Completeness of First-Order Natural Deduction

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1 Completeness of First-Order Natural Deduction

In this section we show the completeness of the first-order natural deduction calculus using the abstract consistency/model-existence method. This method has the advantage that it provides a model-existence theorem that can be re-used for multiple calculi for a given logical system and allows to re-use work in situations where logical systems extend each other (like propositional logic and PREDLOG).

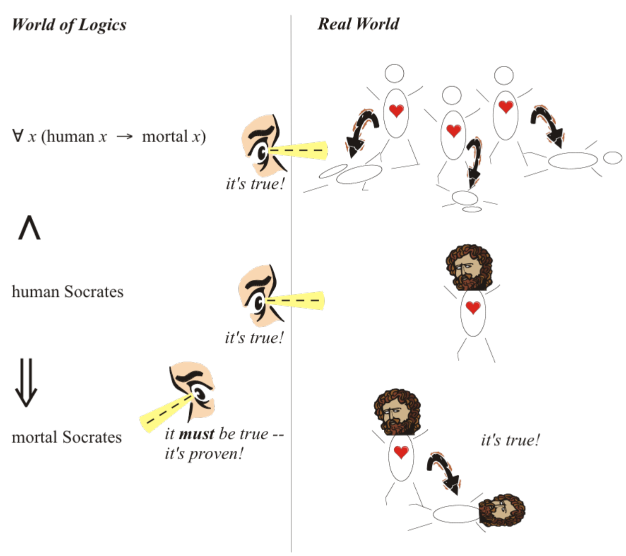
We will first recap propositional ND and show its completeness, essentially re-packaging the ideas from the proof earlier in this lecture and then extend it of first-order ND. In this we only to have to look at the four inference rules that $\mathcal{ND}^1$ introduces on top of $\mathcal{ND}_0$. That makes the completeness proof relatively easy and manageable.

1.1 Soundness and Completeness in Logic

Within the world of logics, one can derive new propositions (the *conclusions*, here: *Socrates is mortal*) from given ones (the *premises*, here: *Every human is mortal* and *Sokrates is human*). Such derivations are *proofs*.
In particular, logics can describe the internal structure of real-life facts; e.g. individual things, actions, properties. A famous example, which is in fact as old as it appears, is illustrated in the slide below.

The Miracle of Logic

▷ Purely formal derivations are true in the real world!



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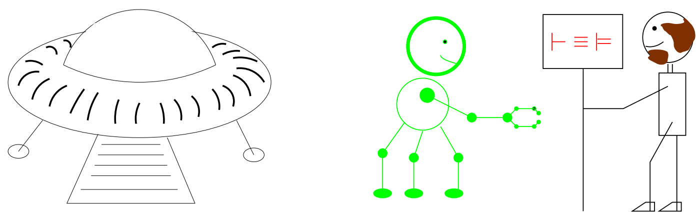
If a formal system is correct, the conclusions one can prove are true (= hold in the real world) whenever the premises are true. This is a miraculous fact (think about it!)
In general, formulae can be used to represent facts about the world as propositions; they have a semantics that is a mapping of formulae into the real world (propositions are mapped to truth values.) We have seen two relations on formulae: the entailment relation and the derivation relation. The first one is defined purely in terms of the semantics, the second one is given by a calculus, i.e. purely syntactically. Is there any relation between these relations?

Soundness and Completeness

- ▷ **Definition 1.1.** Let $\mathcal{L} := \langle \mathcal{L}, \mathcal{M}, \models \rangle$ be a logical system, then we call a calculus $\mathcal{C}$ for $\mathcal{L}$,
  - ▷ **sound** (or **correct**), iff $\mathcal{H} \models \mathbf{A}$, whenever $\mathcal{H} \vdash_{\mathcal{C}} \mathbf{A}$, and
  - ▷ **complete**, iff $\mathcal{H} \vdash_{\mathcal{C}} \mathbf{A}$, whenever $\mathcal{H} \models \mathbf{A}$.
- ▷ **Goal:** Find calculi C, such that $\vdash_C \mathbf{A}$ iff $\models \mathbf{A}$ (provability and validity coincide)
  - ▷ To TRUTH through PROOF (CALCULEMUS [Leibniz ~1680])

▷ 

Ideally, both relations would be the same, then the calculus would allow us to infer all facts that can be represented in the given formal language and that are true in the real world, and only those. In other words, our representation and inference is faithful to the world.

A consequence of this is that we can rely on purely syntactical means to make predictions about the world. Computers rely on formal representations of the world; if we want to solve a problem on our computer, we first represent it in the computer (as data structures, which can be seen as a formal language) and do syntactic manipulations on these structures (a form of calculus). Now, if the provability relation induced by the calculus and the validity relation coincide (this will be quite difficult to establish in general), then the solutions of the program will be correct, and we will find all possible ones.

1.2 Recap: First-Order Natural Deduction

Natural Deduction in Sequent Calculus Formulation

- ▷ **Idea:** Represent hypotheses explicitly. (lift calculus to judgments)
- ▷ **Definition 1.2.** A **judgment** is a meta-statement about the provability of propositions.
- ▷ **Definition 1.3.** A **sequent** is a judgment of the form $\mathcal{H}\ \mathbf{A}$ about the provability of the formula $\mathbf{A}$ from the set $\mathcal{H}$ of hypotheses. We write $\mathbf{A}$ for $\emptyset\ \mathbf{A}$.
- ▷ **Idea:** Reformulate $\mathcal{ND}_0$ inference rules so that they act on sequents.
- ▷ **Example 1.4.** We give the sequent style version of ???:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{A}\land\mathbf{B}\ \mathbf{A}\land\mathbf{B}}\,\text{Ax}}{\mathbf{A}\land\mathbf{B}\ \mathbf{B}}\,\land E_r \qquad \dfrac{\dfrac{}{\mathbf{A}\land\mathbf{B}\ \mathbf{A}\land\mathbf{B}}\,\text{Ax}}{\mathbf{A}\land\mathbf{B}\ \mathbf{A}}\,\land E_l}{\mathbf{A}\land\mathbf{B}\ \mathbf{B}\land\mathbf{A}}\,\land I}{\mathbf{A}\land\mathbf{B}\Rightarrow\mathbf{B}\land\mathbf{A}}\,\Rightarrow I \qquad\qquad \dfrac{\dfrac{\dfrac{}{\mathbf{A}\ \mathbf{B}\ \mathbf{A}}\,\text{Ax}}{\mathbf{A}\ \mathbf{B}\Rightarrow\mathbf{A}}\,\Rightarrow I}{\mathbf{A}\Rightarrow\mathbf{B}\Rightarrow\mathbf{A}}\,\Rightarrow I$$

▷ **Note:** Even though the antecedent of a sequent is written like a sequences, it is actually a set. In particular, we can permute and duplicate members at will.

Sequent-Style Rules for Natural Deduction

▷ **Definition 1.5.** The following inference rules make up the **propositional sequent style natural deduction calculus $\mathcal{ND}^0_\vdash$**:

$$\dfrac{}{\Gamma\ \mathbf{A}\ \mathbf{A}}\,\text{Ax} \qquad \dfrac{\Gamma\ \mathbf{B}}{\Gamma\ \mathbf{A}\ \mathbf{B}}\,\text{weaken} \qquad \dfrac{}{\Gamma\ \mathbf{A}\lor\neg\mathbf{A}}\,\text{TND}$$

$$\dfrac{\Gamma\ \mathbf{A}\quad\Gamma\ \mathbf{B}}{\Gamma\ \mathbf{A}\land\mathbf{B}}\,\land I \qquad \dfrac{\Gamma\ \mathbf{A}\land\mathbf{B}}{\Gamma\ \mathbf{A}}\,\land E_l \qquad \dfrac{\Gamma\ \mathbf{A}\land\mathbf{B}}{\Gamma\ \mathbf{B}}\,\land E_r$$

$$\dfrac{\Gamma\ \mathbf{A}}{\Gamma\ \mathbf{A}\lor\mathbf{B}}\,\lor I_l \qquad \dfrac{\Gamma\ \mathbf{B}}{\Gamma\ \mathbf{A}\lor\mathbf{B}}\,\lor I_r \qquad \dfrac{\Gamma\ \mathbf{A}\lor\mathbf{B}\quad\Gamma\ \mathbf{A}\ \mathbf{C}\quad\Gamma\ \mathbf{B}\ \mathbf{C}}{\Gamma\ \mathbf{C}}\,\lor E$$

$$\dfrac{\Gamma\ \mathbf{A}\ \mathbf{B}}{\Gamma\ \mathbf{A}\Rightarrow\mathbf{B}}\,\Rightarrow I \qquad \dfrac{\Gamma\ \mathbf{A}\Rightarrow\mathbf{B}\quad\Gamma\ \mathbf{A}}{\Gamma\ \mathbf{B}}\,\Rightarrow E$$

$$\dfrac{\Gamma\ \mathbf{A}\ F}{\Gamma\ \neg\mathbf{A}}\,\neg I \qquad \dfrac{\Gamma\ \neg\neg\mathbf{A}}{\Gamma\ \mathbf{A}}\,\neg E$$

$$FI\,\dfrac{\Gamma\ \neg\mathbf{A}\quad\Gamma\ \mathbf{A}}{\Gamma\ F} \qquad FE\,\dfrac{\Gamma\ F}{\Gamma\ \mathbf{A}}$$

First-Order Natural Deduction in Sequent Formulation

▷ Rules for connectives from $\mathcal{ND}^0_\vdash$

▷ **Definition 1.6 (New Quantifier Rules).** The inference rules of the **first-order sequent style ND calculus $\mathcal{ND}^1_\vdash$** consist of those from $\mathcal{ND}^0_\vdash$ plus the following quantifier rules:

$$\dfrac{\Gamma\ \mathbf{A}\quad X\notin\text{free}(\Gamma)}{\Gamma\ \forall X.\mathbf{A}}\,\forall I \qquad \dfrac{\Gamma\ \forall X.\mathbf{A}}{\Gamma\ \mathbf{A}[\frac{\mathbf{B}}{X}]}\,\forall E$$

$$\dfrac{\Gamma\ \mathbf{A}[\frac{\mathbf{B}}{X}]}{\Gamma\ \exists X.\mathbf{A}}\,\exists I \qquad \dfrac{\Gamma\ \exists X.\mathbf{A}\quad\Gamma\ \mathbf{A}[\frac{c}{X}]\ \mathbf{C}\quad c\in\Sigma_0^{sk}\ \text{new}}{\Gamma\ \mathbf{C}}\,\exists E$$

1.3 Abstract Consistency and Model Existence (Overview)

We will now come to an important tool in the theoretical study of reasoning calculi: the abstract consistency/model-existence method. This method for analyzing calculi was developed by Jaako Hintikka, Raymond Smullyan, and Peter Andrews in 1950-1970 as an encapsulation of similar constructions that were used in completeness arguments in the decades before. The basis for this method is Smullyan's Observation [Smu63] that completeness proofs based on Hintikka sets only certain properties of consistency and that with little effort one can obtain a generalization "Smullyan's Unifying Principle".
The basic intuition for this method is the following: typically, a logical system $\mathcal{L} := \langle \mathcal{L}, \models \rangle$ has multiple calculi, human-oriented ones like the natural deduction calculi and machine-oriented ones like the automated theorem proving calculi. All of these need to be analyzed for completeness (as a basic quality assurance measure).

A completeness proof for a calculus $\mathcal{C}$ for $\mathcal{S}$ typically comes in two parts: one analyzes $\mathcal{C}$-consistency (sets that cannot be refuted in $\mathcal{C}$), and the other constructs $\models$-models for $\mathcal{C}$-consistent sets.

In this situtation the abstract consistency/model-existence method encapsulates the model construction process into a meta-theorem: the model-existence theorem. This provides a set of syntactic (abstract consistency) conditions for calculi that are sufficient to construct models.

With the model-existence theorem it suffices to show that $\mathcal{C}$-consistency is an abstract consistency property (a purely syntactic task that can be done by a $\mathcal{C}$-proof transformation argument) to obtain a completeness result for $\mathcal{C}$.

Model Existence Method (Overview)

▷ **Recap:** A completeness proof for a calculus $\mathcal{C}$ for a logical system $\mathcal{S} := \langle \mathcal{L}, \models \rangle$ typically comes in two parts:

1. analyzing $\mathcal{C}$-consistency (sets that cannot be refuted in $\mathcal{C}$),
2. constructing $\models$-models for $\mathcal{C}$-consistent sets.

▷ **Idea:** Re-package the argument, so that the model-construction for $\mathcal{S}$ can be re-used for multiple calculi $\rightsquigarrow$ the abstract consistency/model-existence method:

1. **Definition 1.7.** **Abstract consistency class** $\nabla \mathrel{\widehat{=}}$ family of ∇-consistent sets.
2. **Definition 1.8.** A **∇-Hintikka set** is a $\subseteq$-maximally ∇-consistent.
3. **Theorem 1.9 (Hintikka Lemma).** *∇-Hintikka set are satisfiable.*
4. **Theorem 1.10 (Extension Theorem).** *If Φ is ∇-consistent, then Φ can be extended to a ∇-Hintikka set.*
5. **Corollary 1.11 (Henkins theorem).** *If Φ is ∇-consistent, then Φ is satisfiable.*
6. **Lemma 1.12 (Application).** *Let $\mathcal{C}$ be a calculus, if Φ is $\mathcal{C}$-consistent, then Φ is ∇-consistent.*
7. **Corollary 1.13 (Completeness).** *$\mathcal{C}$ is complete.*

▷ **Note:** Only the last two are $\mathcal{C}$-specific, the rest only depend on $\mathcal{S}$.

The proof of the model-existence theorem goes via the notion of a ∇-Hintikka set, a set of formulae with very strong syntactic closure properties, which allow to read off models. Jaako

Hintikka's original idea for completeness proofs was that for every complete calculus $\mathcal{C}$ and every $\mathcal{C}$-consistent set one can induce a ∇-Hintikka set, from which a model can be constructed. This can be considered as a first model-existence theorem. However, the process of obtaining a ∇-Hintikka set for a $\mathcal{C}$-consistent set Φ of propositions usually involves complicated calculus dependent constructions.

In this situation, Raymond Smullyan was able to formulate the sufficient conditions for the existence of ∇-Hintikka set in the form of "abstract consistency properties" by isolating the calculus independent parts of the Hintikka set construction. His technique allows to reformulate ∇-Hintikka set as maximal elements of abstract consistency classes and interpret the Hintikka set construction as a maximizing limit process.

To carry out the abstract consistency/model-existence method, we will first have to look at the notion of consistency.

consistency and refutability are very important notions when studying the completeness for calculi; they form syntactic counterparts of satisfiability.

Consistency and Refutability: Some General Definitions

- ▷ **Definition 1.14.** We call a pair of propositions $\mathbf{A}$ and $\neg\mathbf{A}$ a **contradiction**.
- ▷ A formula set Φ is $\mathcal{C}$-refutable, if $\mathcal{C}$ can derive a contradiction from it.
- ▷ **Definition 1.15.** Let $\mathcal{C}$ be a calculus, then a logsys/proposition set Φ is called $\mathcal{C}$-**consistent**, iff there is a logsys/proposition $\mathbf{B}$, that is not derivable from Φ in $\mathcal{C}$.
- ▷ **Definition 1.16.** We call a calculus $\mathcal{C}$ **reasonable**, iff implication elimination and conjunction introduction are admissible in $\mathcal{C}$ and $\mathbf{A} \wedge \neg\mathbf{A} \Rightarrow \mathbf{B}$ is a $\mathcal{C}$-theorem.
- ▷ **Theorem 1.17.** *$\mathcal{C}$-inconsistency and $\mathcal{C}$-refutability coincide for reasonable calculi.*
- ▷ *Remark 1.18.* We will use that $\mathcal{C}$-irrefutable $\widehat{=}$ $\mathcal{C}$-consistent below.
- ▷ ⚠ $\mathcal{C}$-consistency (syntactic) and satisfiability (semantics) are fundamentally different!
- ▷ Relating them is the meat of the abstract consistency/model-existence method.

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It is very important to distinguish the syntactic $\mathcal{C}$-refutability and $\mathcal{C}$-consistency from satisfiability, which is a property of formulae that is at the heart of semantics. Note that the former have the calculus (a syntactic device) as a parameter, while the latter does not. In fact we should actually say $\mathcal{S}$-satisfiability, where $\langle\mathcal{L},\models\rangle$ is the current logical system.

Even the word "contradiction" has a syntactical flavor to it, it translates to "saying against each other" from its Latin root.

1.4 Abstract Consistency and Model Existence for Propositional Logic

Abstract Consistency

- ▷ **Definition 1.19.** Let ∇ be a collection of sets. We call ∇ **closed under subsets**, iff for each $\Phi \in \nabla$, all subsets $\Psi \subseteq \Phi$ are elements of ∇.
- ▷ **Definition 1.20 (Notation).** We will use $\Phi{*}\mathbf{A}$ for $\Phi \cup \{\mathbf{A}\}$.

▷ **Definition 1.21.** A collection ∇ of sets of propositional formulae is called an **propositional abstract consistency class** (**ACC^0**), iff it is closed under subsets, and for each $\Phi \in \nabla$

∇_c) $P \notin \Phi$ or $\neg P \notin \Phi$ for $P \in \mathcal{V}_0$

$\nabla_\neg$) $\neg\neg\mathbf{A} \in \Phi$ implies $\Phi * \mathbf{A} \in \nabla$

$\nabla_\vee$) $\mathbf{A} \vee \mathbf{B} \in \Phi$ implies $\Phi * \mathbf{A} \in \nabla$ or $\Phi * \mathbf{B} \in \nabla$

$\nabla_\wedge$) $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ implies $\Phi \cup \{\neg\mathbf{A}, \neg\mathbf{B}\} \in \nabla$

▷ **Example 1.22.** The empty collection is an ACC^0.

▷ **Example 1.23.** The collection $\{\emptyset, \{Q\}, \{P \vee Q\}, \{P \vee Q, Q\}\}$ is an ACC^0.

▷ **Example 1.24.** The collection of satisfiable sets is an ACC^0.

So a collection of sets (we call it a collection, so that we do not have to say "set of sets" and we can distinguish the levels) is an abstract consistency class, iff it fulfills five simple conditions, of which the last three are closure conditions.

Think of an abstract consistency class as a collection of "consistent" sets (e.g. $\mathcal{C}$-consistent for some calculus $\mathcal{C}$), then the properties make perfect sense: They are naturally closed under subsets — if we cannot derive a contradiction from a large set, we certainly cannot from a subset, furthermore,

∇_c) If both $P \in \Phi$ and $\neg P \in \Phi$, then Φ cannot be "consistent".

$\nabla_\neg$) If we cannot derive a contradiction from Φ with $\neg\neg\mathbf{A} \in \Phi$ then we cannot from $\Phi * \mathbf{A}$, since they are logically equivalent.

The other two conditions are motivated similarly. We will carry out the proof here, since it gives us practice in dealing with the abstract consistency properties.

The main result here is that abstract consistency classes can be extended to compact ones. The proof is quite tedious, but relatively straightforward. It allows us to assume that all abstract consistency classes are compact in the first place (otherwise we pass to the compact extension). Actually we are after abstract consistency classes that have an even stronger property than just being closed under subsets. This will allow us to carry out a limit construction in the ∇-Hintikka set extension argument later.

Compact Collections

▷ **Definition 1.25.** We call a collection ∇ of sets **compact**, iff for any set Φ we have $\Phi \in \nabla$, iff $\Psi \in \nabla$ for every finite subset Ψ of Φ.

▷ **Lemma 1.26.** *If ∇ is compact, then ∇ is closed under subsets.*

▷ *Proof:*

1. Suppose $S \subseteq T$ and $T \in \nabla$.
2. Every finite subset A of S is a finite subset of T.
3. As ∇ is compact, we know that $A \in \nabla$.
4. Thus $S \in \nabla$.

The property of being closed under subsets is a “downwards-oriented” property: We go from large sets to small sets, compactness (the interesting direction anyways) is also an “upwards-oriented” property. We can go from small (finite) sets to large (infinite) sets. The main application for the compactness condition will be to show that infinite sets of formulae are in a collection ∇ by testing all their finite subsets (which is much simpler).

Compact Abstract Consistency Classes

▷ **Lemma 1.27.** *Any ACC⁰ can be extended to a compact one.*

▷ *Proof:*

1. We choose $\nabla' := \{\Phi \subseteq \mathit{wff}_0(\mathcal{V}_0) \,|\, \text{every finite subset of } \Phi \text{ is in } \nabla\}$.
2. Now suppose that $\Phi \in \nabla$. ∇ is closed under subsets, so every finite subset of Φ is in ∇ and thus $\Phi \in \nabla'$. Hence $\nabla \subseteq \nabla'$.
3. Next let us show that ∇' is compact.
   3.1. Suppose $\Phi \in \nabla'$ and Ψ is an arbitrary finite subset of Φ.
   3.2. By definition of ∇' all finite subsets of Φ are in ∇ and therefore $\Psi \in \nabla'$.
   3.3. Thus all finite subsets of Φ are in ∇' whenever Φ is in ∇'.
   3.4. On the other hand, suppose all finite subsets of Φ are in ∇'.
   3.5. Then by the definition of ∇' the finite subsets of Φ are also in ∇, so $\Phi \in \nabla'$. Thus ∇' is compact.
4. Note that ∇' is closed under subsets by the Lemma above.
5. Now we show that if ∇ satisfies ∇_*, then ∇' does too.
   5.1. To show ∇_c, let $\Phi \in \nabla'$ and suppose there is an atom $\mathbf{A}$, such that $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$. Then $\{\mathbf{A}, \neg\mathbf{A}\} \in \nabla$ contradicting ∇_c.
   5.2. To show $\nabla_\neg$, let $\Phi \in \nabla'$ and $\neg\neg\mathbf{A} \in \Phi$, then $\Phi{*}\mathbf{A} \in \nabla'$.
      5.2.1. Let Ψ be any finite subset of $\Phi{*}\mathbf{A}$, and $\Theta := (\Psi \backslash \{\mathbf{A}\}){*}\neg\neg\mathbf{A}$.
      5.2.2. Θ is a finite subset of Φ, so $\Theta \in \nabla$.
      5.2.3. Since ∇ is an abstract consistency class and $\neg\neg\mathbf{A} \in \Theta$, we get $\Theta{*}\mathbf{A} \in \nabla$ by $\nabla_\neg$.
      5.2.4. We know that $\Psi \subseteq \Theta{*}\mathbf{A}$ and ∇ is closed under subsets, so $\Psi \in \nabla$.
      5.2.5. Thus every finite subset Ψ of $\Phi{*}\mathbf{A}$ is in ∇ and therefore by definition $\Phi{*}\mathbf{A} \in \nabla'$.
   5.3. the other cases are analogous to that of $\nabla_\neg$.

Hintikka sets are sets of formulae with very strong analytic closure conditions. These are motivated as maximally consistent sets i.e. sets that already contain everything that can be consistently added to them.

∇-Hintikka set

▷ **Definition 1.28.** Let ∇ be an abstract consistency class, then we call a set $\mathcal{H} \in \nabla$ a **∇-Hintikka set**, iff $\mathcal{H}$ is $\subseteq$-maximal in ∇, i.e. for all $\mathbf{A}$ with $\mathcal{H}{*}\mathbf{A} \in \nabla$ we already have $\mathbf{A} \in \mathcal{H}$.

▷ **Theorem 1.29 (Hintikka Properties).** *Let ∇ be an abstract consistency class and $\mathcal{H}$ be a ∇-Hintikka set then*

$\mathcal{H}_c$) For all $\mathbf{A} \in \mathit{wff}_0(\mathcal{V}_0)$ we have $\mathbf{A} \notin \mathcal{H}$ or $\neg\mathbf{A} \notin \mathcal{H}$

$\mathcal{H}_\neg$) *If* $\neg\neg\mathbf{A} \in \mathcal{H}$ *then* $\mathbf{A} \in \mathcal{H}$

$\mathcal{H}_\vee$) *If* $\mathbf{A} \vee \mathbf{B} \in \mathcal{H}$ *then* $\mathbf{A} \in \mathcal{H}$ *or* $\mathbf{B} \in \mathcal{H}$

$\mathcal{H}_\wedge$) *If* $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$ *then* $\neg\mathbf{A}, \neg\mathbf{B} \in \mathcal{H}$

▷ **Remark:** Hintikka sets are usually *defined* by the properties $\mathcal{H}_*$ above, but here we (more generally) characterize them by $\subseteq$-maximality and regain the same properties.

∇-Hintikka set

▷ *Proof:* We prove the properties in turn

1. $\mathcal{H}_c$ goes by induction on the structure of $\mathbf{A}$
   1.1. $\mathbf{A} \in \mathcal{V}_0$ Then $\mathbf{A} \notin \mathcal{H}$ or $\neg\mathbf{A} \notin \mathcal{H}$ by ∇_c.
   1.2. $\mathbf{A} = \neg\mathbf{B}$
      1.2.1. Let us assume that $\neg\mathbf{B} \in \mathcal{H}$ and $\neg\neg\mathbf{B} \in \mathcal{H}$,
      1.2.2. then $\mathcal{H}*\mathbf{B} \in \nabla$ by $\nabla_\neg$, and therefore $\mathbf{B} \in \mathcal{H}$ by maximality.
      1.2.3. So both $\mathbf{B}$ and $\neg\mathbf{B}$ are in $\mathcal{H}$, which contradicts the induction hypothesis.
   1.3. $\mathbf{A} = \mathbf{B} \vee \mathbf{C}$ *is similar to the previous case*
2. We prove $\mathcal{H}_\neg$ by maximality of $\mathcal{H}$ in ∇.
   2.1. If $\neg\neg\mathbf{A} \in \mathcal{H}$, then $\mathcal{H}*\mathbf{A} \in \nabla$ by $\nabla_\neg$.
   2.2. The maximality of $\mathcal{H}$ now gives us that $\mathbf{A} \in \mathcal{H}$.
3. The other $\mathcal{H}_*$ can be proven analogously.

The following theorem is one of the main results in the abstract consistency/model-existence method. For any ∇-consistentset Φ it allows us to construct a ∇-Hintikka set$\mathcal{H}$ with $\Phi \in \mathcal{H}$.

Extension Theorem

▷ **Theorem 1.30.** *If* ∇ *is an abstract consistency class and* $\Phi \in \nabla$, *then there is a* ∇*-Hintikka set*$\mathcal{H}$ *with* $\Phi \subseteq \mathcal{H}$.

▷ *Proof:*

1. Wlog. we assume that ∇ is compact (otherwise pass to compact extension)
2. We choose an enumeration $\mathbf{A}_1, \ldots$ of the set $\mathit{wff}_0(\mathcal{V}_0)$
3. and construct a sequence of sets $\mathbf{H}_i$ with $\mathbf{H}_0 := \Phi$ and

$$\mathbf{H}_{n+1} := \begin{cases} \mathbf{H}_n & \text{if } \mathbf{H}_n*\mathbf{A}_n \notin \nabla \\ \mathbf{H}_n*\mathbf{A}_n & \text{if } \mathbf{H}_n*\mathbf{A}_n \in \nabla \end{cases}$$

4. Note that all $\mathbf{H}_i \in \nabla$, choose $\mathcal{H} := \bigcup_{i\in\mathbb{N}} \mathbf{H}_i$
5. $\Psi \subseteq \mathcal{H}$ finite implies there is a $j \in \mathbb{N}$ such that $\Psi \subseteq \mathbf{H}_j$,
6. so $\Psi \in \nabla$ as ∇ is closed under subsets and $\mathcal{H} \in \nabla$ as ∇ is compact.
7. Let $\mathcal{H}*\mathbf{B} \in \nabla$, then there is a $j \in \mathbb{N}$ with $\mathbf{B} = \mathbf{A}_j$, so that $\mathbf{B} \in \mathbf{H}_{j+1}$ and $\mathbf{H}_{j+1} \subseteq \mathcal{H}$
8. Thus $\mathcal{H}$ is ∇-maximal

Note that the construction in the proof above is non-trivial in two respects. First, the limit construction for $\mathcal{H}$ is not executed in our original abstract consistency class ∇, but in a suitably extended one to make it compact — the original would not have contained $\mathcal{H}$ in general. Second, the set $\mathcal{H}$ is not unique for Φ, but depends on the choice of the enumeration of $wff_0(\mathcal{V}_0)$. If we pick a different enumeration, we will end up with a different $\mathcal{H}$. Say if $\mathbf{A}$ and $\neg\mathbf{A}$ are both ∇-consistent with Φ, then depending on which one is first in the enumeration $\mathcal{H}$, will contain that one; with all the consequences for subsequent choices in the construction process.

Valuation

- **Definition 1.31.** A function $\nu\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is called a **(propositional) valuation**, iff
  - $\nu(\neg\mathbf{A}) = \mathsf{T}$, iff $\nu(\mathbf{A}) = \mathsf{F}$
  - $\nu(\mathbf{A} \wedge \mathbf{B}) = \mathsf{T}$, iff $\nu(\mathbf{A}) = \mathsf{T}$ and $\nu(\mathbf{B}) = \mathsf{T}$
- **Lemma 1.32.** *If $\nu\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is a valuation and $\Phi \subseteq wff_0(\mathcal{V}_0)$ with $\nu(\Phi) = \{\mathsf{T}\}$, then Φ is satisfiable.*
- *Proof sketch:* $\nu|_{\mathcal{V}_0} : \mathcal{V}_0 \to \mathcal{D}_0$ is a satisfying variable assignment.
- **Lemma 1.33.** *If $\varphi\colon \mathcal{V}_0 \to \mathcal{D}_0$ is a variable assignment, then $\mathcal{I}_\varphi\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is a valuation.*

Now, we only have to put the pieces together to obtain the model existence theorem we are after.

Model Existence

- **Lemma 1.34 (Hintikka-Lemma).** *If ∇ is an abstract consistency class and $\mathcal{H}$ a ∇-Hintikka set, then $\mathcal{H}$ is satisfiable.*
- *Proof:*
  1. We define $\nu(\mathbf{A}) := \mathsf{T}$, iff $\mathbf{A} \in \mathcal{H}$
  2. then ν is a valuation by the Hintikka properties
  3. and thus $\nu|_{\mathcal{V}_0}$ is a satisfying assignment.
- **Theorem 1.35 (Model Existence).** *If ∇ is an abstract consistency class and $\Phi \in \nabla$, then Φ is satisfiable.*

  Proof:

- 1. There is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$ (Extension Theorem)
  2. We know that $\mathcal{H}$ is satisfiable. (Hintikka-Lemma)
  3. In particular, $\Phi \subseteq \mathcal{H}$ is satisfiable.

1.5 A Completeness Proof for Propositional ND

With the model existence proof we have introduced in the last subsection, the completeness proof for propositional natural deduction is rather simple, we only have to check that ND-consistency is an ACC^0.

Consistency, Refutability and ∇-Consistency

▷ **Theorem 1.36 (Non-Refutability is an ACC^0).**
$\Gamma := \{\Phi \subseteq \mathit{wff}_0(\mathcal{V}_0) \,|\, \Phi$ *is not* $\mathcal{ND}_0$*-refutable*$\}$ *is an* ACC^0.

▷ *Proof:* We check the properties of an ACC^0

1. If Φ is non-refutable, then any subset is as well, so Γ is closed under subsets.
   We show the abstract consistency properties ∇_* *for* $\Phi \in \Gamma$.
2. ∇_c
   2.1. We have to show that $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$ for atomic $\mathbf{A} \in \mathit{wff}_0(\mathcal{V}_0)$.
   2.2. Equivalently, we show the contrapositive: If $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$, then $\Phi \notin \Gamma$.
   2.3. So let $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$, then Φ is $\mathcal{ND}_0$-refutable by construction.
   2.4. So $\Phi \notin \Gamma$.
3. $\nabla_\neg$ *We show the contrapositive again*
   3.1. Let $\neg\neg\mathbf{A} \in \Phi$ and $\Phi * \mathbf{A} \notin \Gamma$
   3.2. Then we have a refutation $\mathcal{D}\colon \Phi * \mathbf{A} \vdash_{\mathcal{ND}_0} F$
   3.3. By prepending an application of $\neg E$ for $\neg\neg\mathbf{A}$ to $\mathcal{D}$, we obtain a refutation $\mathcal{D}'\colon \Phi \vdash_{\mathcal{ND}_0} F$.
   3.4. Thus $\Phi \notin \Gamma$.
4. The other ∇_* can be proven analogously.

This directly yields two important results that we will use for the completeness analysis.

Henkin's Theorem

▷ **Corollary 1.37 (Henkin's Theorem).** *Every* $\mathcal{ND}_0$*-consistent set of propositions is satisfiable.*

▷ *Proof:*

1. Let Φ be a $\mathcal{ND}_0$-consistent set of propositions.
2. The collection of sets of $\mathcal{ND}_0$-consistent propositions constitute an ACC^0.
3. Thus the model existence theorem guarantees a variable assignment that satisfies Φ.

Now, the completeness result for propositional natural deduction is just a simple argument away. We also get a compactness theorem (almost) for free: logical systems with a complete calculus are always compact.

Completeness of $\mathcal{ND}_0$

▷ **Theorem 1.38 (Completeness Theorem for $\mathcal{ND}_0$).** *If* $\Phi \models \mathbf{A}$*, then* $\Phi \vdash_{\mathcal{ND}_0} \mathbf{A}$.

▷ *Proof:* We prove the result by playing with negations.

1. If $\Phi \models \mathbf{A}$, then (by definition) $\mathbf{A}$ is satisfied by all variable assignment that

satisfy Φ

2. So $\Phi * \neg \mathbf{A}$ has no satisfying assignment.
3. Thus $\Phi * \neg \mathbf{A}$ is inconsistent by (the contrapositive of) Henkins Theorem.
4. So $\Phi \vdash_{\mathcal{ND}_0} \neg\neg \mathbf{A}$ by $\mathcal{ND}_0 \neg I$ and thus $\Phi \vdash_{\mathcal{ND}_0} \mathbf{A}$ by $\neg E$.

1.6 Completeness of Propositional Tableaux

Just to show that the model existence theorem helps us with other calculi, we now introduce the propositional tableau calculus, a calculus for propositional logic that is optimized for ease of implementation.

Test Calculi: Tableaux and Model Generation

- **Idea:** A tableau calculus is a test calculus that
  - analyzes a labeled formulae in a tree to determine satisfiability,
  - its branches correspond to valuations ($\leadsto$ models).
- **Example 1.39.** Tableau calculi try to construct models for labeled formulae: E.g. the propositional tableau calculus for PL^0

Tableau refutation (Validity)	Model generation (Satisfiability)
$\models P \wedge Q \Rightarrow Q \wedge P$	$\models P \wedge (Q \vee \neg R) \wedge \neg Q$
$(P \wedge Q \Rightarrow Q \wedge P)^{\mathsf{F}}$ $(P \wedge Q)^{\mathsf{T}}$ $(Q \wedge P)^{\mathsf{F}}$ P^{T} Q^{T} $P^{\mathsf{F}} \mid Q^{\mathsf{F}}$ $\bot \mid \bot$	$(P \wedge (Q \vee \neg R) \wedge \neg Q)^{\mathsf{T}}$ $(P \wedge (Q \vee \neg R))^{\mathsf{T}}$ $\neg Q^{\mathsf{T}}$ Q^{F} P^{T} $(Q \vee \neg R)^{\mathsf{T}}$ $Q^{\mathsf{T}} \mid \neg R^{\mathsf{T}}$ $\bot \mid R^{\mathsf{F}}$
No Model	Herbrand valuation $\{P^{\mathsf{T}}, Q^{\mathsf{F}}, R^{\mathsf{F}}\}$ $\varphi := \{P \mapsto \mathsf{T}, Q \mapsto \mathsf{F}, R \mapsto \mathsf{F}\}$

- **Idea:** Open branches in saturated tableaux yield satisfying assignments.
- **Algorithm:** Fully expand all possible tableaux, (no rule can be applied)
  - Satisfiable, iff there are open branches (correspond to models)

Tableau calculi develop a formula in a tree-shaped arrangement that represents a case analysis on when a formula can be made true (or false). Therefore the formulae are decorated with upper indices that hold the intended truth value.

On the left we have a refutation tableau that analyzes a negated formula (it is decorated with the intended truth value F). Both branches contain an elementary contradiction $\bot$.

On the right we have a model generation tableau, which analyzes a positive formula (it is decorated with the intended truth value T). This tableau uses the same rules as the refutation tableau, but makes a case analysis of when this formula can be satisfied. In this case we have a closed branch and an open one. The latter corresponds a model.

Now that we have seen the examples, we can write down the tableau rules formally.

Analytical Tableaux (Formal Treatment of $\mathcal{T}_0$)

- **Idea:** A test calculus where
  - A labeled formula is analyzed in a tree to determine satisfiability,
  - branches correspond to valuations (models)
- **Definition 1.40.** The **propositional tableau calculus** $\mathcal{T}_0$ has two inference rules per connective (one for each possible label)

$$\frac{(\mathbf{A}\wedge\mathbf{B})^{\mathsf{T}}}{\begin{array}{c}\mathbf{A}^{\mathsf{T}}\\ \mathbf{B}^{\mathsf{T}}\end{array}}\mathcal{T}_0\wedge \qquad \frac{(\mathbf{A}\wedge\mathbf{B})^{\mathsf{F}}}{\mathbf{A}^{\mathsf{F}}\mid\mathbf{B}^{\mathsf{F}}}\mathcal{T}_0\vee \qquad \frac{\neg\mathbf{A}^{\mathsf{T}}}{\mathbf{A}^{\mathsf{F}}}\mathcal{T}_0\neg^{\mathsf{T}} \qquad \frac{\neg\mathbf{A}^{\mathsf{F}}}{\mathbf{A}^{\mathsf{T}}}\mathcal{T}_0\neg^{\mathsf{F}} \qquad \frac{\begin{array}{c}\mathbf{A}^{\alpha}\\ \mathbf{A}^{\beta}\end{array}\quad \alpha\neq\beta}{\bot}\mathcal{T}_0\bot$$

  Use rules exhaustively as long as they contribute new material ($\rightsquigarrow$ termination)
- **Definition 1.41.** We call any tree (| introduces branches) produced by the $\mathcal{T}_0$ inference rules from a set Φ of labeled formulae a **tableau** for Φ.
- **Definition 1.42.** Call a tableau **saturated**, iff no rule adds new material and a branch **closed**, iff it ends in $\bot$, else **open**. A tableau is **closed**, iff all of its branches are.

  In analogy to the $\bot$ at the end of closed branches, we sometimes decorate open branches with a $\Box$ symbol.

These inference rules act on tableaux have to be read as follows: if the formulae over the line appear in a tableau branch, then the branch can be extended by the formulae or branches below the line. There are two rules for each primary connective, and a branch closing rule that adds the special symbol $\bot$ (for unsatisfiability) to a branch.

We use the tableau rules with the convention that they are only applied, if they contribute new material to the branch. This ensures termination of the tableau procedure for propositional logic (every rule eliminates one primary connective).

Definition 1.43. We will call a closed tableau with the labeled formula $\mathbf{A}^{\alpha}$ at the root a **tableau refutation** for $\mathcal{A}^{\alpha}$.

The saturated tableau represents a full case analysis of what is necessary to give $\mathbf{A}$ the truth value α; since all branches are closed (contain contradictions) this is impossible.

Analytical Tableaux ($\mathcal{T}_0$ continued)

- **Definition 1.44 ($\mathcal{T}_0$-Theorem/Derivability).** $\mathbf{A}$ is a **$\mathcal{T}_0$-theorem** ($\vdash_{\mathcal{T}_0}\mathbf{A}$), iff there is a closed tableau with $\mathbf{A}^{\mathsf{F}}$ at the root.

  $\Phi\subseteq wff_0(\mathcal{V}_0)$ **derives** $\mathbf{A}$ in $\mathcal{T}_0$ ($\Phi\vdash_{\mathcal{T}_0}\mathbf{A}$), iff there is a closed tableau starting with $\mathbf{A}^{\mathsf{F}}$ and Φ^{T}. The tableau with only a branch of $\mathbf{A}^{\mathsf{F}}$ and Φ^{T} is called **initial** for $\Phi\vdash_{\mathcal{T}_0}\mathbf{A}$.

Definition 1.45. We will call a tableau refutation for $\mathbf{A}^{\mathsf{F}}$ a **tableau proof** for $\mathbf{A}$, since it refutes the possibility of finding a model where $\mathbf{A}$ evaluates to F. Thus $\mathbf{A}$ must evaluate to T in all models, which is just our definition of validity.

Thus the tableau procedure can be used as a calculus for propositional logic. In contrast to the propositional Hilbert calculus it does not prove a theorem $\mathbf{A}$ by deriving it from a set of axioms, but it proves it by refuting its negation – here in form of a F label. Such calculi are called negative or test calculi. Generally test calculi have computational advantages over positive ones, since they have a built-in sense of direction.

We have rules for all the necessary connectives (we restrict ourselves to $\wedge$ and $\neg$, since the others can be expressed in terms of these two via the propositional identities above. For instance, we can write $\mathbf{A}\vee\mathbf{B}$ as $\neg(\neg\mathbf{A}\wedge\neg\mathbf{B})$, and $\mathbf{A}\Rightarrow\mathbf{B}$ as $\neg\mathbf{A}\vee\mathbf{B}$,…)

A more Complex $\mathcal{T}_0$ Tableau

▷ **Example 1.46.** We construct a saturated $\mathcal{T}_0$ tableau for the formula $\neg((A\vee B)\wedge\neg(B\wedge C)\wedge(\neg C\vee\neg A))$:

$$\begin{array}{c}
\neg((A\vee B)\wedge\neg(B\wedge C)\wedge(\neg C\vee\neg A))^{\mathsf{F}}\\
((A\vee B)\wedge\neg(B\wedge C)\wedge(\neg C\vee\neg A))^{\mathsf{T}}\\
(A\vee B)^{\mathsf{T}}\\
(\neg(B\wedge C)\wedge(\neg C\vee\neg A))^{\mathsf{T}}\\
\neg(B\wedge C)^{\mathsf{T}}\\
(\neg C\vee\neg A)^{\mathsf{T}}\\
(B\wedge C)^{\mathsf{F}}\\
\begin{array}{c|c}
\begin{array}{c} A^{\mathsf{T}}\\ \begin{array}{c|c} \begin{array}{c}\neg C^{\mathsf{T}}\\ C^{\mathsf{F}}\\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}}\\ \Box & \Box\end{array}\end{array} & \begin{array}{c}\neg A^{\mathsf{T}}\\ A^{\mathsf{F}}\\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}}\\ \bot & \bot\end{array}\end{array}\end{array}\end{array}
&
\begin{array}{c} B^{\mathsf{T}}\\ \begin{array}{c|c} \begin{array}{c}\neg C^{\mathsf{T}}\\ C^{\mathsf{F}}\\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}}\\ \bot & \Box\end{array}\end{array} & \begin{array}{c}\neg A^{\mathsf{T}}\\ A^{\mathsf{F}}\\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}}\\ \bot & \Box\end{array}\end{array}\end{array}\end{array}
\end{array}
\end{array}$$

So we have four closed branches (they end in $\bot$), and four open ones (decorated by $\Box$), these correspond to counter-examples to validity.

We encapsulate all of the technical difficulties of the problem in a technical Lemma. From that, the completeness proof is just an application of the high-level theorems we have just proven.

Abstract Consistency for $\mathcal{T}_0$

▷ **Lemma 1.47.** $\nabla := \{\Phi \,|\, \Phi^{\mathsf{T}}$ *has no closed* $\mathcal{T}_0$*-tableau*$\}$ *is an ACC⁰.*

▷ *Proof:* We convince ourselves of the abstract consistency properties

1. For ∇_c, let $P,\neg P\in\Phi$ implies $P^{\mathsf{F}},P^{\mathsf{T}}\in\Phi^{\mathsf{T}}$.
   1.1. So a single application of $\mathcal{T}_0\bot$ yields a closed tableau for Φ^{T}
2. For $\nabla_\neg$, let $\neg\neg\mathbf{A}\in\Phi$.
   2.1. For the proof of the contrapositive we assume that $\Phi*\mathbf{A}$ has a closed tableau $\mathcal{T}$ and show that already Φ has one:
   2.2. Applying each of $\mathcal{T}_0\neg^{\mathsf{T}}$ and $\mathcal{T}_0\neg^{\mathsf{F}}$ once allows to extend any tableau branch that contains $\neg\neg\mathbf{B}^\alpha$ by $\mathbf{B}^\alpha$.
   2.3. Any branch in $\mathcal{T}$ that is closed with $\neg\neg\mathbf{A}^\alpha$, can be closed by $\mathbf{A}^\alpha$.
3. $\nabla_\vee$ *Suppose* $\mathbf{A}\vee\mathbf{B}\in\Phi$ *and both* $\Phi*\mathbf{A}$ *and* $\Phi*\mathbf{B}$ *have closed tableaux*

3.1. Consider the tableaux:

$$\begin{array}{c} \Phi^{\mathsf{T}} \\ \mathbf{A}^{\mathsf{T}} \\ Rest^1 \end{array} \qquad \begin{array}{c} \Phi^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}} \\ Rest^2 \end{array} \qquad \begin{array}{c} \Psi^{\mathsf{T}} \\ (\mathbf{A} \vee \mathbf{B})^{\mathsf{T}} \\ \begin{array}{c|c} \mathbf{A}^{\mathsf{T}} & \mathbf{B}^{\mathsf{T}} \\ Rest^1 & Rest^2 \end{array} \end{array}$$

4. $\nabla_\wedge$ *Suppose,* $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ *and* $\Phi\{\neg\mathbf{A}, \neg\mathbf{B}\}$ *have closed tableau* $\mathcal{T}$*.*
   4.1. We consider

$$\begin{array}{c} \Phi^{\mathsf{T}} \\ \mathbf{A}^{\mathsf{F}} \\ \mathbf{B}^{\mathsf{F}} \\ Rest \end{array} \qquad \begin{array}{c} \Psi^{\mathsf{T}} \\ (\mathbf{A} \vee \mathbf{B})^{\mathsf{F}} \\ \mathbf{A}^{\mathsf{F}} \\ \mathbf{B}^{\mathsf{F}} \\ Rest \end{array}$$

where $\Phi = \Psi * \neg(\mathbf{A} \vee \mathbf{B})$.

Observation: If we look at the completeness proof below, we see that the Lemma above is the only place where we had to deal with specific properties of the $\mathcal{T}_0$.

So if we want to prove completeness of any other calculus with respect to propositional logic, then we only need to prove an analogon to this Lemma and can use the rest of the machinery we have already established "off the shelf".

This is one great advantage of the "abstract consistency/model-existence method"; the other is that the method can be applied to other logics as well. In particular, if these logic are extensions, then we can re-use the work we did already and only cover the additions.

Completeness of $\mathcal{T}_0$

▷ **Corollary 1.48.** $\mathcal{T}_0$ *is complete.*

▷ *Proof:* by contradiction

1. We assume that $\mathbf{A} \in wff_0(\mathcal{V}_0)$ is valid, but there is no closed tableau for $\mathbf{A}^{\mathsf{F}}$.
2. We have $\{\neg\mathbf{A}\} \in \nabla$ as $\neg\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{\mathsf{F}}$.
3. So $\neg\mathbf{A}$ is satisfiable by the model-existence theorem (which is applicable as ∇ is an abstract consistency class by our Lemma above).
4. This contradicts our assumption that $\mathbf{A}$ is valid.

We leave the soundness result for the first order natural deduction calculus to the reader and turn to the completeness result, which is much more involved and interesting.

1.7 Abstract Consistency and Model Existence for First-Order Logic

We will now extend the notion of abstract consistency class from propositional logic to PREDLOG. For that we will have to introduce abstract consistency properties for the quantifiers the characterize PREDLOG.

Abstract Consistency

▷ **Definition 1.49.** A collection $\nabla \subseteq wff_o(\Sigma_\iota, \mathcal{V}_\iota)$ of sets of formulae is called a

first-order abstract consistency class (ACC^1), iff it is a ACC^0 and additionally

$\nabla_\forall$) If $\forall X.\mathbf{A} \in \Phi$, then $\Phi * (\mathbf{A}[\frac{\mathbf{B}}{X}]) \in \nabla$ for each closed term $\mathbf{B}$.

$\nabla_\exists$) If $\neg(\forall X.\mathbf{A}) \in \Phi$ and c is an individual constant that does not occur in Φ, then $\Phi * \neg(\mathbf{A}[\frac{c}{X}]) \in \nabla$

▷ **Example 1.50.** The collection $\{\emptyset, \{\forall x.p(x)\}\}$ is an ACC^1. (no closed terms)

▷ **Example 1.51.** The collection $\Phi := \{\emptyset, \{p(a)\}, \{\forall x.p(x)\}\}$ is not an ACC^1.
⇝ $\{p(a), \forall x.p(x)\}$ is missing from Φ.

▷ **Example 1.52.** The collection $\Phi := \{\emptyset, \{\exists x.p(x)\}\}$ is not an ACC^1.
⇝ $\{p(c), \exists x.p(x)\}$ is missing from Φ or some individual constant c

Again, the conditions are very natural: Take for instance $\nabla_\exists$, it says that if a set Φ that contains a sentence $\neg(\forall X.\mathbf{A})$ is "consistent", then we should be able to extend it by $\neg(\mathbf{A}[\frac{c}{X}])$ for any new individual constant c without losing this property; in other words, a complete calculus should be able to recognize $\neg(\forall X.\mathbf{A})$ and $\neg(\mathbf{A}[\frac{c}{X}])$ to be equivalent.

Compact Abstract Consistency Classes

▷ **Lemma 1.53.** *Any ACC^1 can be extended to a compact one.*

▷ *Proof:* We extend the proof for propositional logic; we only have to look at the two new abstract consistency properties.

1. Again, we choose $\nabla' := \{\Phi \subseteq cwff_o(\Sigma_\iota) \mid$ every finite subset of Φ is in $\nabla\}$. This can be seen to be closed under subsets and compact by the same argument as above.
2. To show $\nabla_\forall$ for ∇', let $\Phi \in \nabla'$ and $\forall X.\mathbf{A} \in \Phi$.
   2.1. Let Ψ be any finite subset of $\Phi * (\mathbf{A}[\frac{\mathbf{B}}{X}])$, and $\Theta := (\Psi \backslash \{\mathbf{A}[\frac{\mathbf{B}}{X}]\}) * (\forall X.\mathbf{A})$.
   2.2. Θ is a finite subset of Φ, so $\Theta \in \nabla$.
   2.3. Since ∇ is a ACC^1 and $\mathbf{A}[\frac{\mathbf{B}}{X}] \in \Theta$, we get $\Theta * (\forall X.\mathbf{A}) \in \nabla$ by $\nabla_\forall$.
   2.4. We know that $\Psi \subseteq \Theta * (\mathbf{A}[\frac{\mathbf{B}}{X}])$ and ∇ is closed under subsets, so $\Psi \in \nabla$.
   2.5. Thus every finite subset Ψ of $\Phi * (\mathbf{A}[\frac{\mathbf{B}}{X}])$ is in ∇ and therefore by definition $\Phi * (\mathbf{A}[\frac{\mathbf{B}}{X}]) \in \nabla'$.
3. The $\nabla_\exists$ case are analogous to that for $\nabla_\forall$.

Hintikka sets are sets of sentences with very strong analytic closure conditions. These are motivated as maximally consistent sets i.e. sets that already contain everything that can be consistently added to them.

∇-Hintikka set

▷ **Theorem 1.54 (Hintikka Properties).** *Let ∇ be a ACC^1 and $\mathcal{H}$ be a ∇-Hintikka set, then $\mathcal{H}$ has all the propositional Hintikka properties plus*

$\mathcal{H}_\forall$) *If $\forall X.\mathbf{A} \in \mathcal{H}$, then $\mathbf{A}[\frac{\mathbf{B}}{X}] \in \mathcal{H}$ for each closed term $\mathbf{B}$.*

$\mathcal{H}_\exists$) *If $\neg(\forall X.\mathbf{A}) \in \mathcal{H}$ then $\neg(\mathbf{A}[\frac{\mathbf{B}}{X}]) \in \mathcal{H}$ for some closed term $\mathbf{B}$.*

▷ *Proof:* We prove the two new cases

1. We prove $\mathcal{H}_\forall$ by maximality of $\mathcal{H}$ in ∇.
   1.1. If $\forall X.\mathbf{A} \in \mathcal{H}$, then $\mathcal{H}*(\mathbf{A}[\frac{\mathbf{B}}{X}]) \in \nabla$ by $\nabla_\forall$.
   1.2. The maximality of $\mathcal{H}$ now gives us that $\mathbf{A}[\frac{\mathbf{B}}{X}] \in \mathcal{H}$.
2. The proof of $\mathcal{H}_\exists$ is similar

The following theorem is one of the main results in the abstract consistency/model-existence method. For any ∇-consistent set Φ it allows us to construct a ∇-Hintikka set $\mathcal{H}$ with $\Phi \in \mathcal{H}$.

Extension Theorem

▷ **Theorem 1.55.** *If ∇ is a ACC[1] and $\Phi \in \nabla$ finite, then there is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$.*

▷ *Proof:*

1. Wlog. assume that ∇ compact (else use compact extension)
2. Choose an enumeration $\mathbf{A}_1, \ldots$ of $\mathit{cwff}_o(\Sigma_\iota)$ and $c_1, c_2, \ldots$ of Σ_0^{sk}.
3. and construct a sequence of sets $\mathbf{H}_i$ with $\mathbf{H}_0 := \Phi$ and

$$\mathbf{H}_{n+1} := \begin{cases} \mathbf{H}_n & \text{if } \mathbf{H}_n*\mathbf{A}_n \notin \nabla \\ \mathbf{H}_n \cup \{\mathbf{A}_n, \neg(\mathbf{B}[\frac{c_n}{X}])\} & \text{if } \mathbf{H}_n*\mathbf{A}_n \in \nabla \text{ and } \mathbf{A}_n = \neg(\forall X.\mathbf{B}) \\ \mathbf{H}_n*\mathbf{A}_n & \text{else} \end{cases}$$

4. Note that all $\mathbf{H}_i \in \nabla$, choose $\mathcal{H} := \bigcup_{i\in\mathbb{N}} \mathbf{H}_i$
5. $\Psi \subseteq \mathcal{H}$ finite implies there is a $j \in \mathbb{N}$ such that $\Psi \subseteq \mathbf{H}_j$,
6. so $\Psi \in \nabla$ as ∇ closed under subsets and $\mathcal{H} \in \nabla$ as ∇ is compact.
7. Let $\mathcal{H}*\mathbf{B} \in \nabla$, then there is a $j \in \mathbb{N}$ with $\mathbf{B} = \mathbf{A}_j$, so that $\mathbf{B} \in \mathbf{H}_{j+1}$ and $\mathbf{H}_{j+1} \subseteq \mathcal{H}$
8. Thus $\mathcal{H}$ is ∇-maximal

Note that the construction in the proof above is non-trivial in two respects. First, the limit construction for $\mathcal{H}$ is not executed in our original abstract consistency class ∇, but in a suitably extended one to make it compact — the original would not have contained $\mathcal{H}$ in general. Second, the set $\mathcal{H}$ is not unique for Φ, but depends on the choice of the enumeration of $\mathit{cwff}_o(\Sigma_\iota)$. If we pick a different enumeration, we will end up with a different $\mathcal{H}$. Say if $\mathbf{A}$ and $\neg\mathbf{A}$ are both ∇-consistent with Φ, then depending on which one is first in the enumeration $\mathcal{H}$, will contain that one; with all the consequences for subsequent choices in the construction process.

What now?

▷ The next step is to take a ∇-Hintikka set – the extension lemma above gives us one – and show that it is satisfiable.

▷ **Problem:** For that we have to conjure a model $\langle\mathcal{A}, \mathcal{I}\rangle$ out of thin air.

▷ **Idea 1:** Maybe the ∇-Hintikka set will help us with the interpretation

⇜ After all it helped us with the variable assignments in PL^0.

▷ **Idea 2:** For the universe we use something that is already lying around:

⇝ The set $cwff_\iota(\Sigma)$ of closed terms!

▷ Again, the notion of a valuation helps write things down, so we start with that.

▷ Tighten your seat belts and hold on.

Valuations

▷ **Definition 1.56.** A function $\nu\colon cwff_o(\Sigma_\iota)\to\mathcal{D}_0$ is called a **(first-order) valuation**, iff ν is a propositional valuation and

▹ $\nu(\forall X.\mathbf{A})=\mathsf{T}$, iff $\nu(\mathbf{A}[\frac{\mathbf{B}}{X}])=\mathsf{T}$ for all closed terms $\mathbf{B}$.

▷ **Lemma 1.57.** *If $\varphi\colon\mathcal{V}_\iota\to U$ is a variable assignment, then $\mathcal{I}_\varphi\colon cwff_o(\Sigma_\iota)\to\mathcal{D}_0$ is a valuation.*

▷ *Proof sketch:* Immediate from the definitions.

Note: A valuation is a weaker notion of evaluation in first-order logic; the other direction is also true, even though the proof of this result is much more involved: The existence of a first-order valuation that makes a set of sentences true entails the existence of a model that satisfies it.

Valuation and Satisfiability

▷ **Lemma 1.58.** *If $\nu\colon cwff_o(\Sigma_\iota)\to\mathcal{D}_0$ is a valuation and $\Phi\subseteq cwff_o(\Sigma_\iota)$ with $\nu(\Phi)=\{\mathsf{T}\}$, then Φ is satisfiable.*

▷ *Proof:* We construct a model $\mathcal{M}:=\langle\mathcal{D}_\iota,\mathcal{I}\rangle$ for Φ.

1. Let $\mathcal{D}_\iota:=cwff_\iota(\Sigma_\iota)$, and
   ▹ $\mathcal{I}(f)\colon {\mathcal{D}_\iota}^k\to\mathcal{D}_\iota\,;\langle\mathbf{A}_1,\ldots,\mathbf{A}_k\rangle\mapsto f(\mathbf{A}_1,\ldots,\mathbf{A}_k)$ for $f\in\Sigma^f$
   ▹ $\mathcal{I}(p)\colon {\mathcal{D}_\iota}^k\to\mathcal{D}_0\,;\langle\mathbf{A}_1,\ldots,\mathbf{A}_k\rangle\mapsto\nu(p(\mathbf{A}_1,\ldots,\mathbf{A}_k))$ for $p\in\Sigma^p$.
2. Then variable assignments into $\mathcal{D}_\iota$ are ground substitutions.
3. We show $\mathcal{I}_\varphi(\mathbf{A})=\mathbf{A}\varphi$ for $\mathbf{A}\in wff_\iota(\Sigma_\iota,\mathcal{V}_\iota)$ by induction on $\mathbf{A}$:
   3.1. If $\mathbf{A}=X$, then $\mathcal{I}_\varphi(\mathbf{A})=X\varphi$ by definition.
   3.2. If $\mathbf{A}=f(\mathbf{A}_1,\ldots,\mathbf{A}_k)$, then $\mathcal{I}_\varphi(\mathbf{A})=\mathcal{I}(f)(\mathcal{I}_\varphi(\mathbf{A}_1),\ldots,\mathcal{I}_\varphi(\mathbf{A}_n))=\mathcal{I}(f)(\mathbf{A}_1\varphi,\ldots,\mathbf{A}_n\varphi)=f(\mathbf{A}_1\varphi,\ldots,\mathbf{A}_n\varphi)=f(\mathbf{A}_1,\ldots,\mathbf{A}_k)\varphi=\mathbf{A}\varphi$
4. We show $\mathcal{I}_\varphi(\mathbf{A})=\nu(\mathbf{A}\varphi)$ for $\mathbf{A}\in wff_o(\Sigma_\iota,\mathcal{V}_\iota)$ by induction on $\mathbf{A}$.
   4.1. If $\mathbf{A}=p(\mathbf{A}_1,\ldots,\mathbf{A}_k)$ then $\mathcal{I}_\varphi(\mathbf{A})=\mathcal{I}(p)(\mathcal{I}_\varphi(\mathbf{A}_1),\ldots,\mathcal{I}_\varphi(\mathbf{A}_n))=\mathcal{I}(p)(\mathbf{A}_1\varphi,\ldots,\mathbf{A}_n\varphi)=\nu(p(\mathbf{A}_1\varphi,\ldots,\mathbf{A}_n\varphi))=\nu(p(\mathbf{A}_1,\ldots,\mathbf{A}_k)\varphi)=\nu(\mathbf{A}\varphi)$
   4.2. If $\mathbf{A}=\neg\mathbf{B}$ then $\mathcal{I}_\varphi(\mathbf{A})=\mathsf{T}$, iff $\mathcal{I}_\varphi(\mathbf{B})=\nu(\mathbf{B}\varphi)=\mathsf{F}$, iff $\nu(\mathbf{A}\varphi)=\mathsf{T}$.
   4.3. $\mathbf{A}=\mathbf{B}\wedge\mathbf{C}$ is similar
   4.4. If $\mathbf{A}=\forall X.\mathbf{B}$ then $\mathcal{I}_\varphi(\mathbf{A})=\mathsf{T}$, iff $\mathcal{I}_\psi(\mathbf{B})=\nu(\mathbf{B}\psi)=\mathsf{T}$, for all $\mathbf{C}\in\mathcal{D}_\iota$, where $\psi=\varphi,[\frac{\mathbf{C}}{X}]$. This is the case, iff $\nu(\mathbf{A}\varphi)=\mathsf{T}$.
5. Thus $\mathcal{I}_\varphi(\mathbf{A})=\nu(\mathbf{A}\varphi)=\nu(\mathbf{A})=\mathsf{T}$ for all $\mathbf{A}\in\Phi$.
6. Hence $\mathcal{M}\models\mathbf{A}$.

Herbrand-Model

▷ **Definition 1.59.** Let $\Sigma := \langle \Sigma^f, \Sigma^p \rangle$ be a first-order signature, then we call $\langle \mathcal{D}, \mathcal{I} \rangle$ a **Herbrand model**, iff

1. $\mathcal{D} = cwff_\iota(\Sigma)$ – i.e. the Herbrand universe over Σ.
2. $\mathcal{I}(f) : \mathcal{D}^k \to \mathcal{D}; \langle \mathbf{A}_1, \ldots, \mathbf{A}_k \rangle \mapsto f(\mathbf{A}_1, \ldots, \mathbf{A}_k)$ for function constants $f \in \Sigma^f_k$, and
3. $\mathcal{I}(p) \subseteq \mathcal{D}^k$ for predicate constants p.

▷ **Note:** Variable assignments into $\mathcal{D} = cwff_\iota(\Sigma)$ are naturally ground substitutions by construction.

▷ **Lemma 1.60.** *$\mathcal{I}_\varphi(t) = t\varphi$ for terms t.*

Proof sketch: By induction on the structure of $\mathbf{A}$.

▷ **Corollary 1.61.** *A Herbrand model $\mathcal{M}$ can be represented by the set $H_\mathcal{M} = \{\mathbf{A} \in cwff(\Sigma) \mid \mathbf{A}$ atomic and $\mathcal{M} \models \Phi\}$ of closed atoms it satisfies.*

Proof: Let $\mathbf{A} = p(t_1, \ldots, t_k)$.

1. $\mathcal{I}_\varphi(\mathbf{A}) = \mathcal{I}_\varphi(p(t_1, \ldots, t_k)) = \mathcal{I}(p)(\langle t_1\varphi, \ldots, t_k\varphi \rangle) = \mathsf{T}$, iff $\mathbf{A} \in H_\mathcal{M}$.
2. In the definition of Herbrand model, only the interpretation of predicate constants is flexible, and $H_\mathcal{M}$ determines that.

▷ **Theorem 1.62 (Herbrand's Theorem).** *A set Φ of first-order propositions is satisfiable, iff it has a Herbrand model.*

Now, we only have to put the pieces together to obtain the model existence theorem we are after.

Model Existence

▷ **Theorem 1.63 (Hintikka-Lemma).** *If ∇ is an ACC[1] and $\mathcal{H}$ a ∇-Hintikka set, then $\mathcal{H}$ is satisfiable.*

▷ *Proof:*

1. we define $\nu(\mathbf{A}) := \mathsf{T}$, iff $\mathbf{A} \in \mathcal{H}$,
2. then ν is a valuation by the Hintikka set properties.
3. We have $\nu(\mathcal{H}) = \{\mathsf{T}\}$, so $\mathcal{H}$ is satisfiable.

▷ **Theorem 1.64 (Model Existence).** *If ∇ is an ACC[1] and $\Phi \in \nabla$, then Φ is satisfiable.*

Proof:

▷ 1. There is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$ (Extension Theorem)
2. We know that $\mathcal{H}$ is satisfiable. (Hintikka-Lemma)
3. In particular, $\Phi \subseteq \mathcal{H}$ is satisfiable.

1.8 A Completeness Proof for First-Order ND

With the model existence proof we have introduced in the last section, the completeness proof for first-order natural deduction is rather simple, we only have to check that ND-consistency is an ACC^1.

Consistency, Refutability and ∇-consistent

▷ **Theorem 1.65 ($\mathcal{ND}^1$-Non-Refutability is an ACC^1).** $\Gamma := \{\Phi \subseteq \mathit{cwff}_o(\Sigma_\iota) \,|\, \Phi$ *is not* $\mathcal{ND}^1$*-refutable*$\}$ *is an* ACC^1.

▷ *Proof:* We check the two additional properties of an ACC^1

1. $\nabla_\forall$: We use the contrapositive
   1.1. So let $\forall X.\mathbf{A} \in \Phi$, $\Phi \in \Gamma$, and $\Phi * (\mathbf{A}[\frac{\mathbf{A}}{X}]) \notin \Gamma$,
   1.2. then there is a $\mathcal{ND}^1$-refutation of $\Phi * (\mathbf{A}[\frac{\mathbf{A}}{X}])$.
   1.3. Prepending $\forall E$ to that, gives us a $\mathcal{ND}^1$-refutation of Φ.
   1.4. So $\Phi \notin \Gamma$, which is what we wanted to show for the contrapositive of $\nabla_\forall$.
2. $\nabla_\exists$ can be proven similarly using $\forall I$

This directly yields two important results that we will use for the completeness analysis.

Henkin's Theorem

▷ **Corollary 1.66 (Henkin's Theorem).** *Every* $\mathcal{ND}^1$*-consistent set of sentences has a model.*

▷ *Proof:*

1. Let Φ be a $\mathcal{ND}^1$-consistent set of sentences.
2. The collection of sets of $\mathcal{ND}^1$-consistent sentences constitute an ACC^1.
3. Thus the model existence theorem guarantees a model for Φ.

▷ **Corollary 1.67 (Löwenheim&Skolem Theorem).** *Any satisfiable set* Φ *of first-order sentences has a countable model.*

Proof sketch: The model we constructed is countable, since the set of ground terms is.

Now, the completeness result for first-order natural deduction is just a simple argument away. We also get a compactness theorem (almost) for free: logical systems with a complete calculus are always compact.

▷ Completeness and Compactness

▷ **Theorem 1.68 (Completeness Theorem for $\mathcal{ND}^1$).** *If* $\Phi \models \mathbf{A}$*, then* $\Phi \vdash_{\mathcal{ND}^1} \mathbf{A}$.

▷ *Proof:* We prove the result by playing with negations.

1. If $\mathbf{A}$ is valid in all models of Φ, then $\Phi * \neg\mathbf{A}$ has no model
2. Thus $\Phi * \neg\mathbf{A}$ is inconsistent by (the contrapositive of) Henkins Theorem.
3. So $\Phi \vdash_{\mathcal{ND}^1} \neg\neg\mathbf{A}$ by $\mathcal{ND}_0\neg I$ and thus $\Phi \vdash_{\mathcal{ND}^1} \mathbf{A}$ by $\neg E$.

▷ **Theorem 1.69 (Compactness Theorem for first-order logic).** *If $\Phi \models \mathbf{A}$, then there is already a finite set $\Psi \subseteq \Phi$ with $\Psi \models \mathbf{A}$.*

Proof: This is a direct consequence of the completeness theorem

▷ 1. We have $\Phi \models \mathbf{A}$, iff $\Phi \vdash_{\mathcal{ND}^1} \mathbf{A}$.
2. As a proof is a finite object, only a finite subset $\Psi \subseteq \Phi$ can appear as leaves in the proof.

1.9 Completeness of First-Order Tableaux

Only because we can, we will now take a a brief excusion into first-order tableaux, briefly introduce the standard tableaux calculus and sketch the completeness proof in one slide. Just to show how easy things become.

Note that the standard tableau calculus for first-order logic $\mathcal{T}_1$ is not what we would normally use for practical automated theorem proving systems – that would need the introduction of unification – but it at least shows another completeness proof easily.
We will now extend the propositional tableau techniques to first-order logic. We only have to add two new rules for the universal quantifier (in positive and negative polarity).

First-Order Standard Tableaux ($\mathcal{T}_1$) are Complete

▷ **Definition 1.70.** The **standard tableau calculus** ($\mathcal{T}_1$) extends $\mathcal{T}_0$ (propositional tableau calculus) with the following quantifier rules:

$$\frac{(\forall X.\mathbf{A})^{\mathsf{T}} \quad \mathbf{C} \in \mathit{cwff}_\iota(\Sigma_\iota)}{(\mathbf{A}[\frac{\mathbf{C}}{X}])^{\mathsf{T}}}\,\mathcal{T}_1\forall \qquad \frac{(\forall X.\mathbf{A})^{\mathsf{F}} \quad c \in \Sigma_0^{sk} \text{ new}}{(\mathbf{A}[\frac{c}{X}])^{\mathsf{F}}}\,\mathcal{T}_1\exists$$

▷ **Theorem 1.71.** $\mathcal{T}_1$ *is refutation complete.*

▷ *Proof:* We show that $\nabla := \{\Phi \mid \Phi^{\mathsf{T}} \text{ has no closed } \mathcal{T}_1\text{tableau}\}$ is an ACC[1]

1. ∇_c, $\nabla_\neg$, $\nabla_\vee$, and $\nabla_\wedge$ as for $\mathcal{T}_0$; $\nabla_\forall$ similar to the next ($\nabla_\exists$) below.
2. $\nabla_\exists$: We prove the contrapositive
   2.1. Let $\Phi = \Psi * (\exists X.\mathbf{A})$, but $\Phi * (\mathbf{A}[\frac{c}{X}]) \notin \nabla$,
   2.2. then $\Phi * (\mathbf{A}[\frac{c}{X}])$ has a closed $\mathcal{T}_1$-tableau (on the left).

Ψ^{T}	Ψ^{T}
$(\exists X.\mathbf{A})^{\mathsf{T}}$	$(\exists X.\mathbf{A})^{\mathsf{T}}$
$(\mathbf{A}[\frac{c}{X}])^{\mathsf{T}}$	$(\mathbf{A}[\frac{c}{X}])^{\mathsf{T}}$
Rest	*Rest*

The right $\mathcal{T}_1$-tableau starts with $\Phi = \Psi * (\exists X.\mathbf{A})$ and applies $\mathcal{T}_1\exists$ and then continues as on the left.

3. We argue from $\nabla \hat{=}$ ACC[1] to completeness as above.

The rule $\mathcal{T}_1\forall$ operationalizes the intuition that a universally quantified formula is true, iff all of the instances of the scope are. To understand the $\mathcal{T}_1\exists$ rule, we have to keep in mind that $\exists X.\mathbf{A}$ abbreviates $\neg(\forall X.\neg\mathbf{A})$, so that we have to read $(\forall X.\mathbf{A})^{\mathsf{F}}$ existentially — i.e. as $(\exists X.\neg\mathbf{A})^{\mathsf{T}}$, stating that there is an object with property $\neg\mathbf{A}$. In this situation, we can simply give this object a name: c, which we take from our (infinite) set of witness constants Σ_0^{sk}, which we have given ourselves expressly for this purpose when we defined first-order syntax. In other words $(\neg\mathbf{A}[\frac{c}{X}])^{\mathsf{T}} = (\mathbf{A}[\frac{c}{X}])^{\mathsf{F}}$ holds, and this is just the conclusion of the $\mathcal{T}_1\exists$ rule.

Problem: The rule $\mathcal{T}_1\forall$ displays a case of "don't know indeterminism": to find a refutation we have to guess a formula $\mathbf{C}$ from the (usually infinite) set $\mathit{cwff}_\iota(\Sigma_\iota)$.

For proof search, this means that we have to systematically try all, so $\mathcal{T}_1\forall$ is infinitely branching in general.

References

[Smu63] Raymond M. Smullyan. "A Unifying Principle for Quantification Theory". In: *Proc. Nat. Acad Sciences* 49 (1963), pp. 828–832.