Overview of E0 205
Mathematical Logic and Theorem Proving

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Mathematical Logic and Theorem Proving

- "Mathematical" Logic (pioneered by Boole, Frege, Russell, Hilbert, Gödel, ...)
  - Goals related to foundations of Mathematics (formalizing set theory and mathematical reasoning techniques)
  - Applications in Math and Theoretical CS (e.g. Büchi's logical characterisations of regular languages)

  as opposed to Philosophical Logic.
- SMT (SAT+Decision Procedures for certain theories) vs Theorem Proving
  - Fully automated vs Interactive.

Why study Logic in Computer Science?

Computability

- Notions of computability were proposed to answer questions in logic
  - Formalizing mathematics (coming up with a complete proof system, deciding truth of logical statements) led to Hilbert proposing the "Entscheidungsproblem" (decision problem for logical validity).
  - Church and Turing separately proposed Lambda Calculus and Turing machines as notions of computability, and showed the Entscheidungsproblem was undecidable.
- Natural computational problems
  - SAT complete for NP, Horn-SAT complete for P
  - FO with fixpoints.

Why study Logic in Computer Science?

Verification and Synthesis

- Specification languages
  - Temporal Logic
  - Floyd-Hoare Logic
- Checking whether a program/system satisfies a specification
  - Program satisfies a pre-post specification if generated Verification Conditions (VCs) are logically valid.
  - Model-Checking procedures for Temporal Logics.
  - Constrained Horn Clauses
- Symbolic Analysis
  - Symbolic Model-Checking
  - Predicate abstraction
  - Controller Synthesis

Others (Proofs as Types, Algorithmic meta theorems, etc)

Course Contents

- Mathematical Logic
  - Propositional and First-Order Logic
  - Definability
  - Normal Forms
  - Sound and complete proof systems (Sequent Calculus)
  - Compactness and Lowenheim-Skolem Theorem
- Decision Procedures
  - DPLL procedure for Propositional Logic (SAT)
  - Equality and Uninterpreted Functions (EUF)
  - Real and Integer Linear Arithmetic
  - Array logic
  - Nelson-Oppen combination

Example FO Logic Formula

$$\forall x \exists y (x < y \wedge op(y, x) = e)$$

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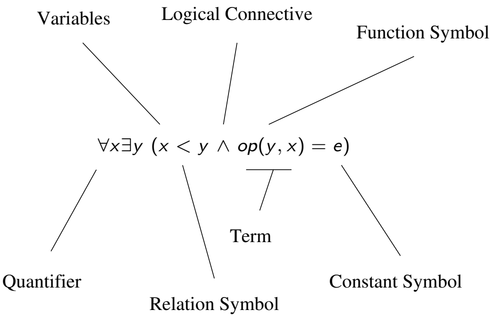


FO Signature

A First-Order signature is a tuple

$$S = (R, F, C)$$

where

- R is a countable set of relation symbols
- F is a countable set of function symbols
- C is a countable set of constant symbols

Each relational/functional symbol comes with an associated "arity".

Example FO signatures

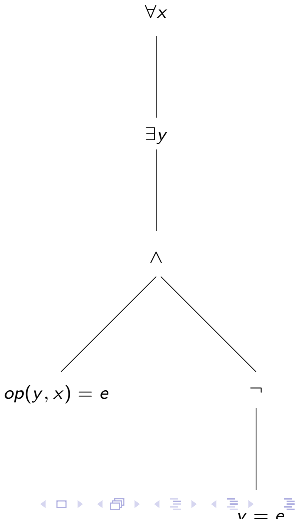
- $S_{gr} = (\{\}, \{op^{(2)}\}, \{e\})$ (Groups)
- $S_{ogr} = (\{<^{(2)}\}, \{op^{(2)}\}, \{e\})$ (Ordered Groups)
- $S_{ar} = (\{\}, \{+^{(2)}, \cdot^{(2)}\}, \{0, 1\})$ (Arithmetic)
- $S_{eq} = (\{r^{(2)}\}, \{\}, \{\})$ (Equivalence Relations)

Semantics: Example

Find the truth of the S_{gr}-formula

$$\forall x \exists y ((op(y,x) = e) \wedge \neg (y = e))$$

in the structure $(\mathbb{Z}, +, 0)$.



Theories

An S-theory is a set of S-sentences T which is satisfiable and closed under logical consequence.

The theory of a set of S-formulas T, written "$Th(T)$", is the set of S-sentences that are logical consequences of T. That is:

$$Th(T) = \{\varphi \in L_0^S \mid T \models \varphi\}.$$

Theory of Groups $Th(\Phi_{gr})$

Let Φ_{gr} be the set of formulas (group axioms):

$$\forall x \forall y \forall z\ (op(op(x,y),z) = op(x, op(y,z))) \quad (1)$$

$$\forall x\ (op(x,e) = x) \quad (2)$$

$$\forall x \exists y\ (op(x,y) = e) \quad (3)$$

Then $Th(\Phi_{gr})$

- Contains $\forall x \exists y (op(y,x) = e)$, but

Example of Group Theory

Group Axioms Φ_{Gr}

$$\forall x \forall y \forall z\ ((x \circ y) \circ z = x \circ (y \circ z)) \tag{4}$$
$$\forall x\ (x \circ e = x) \tag{5}$$
$$\forall x \exists y\ (x \circ y = e) \tag{6}$$

Structures for Φ_{Gr}: $(\mathbb{Z}, +, 0)$ and $(\mathbb{R}, +, 0)$; but not $(\mathbb{R}, \cdot, 1)$.

Theorem: Every element of a group has a left-inverse: $\forall x \exists y (y \circ x = e)$.

Question: is there a complete proof system for Group theory? That is, whenever we have $\Phi_{Gr} \models \varphi$, then we also have $\Phi_{Gr} \vdash \varphi$.

Gödel's Completeness Theorem

Let $\Phi \vdash \varphi$ denote a derivation of φ from Φ using the Sequent Calculus proof system.

Theorem (Completeness)

For any set of first-order logic sentences Φ:

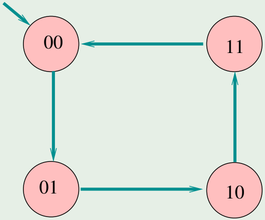
$$\Phi \models \varphi \text{ iff } \Phi \vdash \varphi.$$

Some consequences of the theorem and its proof:

- There is a complete proof system for Group Theory (Sequent Calculus + Φ_{Gr} as axioms).
- (Lowenheim-Skolem) If a set of FO formulas Φ is satisfiable then it is satisfiable in a countable model.
- (Compactness) If a set of formulas Φ is unsatisfiable, then there is a finite subset of Φ which is unsatisfiable.

Boolean SAT solving

Does the system satisfy the temporal logic formula $G(b \implies X(\neg b))$?



In bounded model-checking we could ask for a path of length 2 that violates the specification: Is

$$\neg a_0 \wedge \neg b_0 \wedge T(a_0, b_0, a_1, b_1) \wedge T(a_1, b_1, a_2, b_2) \wedge b_1 \wedge b_2,$$

where $T(a, b, a', b') = (\neg a \wedge a' \wedge b \iff b') \vee (a \wedge \neg a' \wedge b \iff \neg b')$, satisfiable?

Linear Arithmetic

Bounded model-checking for programs:

```
int x = 19;
int y = 15;
while (x >= 10) {
  int z = -1;
  x = x + z;
}
assert(y >= x);
```

Does there exist zero-iteration execution violating the assertion: Is

$$x_1 = 19 \land y_1 = 15 \land x_1 < 10 \land y_1 < x_1$$

satisfiable?

Linear Arithmetic

Floyd-Hoare style verification of programs:

```
int x = 19;
int y = 15;
// inv: x <= 20
while (x >= 10) {
  int z = -1;
  x = x + z;
}
assert(y >= x);
```

Are the constraints: $\forall x, y, z, x'$:

$$x = 19 \implies x \leq 20$$

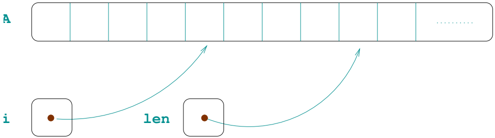
$$x \leq 20 \wedge x \geq 10 \wedge z = -1 \wedge x' = x + z \implies x' \leq 20$$

$$x \leq 20 \wedge y = 15 \wedge \neg(x \geq 10) \implies y \geq x'$$

valid?

Array Logic

```
ainit(int A[], int len) {
// Pre: 0 <= len
  int i = 0;
  while (i < len) {
    A[i] = 0;
    i = i + 1;
  }
}
// Post:
forall k: ((0 <= k < len)
              => A[k] = 0)
```



Loop invariant:

$$(0 \leq i \leq len) \wedge \forall k((0 \leq k < i) \implies A[k] = 0)$$

Verification condition:

$$[(0 \leq i \leq len) \wedge \forall k((0 \leq k < i) \implies A[k] = 0) \wedge \neg(i < len)] \implies$$

$$\forall k : ((0 \leq k < len) \implies A[k] = 0).$$

Program Transformation

Example: Are these programs equivalent?

```
S1: z := (x1 + y1) * (x2 + y2);
```

```
T1: u1 := (x1 + y1);
T2: u2 := (x2 + y2);
T3: z := u1 * u2;
```

We want to check whether (forall $x_1, x_2, y_1, y_2, z_1, z_2, u_1, u_2$)

$$
\begin{aligned}
&(z_1 = (x_1 + y_1) * (x_2 + y_2) \wedge \\
&\; u_1 = x_1 + y_1 \;\wedge u_2 = x_2 + y_2 \;\wedge z_2 = u_1 * u_2) \\
&\rightarrow z_1 = z_2.
\end{aligned}
$$

Since reasoning about 32 bit ints and addition and multiplication is difficult, We could instead check whether the EUF formula:

$$
\begin{aligned}
&(z_1 = G(F(x_1, y_1), F(x_2, y_2)) \wedge \\
&\; u_1 = F(x_1, y_1) \;\wedge u_2 = F(x_2 + y_2) \;\wedge z_2 = G(u_1, u_2)) \\
&\rightarrow z_1 = z_2.
\end{aligned}
$$

is valid.

Nelson-Oppen Combination

Example: Is this sentence satisfiable?

$$f(f(x) - f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge z \geq 0$$

Nelson-Oppen Combination

Example: Is this sentence satisfiable?

$$f(f(x) - f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge z \geq 0$$

No, because the arithmetic constraints imply that $x = y$ and $z = 0$; and the functional constraints must then imply that $f(f(x) - f(y)) = f(0) = f(z)$.

Nelson-Oppen Combination

Shows how we can combine decision procedures for two theories into a decision procedure for their union.

"Equality Sharing" Procedure:

Is this sentence satisfiable?

$$f(f(x) - f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge z \geq 0$$

Arithmetic Constraints

$$\begin{array}{rcl} x & \leq & y \\ y + z & \leq & x \\ z & \geq & 0 \\ g_2 - g_3 & = & g_1 \end{array}$$

Function Constraints

$$\begin{array}{rcl} f(g_1) & \neq & f(z) \\ f(x) & = & g_2 \\ f(y) & = & g_3 \end{array}$$

Constrained Horn Clauses

```
int x = 19;
while (*) {
  int z = f();
  x = x + z;
}
int y = g();
assert(y >= x);
```

Find unary relations f, g and *inv* such that:

$$x = 19 \implies inv(x)$$
$$inv(x) \wedge \mathrm{f}(z) \wedge x' = x + z \implies inv(x')$$
$$inv(x) \wedge \mathrm{g}(y) \implies y \geq x$$

Constrained Horn Clauses

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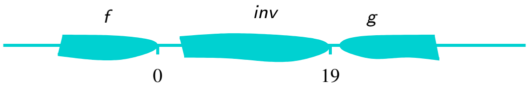
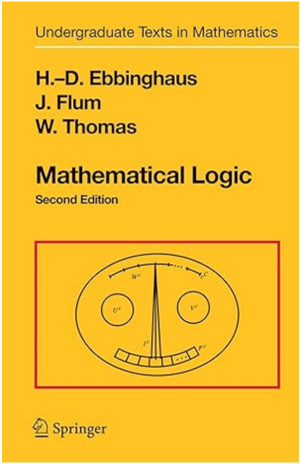
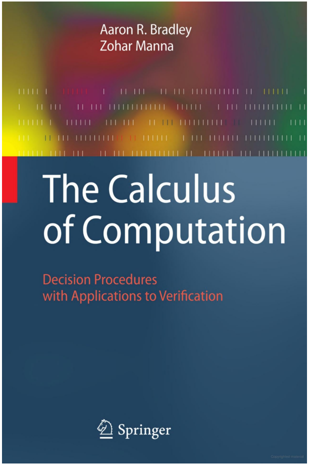
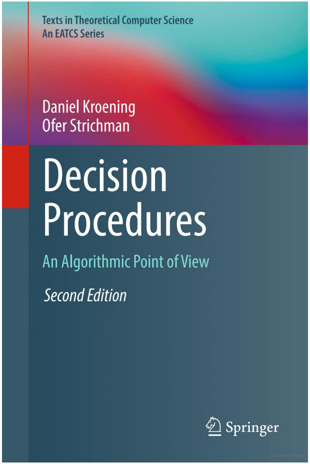


Course Textbooks







Course Details

- Weightage: 40% assignments + seminar, 20% midsem exam, 40% final exam.
- Assignments to be done on your own.
- Dishonesty Policy: Any instance of copying in an assignment will fetch you a 0 in that assignment + one grade reduction + report to DCC.
- Seminar (in pairs) can be chosen from list on course webpage or your own topic.
- Course webpage:
  `www.csa.iisc.ac.in/~deepakd/logic-2025`
- Teaching assistants for the course: Alan Jojo and Abhishek Uppar
- Those interested in crediting/auditing please send me an email so that I can add you to the course Teams / mailing list.