Completeness of First-Order Natural Deduction

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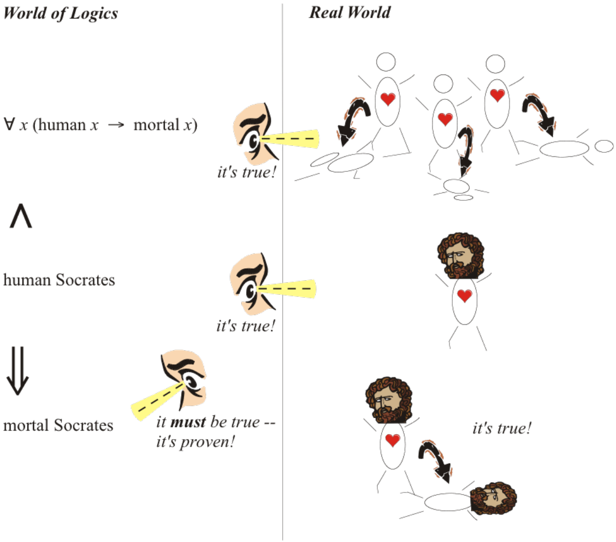
Chapter 1
Completeness of First-Order Natural Deduction

1.1 Soundness and Completeness in Logic

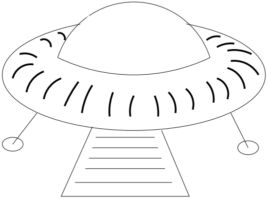
The Miracle of Logic

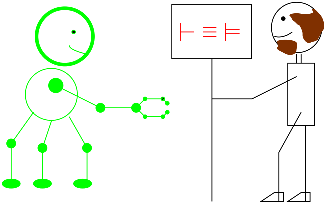
- Purely formal derivations are true in the real world!



Soundness and Completeness

- **Definition 1.1.** Let $\mathcal{L} := \langle \mathcal{L}, \mathcal{M}, \models \rangle$ be a logical system, then we call a calculus $\mathcal{C}$ for $\mathcal{L}$,
  - **sound** (or **correct**), iff $\mathcal{H} \models A$, whenever $\mathcal{H} \vdash_{\mathcal{C}} A$, and
  - **complete**, iff $\mathcal{H} \vdash_{\mathcal{C}} A$, whenever $\mathcal{H} \models A$.
- **Goal:** Find calculi C, such that $\vdash_C A$ iff $\models A$ (provability and validity coincide)
  - To TRUTH through PROOF (CALCULEMUS [Leibniz ∼1680])

- 



1.2 Recap: First-Order Natural Deduction

Natural Deduction in Sequent Calculus Formulation

- **Idea:** Represent hypotheses explicitly. (lift calculus to judgments)
- **Definition 2.1.** A **judgment** is a meta-statement about the provability of propositions.
- **Definition 2.2.** A **sequent** is a judgment of the form $\mathcal{H}$ A about the provability of the formula A from the set $\mathcal{H}$ of hypotheses. We write A for $\emptyset$ A.
- **Idea:** Reformulate $\mathcal{ND}_0$ inference rules so that they act on sequents.
- **Example 2.3.** We give the sequent style version of ???:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \land B \quad A \land B}\,\mathrm{Ax}}{A \land B \quad B}\,\land E_r \qquad \dfrac{\dfrac{}{A \land B \quad A \land B}\,\mathrm{Ax}}{A \land B \quad A}\,\land E_l}{A \land B \quad B \land A}\,\land I}{A \land B \Rightarrow B \land A}\,\Rightarrow I \qquad\qquad \dfrac{\dfrac{\dfrac{}{A \quad B \quad A}\,\mathrm{Ax}}{A \quad B \Rightarrow A}\,\Rightarrow I}{A \Rightarrow B \Rightarrow A}\,\Rightarrow I$$

- **Note:** Even though the antecedent of a sequent is written like a sequences, it is actually a set. In particular, we can permute and duplicate members at will.

Sequent-Style Rules for Natural Deduction

- **Definition 2.4.** The following inference rules make up the **propositional sequent style natural deduction calculus** $\mathcal{ND}^0_\vdash$:

$$\frac{}{\Gamma \vdash A \vdash A}\;\mathrm{Ax} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vdash B}\;\mathrm{weaken} \qquad \frac{}{\Gamma \vdash A \vee \neg A}\;\mathrm{TND}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\;\wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\;\wedge E_l \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\;\wedge E_r$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\;\vee I_l \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\;\vee I_r \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\;\vee E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\;\Rightarrow I \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\;\Rightarrow E$$

$$\frac{\Gamma, A \vdash F}{\Gamma \vdash \neg A}\;\neg I \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\;\neg E$$

$$FI\;\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash F} \qquad FE\;\frac{\Gamma \vdash F}{\Gamma \vdash A}$$

First-Order Natural Deduction in Sequent Formulation

- Rules for connectives from $\mathcal{ND}^0_\vdash$
- **Definition 2.5 (New Quantifier Rules).** The inference rules of the **first-order sequent style ND calculus** $\mathcal{ND}^1_\vdash$ consist of those from $\mathcal{ND}^0_\vdash$ plus the following quantifier rules:

$$\frac{\Gamma \vdash \mathbf{A} \quad X \notin \mathrm{free}(\Gamma)}{\Gamma \vdash \forall X.\mathbf{A}} \forall I \qquad \frac{\Gamma \vdash \forall X.\mathbf{A}}{\Gamma \vdash [\mathbf{B}/X](\mathbf{A})} \forall E$$

$$\frac{\Gamma \vdash [\mathbf{B}/X](\mathbf{A})}{\Gamma \vdash \exists X.\mathbf{A}} \exists I \qquad \frac{\Gamma \vdash \exists X.\mathbf{A} \quad \Gamma, [c/X](\mathbf{A}) \vdash \mathbf{C} \quad c \in \Sigma_0^{sk} \text{ new}}{\Gamma \vdash \mathbf{C}} \exists E$$

1.3 Abstract Consistency and Model Existence (Overview)

Model Existence Method (Overview)

- **Recap:** A completeness proof for a calculus $\mathcal{C}$ for a logical system $\mathcal{S} := \langle\mathcal{L}, \models\rangle$ typically comes in two parts:
  1. analyzing $\mathcal{C}$-consistency (sets that cannot be refuted in $\mathcal{C}$),
  2. constructing $\models$-models for $\mathcal{C}$-consistent sets.

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  1. **Definition 3.15.** **Abstract consistency class** $\nabla \mathrel{\widehat{=}}$ family of ∇-consistent sets.
  2. **Definition 3.16.** A **∇-Hintikka set** is a $\subseteq$-maximally ∇-consistent.
  3. **Theorem 3.17 (Hintikka Lemma).** *∇-Hintikka set are satisfiable.*

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- **Idea:** Re-package the argument, so that the model-construction for $\mathcal{S}$ can be re-used for multiple calculi $\leadsto$ the abstract consistency/model-existence method:
  1. **Definition 3.22.** **Abstract consistency class** $\nabla \mathrel{\widehat{=}}$ family of ∇-consistent sets.
  2. **Definition 3.23.** A **∇-Hintikka set** is a $\subseteq$-maximally ∇-consistent.
  3. **Theorem 3.24 (Hintikka Lemma).** *∇-Hintikka set are satisfiable.*
  4. **Theorem 3.25 (Extension Theorem).** *If Φ is ∇-consistent, then Φ can be extended to a ∇-Hintikka set.*
  5. **Corollary 3.26 (Henkins theorem).** *If Φ is ∇-consistent, then Φ is satisfiable.*

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  1. **Definition 3.29.** **Abstract consistency class** $\nabla \mathrel{\widehat{=}}$ family of ∇-consistent sets.
  2. **Definition 3.30.** A **∇-Hintikka set** is a $\subseteq$-maximally ∇-consistent.
  3. **Theorem 3.31 (Hintikka Lemma).** *∇-Hintikka set are satisfiable.*
  4. **Theorem 3.32 (Extension Theorem).** *If Φ is ∇-consistent, then Φ can be extended to a ∇-Hintikka set.*
  5. **Corollary 3.33 (Henkins theorem).** *If Φ is ∇-consistent, then Φ is satisfiable.*
  6. **Lemma 3.34 (Application).** *Let $\mathcal{C}$ be a calculus, if Φ is $\mathcal{C}$-consistent, then Φ is ∇-consistent.*
  7. **Corollary 3.35 (Completeness).** *$\mathcal{C}$ is complete.*

Model Existence Method (Overview)

- **Recap:** A completeness proof for a calculus $\mathcal{C}$ for a logical system $\mathcal{S} := \langle\mathcal{L}, \models\rangle$ typically comes in two parts:
  1. analyzing $\mathcal{C}$-consistency (sets that cannot be refuted in $\mathcal{C}$),
  2. constructing $\models$-models for $\mathcal{C}$-consistent sets.
- **Idea:** Re-package the argument, so that the model-construction for $\mathcal{S}$ can be re-used for multiple calculi $\rightsquigarrow$ the abstract consistency/model-existence method:
  1. **Definition 3.36.** **Abstract consistency class** $\nabla \mathrel{\widehat{=}}$ family of ∇-consistent sets.
  2. **Definition 3.37.** A **∇-Hintikka set** is a $\subseteq$-maximally ∇-consistent.
  3. **Theorem 3.38 (Hintikka Lemma).** *∇-Hintikka set are satisfiable.*
  4. **Theorem 3.39 (Extension Theorem).** *If Φ is ∇-consistent, then Φ can be extended to a ∇-Hintikka set.*
  5. **Corollary 3.40 (Henkins theorem).** *If Φ is ∇-consistent, then Φ is satisfiable.*
  6. **Lemma 3.41 (Application).** *Let $\mathcal{C}$ be a calculus, if Φ is $\mathcal{C}$-consistent, then Φ is ∇-consistent.*
  7. **Corollary 3.42 (Completeness).** *$\mathcal{C}$ is complete.*
- **Note:** Only the last two are $\mathcal{C}$-specific, the rest only depend on $\mathcal{S}$.

Consistency and Refutability: Some General Definitions

- **Definition 3.43.** We call a pair of propositions A and $\neg A$ a **contradiction**.
- A formula set Φ is $\mathcal{C}$-refutable, if $\mathcal{C}$ can derive a contradiction from it.

Consistency and Refutability: Some General Definitions

- **Definition 3.48.** We call a pair of propositions A and $\neg$A a **contradiction**.
- A formula set Φ is $\mathcal{C}$-refutable, if $\mathcal{C}$ can derive a contradiction from it.
- **Definition 3.49.** Let $\mathcal{C}$ be a calculus, then a logsys/proposition set Φ is called $\mathcal{C}$-**consistent**, iff there is a logsys/proposition B, that is not derivable from Φ in $\mathcal{C}$.

Consistency and Refutability: Some General Definitions

- **Definition 3.53.** We call a pair of propositions A and ¬A a **contradiction**.
- A formula set Φ is $\mathcal{C}$-refutable, if $\mathcal{C}$ can derive a contradiction from it.
- **Definition 3.54.** Let $\mathcal{C}$ be a calculus, then a logsys/proposition set Φ is called $\mathcal{C}$-**consistent**, iff there is a logsys/proposition B, that is not derivable from Φ in $\mathcal{C}$.
- **Definition 3.55.** We call a calculus $\mathcal{C}$ **reasonable**, iff implication elimination and conjunction introduction are admissible in $\mathcal{C}$ and $A \wedge \neg A \Rightarrow B$ is a $\mathcal{C}$-theorem.
- **Theorem 3.56.** *$\mathcal{C}$-inconsistency and $\mathcal{C}$-refutability coincide for reasonable calculi.*
- *Remark 3.57.* We will use that $\mathcal{C}$-irrefutable $\widehat{=}$ $\mathcal{C}$-consistent below.

Consistency and Refutability: Some General Definitions

- **Definition 3.58.** We call a pair of propositions A and $\neg$A a **contradiction**.
- A formula set Φ is $\mathcal{C}$-refutable, if $\mathcal{C}$ can derive a contradiction from it.
- **Definition 3.59.** Let $\mathcal{C}$ be a calculus, then a logsys/proposition set Φ is called $\mathcal{C}$-**consistent**, iff there is a logsys/proposition B, that is not derivable from Φ in $\mathcal{C}$.
- **Definition 3.60.** We call a calculus $\mathcal{C}$ **reasonable**, iff implication elimination and conjunction introduction are admissible in $\mathcal{C}$ and $A \wedge \neg A \Rightarrow B$ is a $\mathcal{C}$-theorem.
- **Theorem 3.61.** *$\mathcal{C}$-inconsistency and $\mathcal{C}$-refutability coincide for reasonable calculi.*
- *Remark 3.62.* We will use that $\mathcal{C}$-irrefutable $\widehat{=}$ $\mathcal{C}$-consistent below.
- $\mathcal{C}$-consistency (syntactic) and satisfiability (semantics) are fundamentally different!
- Relating them is the meat of the abstract consistency/model-existence method.

1.4 Abstract Consistency and Model Existence for Propositional Logic

Abstract Consistency

- **Definition 4.1.** Let ∇ be a collection of sets. We call ∇ **closed under subsets**, iff for each $\Phi \in \nabla$, all subsets $\Psi \subseteq \Phi$ are elements of ∇.
- **Definition 4.2 (Notation).** We will use $\Phi * \mathrm{A}$ for $\Phi \cup \{\mathrm{A}\}$.
- **Definition 4.3.** A collection ∇ of sets of propositional formulae is called an **propositional abstract consistency class** (**ACC^0**), iff it is closed under subsets, and for each $\Phi \in \nabla$

  ∇_c) $P \notin \Phi$ or $\neg P \notin \Phi$ for $P \in \mathcal{V}_0$
  $\nabla_\neg$) $\neg\neg \mathrm{A} \in \Phi$ implies $\Phi * \mathrm{A} \in \nabla$
  $\nabla_\vee$) $\mathrm{A} \vee \mathrm{B} \in \Phi$ implies $\Phi * \mathrm{A} \in \nabla$ or $\Phi * \mathrm{B} \in \nabla$
  $\nabla_\wedge$) $\neg(\mathrm{A} \vee \mathrm{B}) \in \Phi$ implies $\Phi \cup \{\neg \mathrm{A}, \neg \mathrm{B}\} \in \nabla$
- **Example 4.4.** The empty collection is an ACC^0.
- **Example 4.5.** The collection $\{\emptyset, \{Q\}, \{P \vee Q\}, \{P \vee Q, Q\}\}$ is an ACC^0.
- **Example 4.6.** The collection of satisfiable sets is an ACC^0.

Compact Collections

- **Definition 4.7.** We call a collection ∇ of sets **compact**, iff for any set Φ we have
  $\Phi \in \nabla$, iff $\Psi \in \nabla$ for every finite subset Ψ of Φ.
- **Lemma 4.8.** *If ∇ is compact, then ∇ is closed under subsets.*
- *Proof:*
  1. Suppose $S \subseteq T$ and $T \in \nabla$.
  2. Every finite subset A of S is a finite subset of T.
  3. As ∇ is compact, we know that $A \in \nabla$.
  4. Thus $S \in \nabla$.

Compact Abstract Consistency Classes I

- **Lemma 4.9.** *Any ACC^0 can be extended to a compact one.*
- *Proof:*
  1. We choose $\nabla' := \{\Phi \subseteq wff_0(\mathcal{V}_0) \,|\, \text{every finite subset of } \Phi \text{ is in } \nabla\}$.
  2. Now suppose that $\Phi \in \nabla$. ∇ is closed under subsets, so every finite subset of Φ is in ∇ and thus $\Phi \in \nabla'$. Hence $\nabla \subseteq \nabla'$.
  3. Next let us show that ∇' is compact.
     - 3.1. Suppose $\Phi \in \nabla'$ and Ψ is an arbitrary finite subset of Φ.
     - 3.2. By definition of ∇' all finite subsets of Φ are in ∇ and therefore $\Psi \in \nabla'$.
     - 3.3. Thus all finite subsets of Φ are in ∇' whenever Φ is in ∇'.
     - 3.4. On the other hand, suppose all finite subsets of Φ are in ∇'.
     - 3.5. Then by the definition of ∇' the finite subsets of Φ are also in ∇, so $\Phi \in \nabla'$. Thus ∇' is compact.
  4. Note that ∇' is closed under subsets by the Lemma above.

Compact Abstract Consistency Classes II

5. Now we show that if ∇ satisfies ∇_*, then ∇' does too.
   5.1. To show ∇_c, let $\Phi \in \nabla'$ and suppose there is an atom A, such that $\{A, \neg A\} \subseteq \Phi$. Then $\{A, \neg A\} \in \nabla$ contradicting ∇_c.
   5.2. To show $\nabla_\neg$, let $\Phi \in \nabla'$ and $\neg\neg A \in \Phi$, then $\Phi * A \in \nabla'$.
   5.2.1. Let Ψ be any finite subset of $\Phi * A$, and $\Theta := (\Psi \backslash \{A\}) * \neg\neg A$.
   5.2.2. Θ is a finite subset of Φ, so $\Theta \in \nabla$.
   5.2.3. Since ∇ is an abstract consistency class and $\neg\neg A \in \Theta$, we get $\Theta * A \in \nabla$ by $\nabla_\neg$.
   5.2.4. We know that $\Psi \subseteq \Theta * A$ and ∇ is closed under subsets, so $\Psi \in \nabla$.
   5.2.5. Thus every finite subset Ψ of $\Phi * A$ is in ∇ and therefore by definition $\Phi * A \in \nabla'$.
   5.3. the other cases are analogous to that of $\nabla_\neg$.

∇-Hintikka set

- **Definition 4.10.** Let ∇ be an abstract consistency class, then we call a set $\mathcal{H} \in \nabla$ a **∇-Hintikka set**, iff $\mathcal{H}$ is $\subseteq$-maximal in ∇, i.e. for all $\mathbf{A}$ with $\mathcal{H} * \mathbf{A} \in \nabla$ we already have $\mathbf{A} \in \mathcal{H}$.
- **Theorem 4.11 (Hintikka Properties).** *Let ∇ be an abstract consistency class and $\mathcal{H}$ be a ∇-Hintikka set then*

  $\mathcal{H}_c$) *For all* $\mathbf{A} \in wff_0(\mathcal{V}_0)$ *we have* $\mathbf{A} \notin \mathcal{H}$ *or* $\neg\mathbf{A} \notin \mathcal{H}$

  $\mathcal{H}_\neg$) *If* $\neg\neg\mathbf{A} \in \mathcal{H}$ *then* $\mathbf{A} \in \mathcal{H}$

  $\mathcal{H}_\vee$) *If* $\mathbf{A} \vee \mathbf{B} \in \mathcal{H}$ *then* $\mathbf{A} \in \mathcal{H}$ *or* $\mathbf{B} \in \mathcal{H}$

  $\mathcal{H}_\wedge$) *If* $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$ *then* $\neg\mathbf{A}, \neg\mathbf{B} \in \mathcal{H}$
- **Remark:** Hintikka sets are usually *defined* by the properties $\mathcal{H}_*$ above, but here we (more generally) characterize them by $\subseteq$-maximality and regain the same properties.

∇-Hintikka set

- *Proof:* We prove the properties in turn
  1. $\mathcal{H}_c$ goes by induction on the structure of A
     1.1. $A \in \mathcal{V}_0$ Then $A \notin \mathcal{H}$ or $\neg A \notin \mathcal{H}$ by ∇_c.
     1.2. $A = \neg B$
     1.2.1. Let us assume that $\neg B \in \mathcal{H}$ and $\neg\neg B \in \mathcal{H}$,
     1.2.2. then $\mathcal{H}*B \in \nabla$ by $\nabla_\neg$, and therefore $B \in \mathcal{H}$ by maximality.
     1.2.3. So both B and $\neg B$ are in $\mathcal{H}$, which contradicts the induction hypothesis.
     1.3. $A = B \vee C$ *is similar to the previous case*
  2. We prove $\mathcal{H}_\neg$ by maximality of $\mathcal{H}$ in ∇.
     2.1. If $\neg\neg A \in \mathcal{H}$, then $\mathcal{H}*A \in \nabla$ by $\nabla_\neg$.
     2.2. The maximality of $\mathcal{H}$ now gives us that $A \in \mathcal{H}$.
  3. The other $\mathcal{H}_*$ can be proven analogously.

Extension Theorem

- **Theorem 4.12.** *If ∇ is an abstract consistency class and $\Phi \in \nabla$, then there is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$.*
- *Proof:*
  1. Wlog. we assume that ∇ is compact(otherwise pass to compact extension)
  2. We choose an enumeration $A_1, \ldots$ of the set $wff_0(\mathcal{V}_0)$
  3. and construct a sequence of sets H_i with $H_0 := \Phi$ and

$$H_{n+1} := \begin{cases} H_n & \text{if } H_n * A_n \notin \nabla \\ H_n * A_n & \text{if } H_n * A_n \in \nabla \end{cases}$$

  4. Note that all $H_i \in \nabla$, choose $\mathcal{H} := \bigcup_{i \in \mathbb{N}} H_i$
  5. $\Psi \subseteq \mathcal{H}$ finite implies there is a $j \in \mathbb{N}$ such that $\Psi \subseteq H_j$,
  6. so $\Psi \in \nabla$ as ∇ is closed under subsets and $\mathcal{H} \in \nabla$ as ∇ is compact.
  7. Let $\mathcal{H} * B \in \nabla$, then there is a $j \in \mathbb{N}$ with $B = A_j$, so that $B \in H_{j+1}$ and $H_{j+1} \subseteq \mathcal{H}$
  8. Thus $\mathcal{H}$ is ∇-maximal

Valuation

- **Definition 4.13.** A function $\nu\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is called a **(propositional) valuation**, iff
  - $\nu(\neg A) = T$, iff $\nu(A) = F$
  - $\nu(A \wedge B) = T$, iff $\nu(A) = T$ and $\nu(B) = T$
- **Lemma 4.14.** *If $\nu\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is a valuation and $\Phi \subseteq wff_0(\mathcal{V}_0)$ with $\nu(\Phi) = \{T\}$, then Φ is satisfiable.*
- *Proof sketch:* $\nu|_{\mathcal{V}_0} : \mathcal{V}_0 \to \mathcal{D}_0$ is a satisfying variable assignment.
- **Lemma 4.15.** *If $\varphi\colon \mathcal{V}_0 \to \mathcal{D}_0$ is a variable assignment, then $\mathcal{I}_\varphi\colon wff_0(\mathcal{V}_0) \to \mathcal{D}_0$ is a valuation.*

Model Existence

- **Lemma 4.16 (Hintikka-Lemma).** *If ∇ is an abstract consistency class and $\mathcal{H}$ a ∇-Hintikka set, then $\mathcal{H}$ is satisfiable.*
- *Proof:*
  1. We define $\nu(\mathrm{A}) := \mathsf{T}$, iff $\mathrm{A} \in \mathcal{H}$
  2. then ν is a valuation by the Hintikka properties
  3. and thus $\nu|_{\mathcal{V}_o}$ is a satisfying assignment.
- **Theorem 4.17 (Model Existence).** *If ∇ is an abstract consistency class and $\Phi \in \nabla$, then Φ is satisfiable.*

  Proof:
- 1. There is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$ (Extension Theorem)
  2. We know that $\mathcal{H}$ is satisfiable. (Hintikka-Lemma)
  3. In particular, $\Phi \subseteq \mathcal{H}$ is satisfiable.

1.5 A Completeness Proof for Propositional ND

Consistency, Refutability and ∇-Consistency

- **Theorem 5.1 (Non-Refutability is an ACC^0).**
  $\Gamma := \{\Phi \subseteq wff_0(\mathcal{V}_0) \,|\, \Phi$ is not $\mathcal{ND}_0$-refutable$\}$ is an ACC^0.
- *Proof:* We check the properties of an ACC^0
  1. If Φ is non-refutable, then any subset is as well, so Γ is closed under subsets.

  We show the abstract consistency properties ∇_* for $\Phi \in \Gamma$.

  2. ∇_c
     2.1. We have to show that $A \notin \Phi$ or $\neg A \notin \Phi$ for atomic $A \in wff_0(\mathcal{V}_0)$.
     2.2. Equivalently, we show the contrapositive: If $\{A, \neg A\} \subseteq \Phi$, then $\Phi \notin \Gamma$.
     2.3. So let $\{A, \neg A\} \subseteq \Phi$, then Φ is $\mathcal{ND}_0$-refutable by construction.
     2.4. So $\Phi \notin \Gamma$.
  3. $\nabla_\neg$ *We show the contrapositive again*
     3.1. Let $\neg\neg A \in \Phi$ and $\Phi * A \notin \Gamma$
     3.2. Then we have a refutation $\mathcal{D}$: $\Phi * A \vdash_{\mathcal{ND}_0} F$
     3.3. By prepending an application of $\neg E$ for $\neg\neg A$ to $\mathcal{D}$, we obtain a refutation $\mathcal{D}'$: $\Phi \vdash_{\mathcal{ND}_0} F$.
     3.4. Thus $\Phi \notin \Gamma$.
  4. The other ∇_* can be proven analogously.

Henkin's Theorem

- **Corollary 5.2 (Henkin's Theorem).** *Every $\mathcal{ND}_0$-consistent set of propositions is satisfiable.*
- *Proof:*
  1. Let Φ be a $\mathcal{ND}_0$-consistent set of propositions.
  2. The collection of sets of $\mathcal{ND}_0$-consistent propositions constitute an ACC^0.
  3. Thus the model existence theorem guarantees a variable assignment that satisfies Φ.

Completeness of $\mathcal{ND}_0$

- **Theorem 5.3 (Completeness Theorem for $\mathcal{ND}_0$).** *If* $\Phi \models A$, *then* $\Phi \vdash_{\mathcal{ND}_0} A$.
- *Proof:* We prove the result by playing with negations.
  1. If $\Phi \models A$, then (by definition) A is satisfied by all variable assignment that satisfy Φ
  2. So $\Phi * \neg A$ has no satisfying assignment.
  3. Thus $\Phi * \neg A$ is inconsistent by (the contrapositive of) Henkins Theorem.
  4. So $\Phi \vdash_{\mathcal{ND}_0} \neg\neg A$ by $\mathcal{ND}_0 \neg I$ and thus $\Phi \vdash_{\mathcal{ND}_0} A$ by $\neg E$.

1.6 Completeness of Propositional Tableaux

Test Calculi: Tableaux and Model Generation

- **Idea:** A tableau calculus is a test calculus that
  - analyzes a labeled formulae in a tree to determine satisfiability,
  - its branches correspond to valuations ($\leadsto$ models).
- **Example 6.1.** Tableau calculi try to construct models for labeled formulae: E.g. the propositional tableau calculus for PL^0

Tableau refutation (Validity)	Model generation (Satisfiability)
$\models P \wedge Q \Rightarrow Q \wedge P$	$\models P \wedge (Q \vee \neg R) \wedge \neg Q$
$(P \wedge Q \Rightarrow Q \wedge P)^F$ $(P \wedge Q)^T$ $(Q \wedge P)^F$ P^T Q^T $P^F \mid Q^F$ $\bot \mid \bot$	$(P \wedge (Q \vee \neg R) \wedge \neg Q)^T$ $(P \wedge (Q \vee \neg R))^T$ $\neg Q^T$ Q^F P^T $(Q \vee \neg R)^T$ $Q^T \mid \neg R^T$ $\bot \mid R^F$
No Model	Herbrand valuation $\{P^T, Q^F, R^F\}$ $\varphi := \{P \mapsto T, Q \mapsto F, R \mapsto F\}$

- **Idea:** Open branches in saturated tableaux yield satisfying assignments.
- **Algorithm:** Fully expand all possible tableaux, (no rule can be applied)
  - Satisfiable, iff there are open branches (correspond to models)

Analytical Tableaux (Formal Treatment of $\mathcal{T}_0$)

- **Idea:** A test calculus where
  - A labeled formula is analyzed in a tree to determine satisfiability,
  - branches correspond to valuations (models)
- **Definition 6.2.** The **propositional tableau calculus** $\mathcal{T}_0$ has two inference rules per connective (one for each possible label)

$$\frac{(A \wedge B)^{T}}{\begin{array}{c}A^{T}\\ B^{T}\end{array}}\,\mathcal{T}_0\wedge \qquad \frac{(A \wedge B)^{F}}{A^{F} \mid B^{F}}\,\mathcal{T}_0\vee \qquad \frac{\neg A^{T}}{A^{F}}\,\mathcal{T}_0\neg^{T} \qquad \frac{\neg A^{F}}{A^{T}}\,\mathcal{T}_0\neg^{F} \qquad \frac{\begin{array}{c}A^{\alpha}\\ A^{\beta}\end{array} \quad \alpha \neq \beta}{\bot}\,\mathcal{T}_0\bot$$

  Use rules exhaustively as long as they contribute new material ($\rightsquigarrow$ termination)
- **Definition 6.3.** We call any tree ($\mid$ introduces branches) produced by the $\mathcal{T}_0$ inference rules from a set Φ of labeled formulae a **tableau** for Φ.
- **Definition 6.4.** Call a tableau **saturated**, iff no rule adds new material and a branch **closed**, iff it ends in $\bot$, else **open**. A tableau is **closed**, iff all of its branches are.
  In analogy to the $\bot$ at the end of closed branches, we sometimes decorate open branches with a $\Box$ symbol.

Analytical Tableaux ($\mathcal{T}_0$ continued)

- **Definition 6.6 ($\mathcal{T}_0$-Theorem/Derivability).** A is a $\mathcal{T}_0$-**theorem** ($\vdash_{\mathcal{T}_0} A$), iff there is a closed tableau with A^F at the root.
  $\Phi \subseteq wff_0(\mathcal{V}_0)$ **derives** A in $\mathcal{T}_0$ ($\Phi \vdash_{\mathcal{T}_0} A$), iff there is a closed tableau starting with A^F and Φ^T. The tableau with only a branch of A^F and Φ^T is called **initial** for $\Phi \vdash_{\mathcal{T}_0} A$.

A more Complex $\mathcal{T}_0$ Tableau

- **Example 6.8.** We construct a saturated $\mathcal{T}_0$ tableau for the formula $\neg((A \lor B) \land \neg(B \land C) \land (\neg C \lor \neg A))$:

$$\begin{array}{c}
\neg((A \lor B) \land \neg(B \land C) \land (\neg C \lor \neg A))^{\mathsf{F}} \\
((A \lor B) \land \neg(B \land C) \land (\neg C \lor \neg A))^{\mathsf{T}} \\
(A \lor B)^{\mathsf{T}} \\
(\neg(B \land C) \land (\neg C \lor \neg A))^{\mathsf{T}} \\
\neg(B \land C)^{\mathsf{T}} \\
(\neg C \lor \neg A)^{\mathsf{T}} \\
(B \land C)^{\mathsf{F}} \\
\begin{array}{c|c}
A^{\mathsf{T}} & B^{\mathsf{T}} \\
\begin{array}{c|c}
\begin{array}{c} \neg C^{\mathsf{T}} \\ C^{\mathsf{F}} \\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}} \\ \Box & \Box \end{array} \end{array} &
\begin{array}{c} \neg A^{\mathsf{T}} \\ A^{\mathsf{F}} \\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}} \\ \bot & \bot \end{array} \end{array}
\end{array} &
\begin{array}{c|c}
\begin{array}{c} \neg C^{\mathsf{T}} \\ C^{\mathsf{F}} \\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}} \\ \bot & \Box \end{array} \end{array} &
\begin{array}{c} \neg A^{\mathsf{T}} \\ A^{\mathsf{F}} \\ \begin{array}{c|c} B^{\mathsf{F}} & C^{\mathsf{F}} \\ \bot & \Box \end{array} \end{array}
\end{array}
\end{array}
\end{array}$$

So we have four closed branches (they end in $\bot$), and four open ones (decorated by $\Box$), these correspond to counter-examples to validity.

Abstract Consistency for $\mathcal{T}_0$ I

- **Lemma 6.9.** $\nabla := \{\Phi \,|\, \Phi^{\mathsf{T}}$ *has no closed* $\mathcal{T}_0$*-tableau*$\}$ *is an ACC*^{0}.
- *Proof:* We convince ourselves of the abstract consistency properties
  1. For ∇_c, let $P, \neg P \in \Phi$ implies $P^{\mathsf{F}}, P^{\mathsf{T}} \in \Phi^{\mathsf{T}}$.
     1.1. So a single application of $\mathcal{T}_0\bot$ yields a closed tableau for Φ^{T}
  2. For $\nabla_\neg$, let $\neg\neg\mathrm{A} \in \Phi$.
     2.1. For the proof of the contrapositive we assume that $\Phi * \mathrm{A}$ has a closed tableau $\mathcal{T}$ and show that already Φ has one:
     2.2. Applying each of $\mathcal{T}_0\neg^{\mathsf{T}}$ and $\mathcal{T}_0\neg^{\mathsf{F}}$ once allows to extend any tableau branch that contains $\neg\neg\mathrm{B}^\alpha$ by B^α.
     2.3. Any branch in $\mathcal{T}$ that is closed with $\neg\neg\mathrm{A}^\alpha$, can be closed by A^α.

Abstract Consistency for $\mathcal{T}_0$ II

3. $\nabla_\vee$ *Suppose* $A \vee B \in \Phi$ *and both* $\Phi * A$ *and* $\Phi * B$ *have closed tableaux*
   3.1. Consider the tableaux:

$$\begin{array}{c} \Phi^T \\ A^T \\ Rest^1 \end{array} \qquad \begin{array}{c} \Phi^T \\ B^T \\ Rest^2 \end{array} \qquad \begin{array}{c} \Psi^T \\ (A \vee B)^T \\ \begin{array}{c|c} A^T & B^T \\ Rest^1 & Rest^2 \end{array} \end{array}$$

4. $\nabla_\wedge$ *Suppose,* $\neg(A \vee B) \in \Phi$ *and* $\Phi\{\neg A, \neg B\}$ *have closed tableau* $\mathcal{T}$.
   4.1. We consider

$$\begin{array}{c} \Phi^T \\ A^F \\ B^F \\ Rest \end{array} \qquad \begin{array}{c} \Psi^T \\ (A \vee B)^F \\ A^F \\ B^F \\ Rest \end{array}$$

where $\Phi = \Psi * \neg(A \vee B)$.

Completeness of $\mathcal{T}_0$

- **Corollary 6.10.** $\mathcal{T}_0$ *is complete.*
- *Proof:* by contradiction
  1. We assume that $A \in wff_0(\mathcal{V}_0)$ is valid, but there is no closed tableau for A^F.
  2. We have $\{\neg A\} \in \nabla$ as $\neg A^T = A^F$.
  3. So $\neg A$ is satisfiable by the model-existence theorem (which is applicable as ∇ is an abstract consistency class by our Lemma above).
  4. This contradicts our assumption that A is valid.

1.7 Abstract Consistency and Model Existence for First-Order Logic

Abstract Consistency

- **Definition 7.1.** A collection $\nabla \subseteq wff_o(\Sigma_\iota, \mathcal{V}_\iota)$ of sets of formulae is called a **first-order abstract consistency class** (**ACC^1**), iff it is a ACC^0 and additionally
  - $\nabla_\forall$) If $\forall X.A \in \Phi$, then $\Phi * (A[\frac{B}{X}]) \in \nabla$ for each closed term B.
  - $\nabla_\exists$) If $\neg(\forall X.A) \in \Phi$ and c is an individual constant that does not occur in Φ, then $\Phi * \neg(A[\frac{c}{X}]) \in \nabla$
- **Example 7.2.** The collection $\{\emptyset, \{\forall x.p(x)\}\}$ is an ACC^1. (no closed terms)
- **Example 7.3.** The collection $\Phi := \{\emptyset, \{p(a)\}, \{\forall x.p(x)\}\}$ is not an ACC^1.
  $\leadsto$ $\{p(a), \forall x.p(x)\}$ is missing from Φ.
- **Example 7.4.** The collection $\Phi := \{\emptyset, \{\exists x.p(x)\}\}$ is not an ACC^1.
  $\leadsto$ $\{p(c), \exists x.p(x)\}$ is missing from Φ or some individual constant c

Compact Abstract Consistency Classes

- **Lemma 7.5.** *Any ACC^1 can be extended to a compact one.*
- *Proof:* We extend the proof for propositional logic; we only have to look at the two new abstract consistency properties.
  1. Again, we choose $\nabla' := \{\Phi \subseteq cwff_o(\Sigma_\iota) \,|\, \text{every finite subset of } \Phi \text{ is in } \nabla\}$. This can be seen to be closed under subsets and compact by the same argument as above.
  2. To show $\nabla_\forall$ for ∇', let $\Phi \in \nabla'$ and $\forall X.A \in \Phi$.
     2.1. Let Ψ be any finite subset of $\Phi * (A[\frac{B}{X}])$, and $\Theta := (\Psi \backslash \{A[\frac{B}{X}]\}) * (\forall X.A)$.
     2.2. Θ is a finite subset of Φ, so $\Theta \in \nabla$.
     2.3. Since ∇ is a ACC^1 and $A[\frac{B}{X}] \in \Theta$, we get $\Theta * (\forall X.A) \in \nabla$ by $\nabla_\forall$.
     2.4. We know that $\Psi \subseteq \Theta * (A[\frac{B}{X}])$ and ∇ is closed under subsets, so $\Psi \in \nabla$.
     2.5. Thus every finite subset Ψ of $\Phi * (A[\frac{B}{X}])$ is in ∇ and therefore by definition $\Phi * (A[\frac{B}{X}]) \in \nabla'$.
  3. The $\nabla_\exists$ case are analogous to that for $\nabla_\forall$.

∇-Hintikka set

- **Theorem 7.6 (Hintikka Properties).** *Let ∇ be a ACC[1] and $\mathcal{H}$ be a ∇-Hintikka set, then $\mathcal{H}$ has all the propositional Hintikka properties plus*

  $\mathcal{H}_\forall$) *If* $\forall X.A \in \mathcal{H}$, *then* $A[\frac{B}{X}] \in \mathcal{H}$ *for each closed term* B.

  $\mathcal{H}_\exists$) *If* $\neg(\forall X.A) \in \mathcal{H}$ *then* $\neg(A[\frac{B}{X}]) \in \mathcal{H}$ *for some closed term* B.

- *Proof:* We prove the two new cases
  1. We prove $\mathcal{H}_\forall$ by maximality of $\mathcal{H}$ in ∇.
     1.1. If $\forall X.A \in \mathcal{H}$, then $\mathcal{H} * (A[\frac{B}{X}]) \in \nabla$ by $\nabla_\forall$.
     1.2. The maximality of $\mathcal{H}$ now gives us that $A[\frac{B}{X}] \in \mathcal{H}$.
  2. The proof of $\mathcal{H}_\exists$ is similar

Extension Theorem

- **Theorem 7.7.** *If ∇ is a ACC*[1] *and $\Phi \in \nabla$ finite, then there is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$.*
- *Proof:*
  1. Wlog. assume that ∇ compact (else use compact extension)
  2. Choose an enumeration $A_1, \ldots$ of $cwff_o(\Sigma_\iota)$ and $c_1, c_2, \ldots$ of Σ_0^{sk}.
  3. and construct a sequence of sets H_i with $H_0 := \Phi$ and

$$H_{n+1} := \begin{cases} H_n & \text{if } H_n{*}A_n \notin \nabla \\ H_n \cup \{A_n, \neg(B[\frac{c_n}{X}])\} & \text{if } H_n{*}A_n \in \nabla \text{ and } A_n = \neg(\forall X.B) \\ H_n{*}A_n & \text{else} \end{cases}$$

  4. Note that all $H_i \in \nabla$, choose $\mathcal{H} := \bigcup_{i \in \mathbb{N}} H_i$
  5. $\Psi \subseteq \mathcal{H}$ finite implies there is a $j \in \mathbb{N}$ such that $\Psi \subseteq H_j$,
  6. so $\Psi \in \nabla$ as ∇ closed under subsets and $\mathcal{H} \in \nabla$ as ∇ is compact.
  7. Let $\mathcal{H}{*}B \in \nabla$, then there is a $j \in \mathbb{N}$ with $B = A_j$, so that $B \in H_{j+1}$ and $H_{j+1} \subseteq \mathcal{H}$
  8. Thus $\mathcal{H}$ is ∇-maximal

What now?

- The next step is to take a ∇-Hintikka set – the extension lemma above gives us one – and show that it is satisfiable.
- **Problem:** For that we have to conjure a model $\langle\mathcal{A},\mathcal{I}\rangle$ out of thin air.
- **Idea 1:** Maybe the ∇-Hintikka set will help us with the interpretation
  $\leftrightsquigarrow$ After all it helped us with the variable assignments in PL^0.
- **Idea 2:** For the universe we use something that is already lying around:
  $\rightsquigarrow$ The set $cwff_\iota(\Sigma)$ of closed terms!
- Again, the notion of a valuation helps write things down, so we start with that.
- Tighten your seat belts and hold on.

Valuations

- **Definition 7.8.** A function $\nu\colon cwff_o(\Sigma_\iota) \to \mathcal{D}_0$ is called a **(first-order) valuation**, iff ν is a propositional valuation and
  - $\nu(\forall X.\mathrm{A}) = \mathsf{T}$, iff $\nu(\mathrm{A}[\frac{\mathrm{B}}{X}]) = \mathsf{T}$ for all closed terms B.
- **Lemma 7.9.** *If $\varphi\colon \mathcal{V}_\iota \to U$ is a variable assignment, then $\mathcal{I}_\varphi\colon cwff_o(\Sigma_\iota) \to \mathcal{D}_0$ is a valuation.*
- *Proof sketch:* Immediate from the definitions.

Valuation and Satisfiability I

- **Lemma 7.10.** *If* $\nu\colon cwff_o(\Sigma_\iota) \to \mathcal{D}_0$ *is a valuation and* $\Phi \subseteq cwff_o(\Sigma_\iota)$ *with* $\nu(\Phi) = \{\mathsf{T}\}$*, then* Φ *is satisfiable.*
- *Proof:* We construct a model $\mathcal{M} := \langle \mathcal{D}_\iota, \mathcal{I} \rangle$ for Φ.
  1. Let $\mathcal{D}_\iota := cwff_\iota(\Sigma_\iota)$, and
     - $\mathcal{I}(f) : {\mathcal{D}_\iota}^k \to \mathcal{D}_\iota\ ; \langle \mathrm{A}_1, \ldots, \mathrm{A}_k \rangle \mapsto f(\mathrm{A}_1, \ldots, \mathrm{A}_k)$ for $f \in \Sigma^f$
     - $\mathcal{I}(p) : {\mathcal{D}_\iota}^k \to \mathcal{D}_0\ ; \langle \mathrm{A}_1, \ldots, \mathrm{A}_k \rangle \mapsto \nu(p(\mathrm{A}_1, \ldots, \mathrm{A}_k))$ for $p \in \Sigma^p$.
  2. Then variable assignments into $\mathcal{D}_\iota$ are ground substitutions.
  3. We show $\mathcal{I}_\varphi(\mathrm{A}) = \mathrm{A}\varphi$ for $\mathrm{A} \in wff_\iota(\Sigma_\iota, \mathcal{V}_\iota)$ by induction on A:
     - 3.1. If $\mathrm{A} = X$, then $\mathcal{I}_\varphi(\mathrm{A}) = X\varphi$ by definition.
     - 3.2. If $\mathrm{A} = f(\mathrm{A}_1, \ldots, \mathrm{A}_k)$, then $\mathcal{I}_\varphi(\mathrm{A}) = \mathcal{I}(f)(\mathcal{I}_\varphi(\mathrm{A}_1), \ldots, \mathcal{I}_\varphi(\mathrm{A}_n)) = \mathcal{I}(f)(\mathrm{A}_1\varphi, \ldots, \mathrm{A}_n\varphi) = f(\mathrm{A}_1\varphi, \ldots, \mathrm{A}_n\varphi) = f(\mathrm{A}_1, \ldots, \mathrm{A}_k)\varphi = \mathrm{A}\varphi$

Valuation and Satisfiability II

4. We show $\mathcal{I}_\varphi(A) = \nu(A\varphi)$ for $A \in wff_o(\Sigma_\iota, \mathcal{V}_\iota)$ by induction on A.
   4.1. If $A = p(A_1, \ldots, A_k)$ then $\mathcal{I}_\varphi(A) = \mathcal{I}(p)(\mathcal{I}_\varphi(A_1), \ldots, \mathcal{I}_\varphi(A_n)) = \mathcal{I}(p)(A_1\varphi, \ldots, A_n\varphi) = \nu(p(A_1\varphi, \ldots, A_n\varphi)) = \nu(p(A_1, \ldots, A_k)\varphi) = \nu(A\varphi)$
   4.2. If $A = \neg B$ then $\mathcal{I}_\varphi(A) = T$, iff $\mathcal{I}_\varphi(B) = \nu(B\varphi) = F$, iff $\nu(A\varphi) = T$.
   4.3. $A = B \wedge C$ is similar
   4.4. If $A = \forall X.B$ then $\mathcal{I}_\varphi(A) = T$, iff $\mathcal{I}_\psi(B) = \nu(B\psi) = T$, for all $C \in \mathcal{D}_\iota$, where $\psi = \varphi,[\frac{C}{X}]$. This is the case, iff $\nu(A\varphi) = T$.
5. Thus $\mathcal{I}_\varphi(A) = \nu(A\varphi) = \nu(A) = T$ for all $A \in \Phi$.
6. Hence $\mathcal{M} \models A$.

Herbrand-Model

- **Definition 7.11.** Let $\Sigma := \langle \Sigma^f, \Sigma^p \rangle$ be a first-order signature, then we call $\langle \mathcal{D}, \mathcal{I} \rangle$ a **Herbrand model**, iff
  1. $\mathcal{D} = cwff_\iota(\Sigma)$ – i.e. the Herbrand universe over Σ.
  2. $\mathcal{I}(f) : \mathcal{D}^k \to \mathcal{D}\,; \langle \mathrm{A}_1, \ldots, \mathrm{A}_k \rangle \mapsto f(\mathrm{A}_1, \ldots, \mathrm{A}_k)$ for function constants $f \in \Sigma_k^f$, and
  3. $\mathcal{I}(p) \subseteq \mathcal{D}^k$ for predicate constants p.
- **Note:** Variable assignments into $\mathcal{D} = cwff_\iota(\Sigma)$ are naturally ground substitutions by construction.

Herbrand-Model

- **Definition 7.15.** Let $\Sigma := \langle \Sigma^f, \Sigma^p \rangle$ be a first-order signature, then we call $\langle \mathcal{D}, \mathcal{I} \rangle$ a **Herbrand model**, iff
  1. $\mathcal{D} = cwff_\iota(\Sigma)$ – i.e. the Herbrand universe over Σ.
  2. $\mathcal{I}(f) : \mathcal{D}^k \to \mathcal{D} ; \langle \mathrm{A}_1, \ldots, \mathrm{A}_k \rangle \mapsto f(\mathrm{A}_1, \ldots, \mathrm{A}_k)$ for function constants $f \in \Sigma^f_k$, and
  3. $\mathcal{I}(p) \subseteq \mathcal{D}^k$ for predicate constants p.
- **Note:** Variable assignments into $\mathcal{D} = cwff_\iota(\Sigma)$ are naturally ground substitutions by construction.
- **Lemma 7.16.** $\mathcal{I}_\varphi(t) = t\varphi$ *for terms* t.
  Proof sketch: By induction on the structure of A.
- **Corollary 7.17.** *A Herbrand model* $\mathcal{M}$ *can be represented by the set* $H_\mathcal{M} = \{\mathrm{A} \in cwff(\Sigma) \,|\, \mathrm{A}$ *atomic and* $\mathcal{M} \models \Phi\}$ *of closed atoms it satisfies.*
  Proof: Let $\mathrm{A} = p(t_1, \ldots, t_k)$.
  1. $\mathcal{I}_\varphi(\mathrm{A}) = \mathcal{I}_\varphi(p(t_1, \ldots, t_k)) = \mathcal{I}(p)(\langle t_1\varphi, \ldots, t_k\varphi \rangle) = \mathsf{T}$, iff $\mathrm{A} \in H_\mathcal{M}$.
  2. In the definition of Herbrand model, only the interpretation of predicate constants is flexible, and $H_\mathcal{M}$ determines that.

Herbrand-Model

- **Definition 7.19.** Let $\Sigma := \langle \Sigma^f, \Sigma^p \rangle$ be a first-order signature, then we call $\langle \mathcal{D}, \mathcal{I} \rangle$ a **Herbrand model**, iff
  1. $\mathcal{D} = \mathit{cwff}_\iota(\Sigma)$ – i.e. the Herbrand universe over Σ.
  2. $\mathcal{I}(f) : \mathcal{D}^k \to \mathcal{D}\,;\, \langle \mathrm{A}_1, \ldots, \mathrm{A}_k \rangle \mapsto f(\mathrm{A}_1, \ldots, \mathrm{A}_k)$ for function constants $f \in \Sigma_k^f$, and
  3. $\mathcal{I}(p) \subseteq \mathcal{D}^k$ for predicate constants p.
- **Note:** Variable assignments into $\mathcal{D} = \mathit{cwff}_\iota(\Sigma)$ are naturally ground substitutions by construction.
- **Lemma 7.20.** $\mathcal{I}_\varphi(t) = t\varphi$ *for terms* t.

  Proof sketch: By induction on the structure of A.
- **Corollary 7.21.** *A Herbrand model* $\mathcal{M}$ *can be represented by the set* $H_\mathcal{M} = \{\mathrm{A} \in \mathit{cwff}(\Sigma) \,|\, \mathrm{A}$ *atomic and* $\mathcal{M} \models \Phi\}$ *of closed atoms it satisfies.*

  Proof: Let $\mathrm{A} = p(t_1, \ldots, t_k)$.
  1. $\mathcal{I}_\varphi(\mathrm{A}) = \mathcal{I}_\varphi(p(t_1, \ldots, t_k)) = \mathcal{I}(p)(\langle t_1\varphi, \ldots, t_k\varphi \rangle) = \mathsf{T}$, iff $\mathrm{A} \in H_\mathcal{M}$.
  2. In the definition of Herbrand model, only the interpretation of predicate constants is flexible, and $H_\mathcal{M}$ determines that.
- **Theorem 7.22 (Herbrand's Theorem).** *A set* Φ *of first-order propositions is satisfiable, iff it has a Herbrand model.*

Model Existence

- **Theorem 7.23 (Hintikka-Lemma).** *If ∇ is an ACC[1] and $\mathcal{H}$ a ∇-Hintikka set, then $\mathcal{H}$ is satisfiable.*
- *Proof:*
  1. we define $\nu(A) := T$, iff $A \in \mathcal{H}$,
  2. then ν is a valuation by the Hintikka set properties.
  3. We have $\nu(\mathcal{H}) = \{T\}$, so $\mathcal{H}$ is satisfiable.
- **Theorem 7.24 (Model Existence).** *If ∇ is an ACC[1] and $\Phi \in \nabla$, then Φ is satisfiable.*

  Proof:
- 1. There is a ∇-Hintikka set $\mathcal{H}$ with $\Phi \subseteq \mathcal{H}$ (Extension Theorem)
  2. We know that $\mathcal{H}$ is satisfiable. (Hintikka-Lemma)
  3. In particular, $\Phi \subseteq \mathcal{H}$ is satisfiable.

1.8 A Completeness Proof for First-Order ND

Consistency, Refutability and ∇-consistent

- **Theorem 8.1 ($\mathcal{ND}^1$-Non-Refutability is an ACC^1).**
  $\Gamma := \{\Phi \subseteq cwff_o(\Sigma_\iota) \,|\, \Phi$ *is not* $\mathcal{ND}^1$*-refutable*$\}$ *is an ACC^1.*
- *Proof:* We check the two additional properties of an ACC^1
  1. $\nabla_\forall$: We use the contrapositive
     1.1. So let $\forall X.A \in \Phi$, $\Phi \in \Gamma$, and $\Phi * (A[\frac{A}{X}]) \notin \Gamma$,
     1.2. then there is a $\mathcal{ND}^1$-refutation of $\Phi * (A[\frac{A}{X}])$.
     1.3. Prepending $\forall E$ to that, gives us a $\mathcal{ND}^1$-refutation of Φ.
     1.4. So $\Phi \notin \Gamma$, which is what we wanted to show for the contrapositive of $\nabla_\forall$.
  2. $\nabla_\exists$ can be proven similarly using $\forall I$

Henkin's Theorem

- **Corollary 8.2 (Henkin's Theorem).** *Every $\mathcal{ND}^1$-consistent set of sentences has a model.*
- *Proof:*
  1. Let Φ be a $\mathcal{ND}^1$-consistent set of sentences.
  2. The collection of sets of $\mathcal{ND}^1$-consistent sentences constitute an ACC^1.
  3. Thus the model existence theorem guarantees a model for Φ.
- **Corollary 8.3 (Löwenheim&Skolem Theorem).** *Any satisfiable set Φ of first-order sentences has a countable model.*

  Proof sketch: The model we constructed is countable, since the set of ground terms is.

Completeness and Compactness

- **Theorem 8.4 (Completeness Theorem for $\mathcal{ND}^1$).** *If* $\Phi \vDash A$, *then* $\Phi \vdash_{\mathcal{ND}^1} A$.
- *Proof:* We prove the result by playing with negations.
  1. If A is valid in all models of Φ, then $\Phi * \neg A$ has no model
  2. Thus $\Phi * \neg A$ is inconsistent by (the contrapositive of) Henkins Theorem.
  3. So $\Phi \vdash_{\mathcal{ND}^1} \neg\neg A$ by $\mathcal{ND}_0 \neg I$ and thus $\Phi \vdash_{\mathcal{ND}^1} A$ by $\neg E$.
- **Theorem 8.5 (Compactness Theorem for first-order logic).** *If* $\Phi \vDash A$, *then there is already a finite set* $\Psi \subseteq \Phi$ *with* $\Psi \vDash A$.

  Proof: This is a direct consequence of the completeness theorem
- 1. We have $\Phi \vDash A$, iff $\Phi \vdash_{\mathcal{ND}^1} A$.
  2. As a proof is a finite object, only a finite subset $\Psi \subseteq \Phi$ can appear as leaves in the proof.

1.9 Completeness of First-Order Tableaux

First-Order Standard Tableaux ($\mathcal{T}_1$) are Complete

- **Definition 9.1.** The **standard tableau calculus** ($\mathcal{T}_1$) extends $\mathcal{T}_0$ (propositional tableau calculus) with the following quantifier rules:

$$\frac{(\forall X.A)^{\mathsf{T}} \quad C \in cwff_\iota(\Sigma_\iota)}{(A[\frac{C}{X}])^{\mathsf{T}}} \mathcal{T}_1\forall \qquad \frac{(\forall X.A)^{\mathsf{F}} \quad c \in \Sigma_0^{sk} \text{ new}}{(A[\frac{c}{X}])^{\mathsf{F}}} \mathcal{T}_1\exists$$

First-Order Standard Tableaux ($\mathcal{T}_1$) are Complete

- **Definition 9.3.** The **standard tableau calculus** ($\mathcal{T}_1$) extends $\mathcal{T}_0$ (propositional tableau calculus) with the following quantifier rules:

$$\frac{(\forall X.A)^{\mathsf{T}} \quad C \in cwff_\iota(\Sigma_\iota)}{(A[\frac{C}{X}])^{\mathsf{T}}} \mathcal{T}_1\forall \qquad \frac{(\forall X.A)^{\mathsf{F}} \quad c \in \Sigma_0^{sk} \text{ new}}{(A[\frac{c}{X}])^{\mathsf{F}}} \mathcal{T}_1\exists$$

- **Theorem 9.4.** *$\mathcal{T}_1$ is refutation complete.*
- *Proof:* We show that $\nabla := \{\Phi \mid \Phi^{\mathsf{T}}$ has no closed $\mathcal{T}_1$tableau$\}$ is an ACC^1
  1. ∇_c, $\nabla_\neg$, $\nabla_\vee$, and $\nabla_\wedge$ as for $\mathcal{T}_0$; $\nabla_\forall$ similar to the next ($\nabla_\exists$) below.
  2. $\nabla_\exists$: We prove the contrapositive
     - 2.1. Let $\Phi = \Psi * (\exists X.A)$, but $\Phi * (A[\frac{c}{X}]) \notin \nabla$,
     - 2.2. then $\Phi * (A[\frac{c}{X}])$ has a closed $\mathcal{T}_1$-tableau (on the left).

Ψ^{T}	Ψ^{T}
$(\exists X.A)^{\mathsf{T}}$	$(\exists X.A)^{\mathsf{T}}$
$(A[\frac{c}{X}])^{\mathsf{T}}$	$(A[\frac{c}{X}])^{\mathsf{T}}$
Rest	*Rest*

     The right $\mathcal{T}_1$-tableau starts with $\Phi = \Psi * (\exists X.A)$ and applies $\mathcal{T}_1\exists$ and then continues as on the left.
  3. We argue from $\nabla \mathrel{\hat{=}} ACC^1$ to completeness as above.

References I