20)*

   (a) $x = F^6(x) \wedge x = F^3(x) \wedge x \neq F(x)$

   (b) $(G(F(x)) = x \wedge F(G(F(x))) = x \wedge F(G(x)) \neq x) \vee (F(G(x)) = x \wedge F(F(G(x))) = x \wedge F(x) \neq G(x))$