On the Expressiveness of TPTL in the Pointwise and Continuous Semantics

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Abstract. We show that the expressiveness of Timed Propositional Temporal Logic (TPTL) [1] with the since operator, coincides for the pointwise and continuous semantics. We do this by first going over to an equivalent first-order logic with linear constraints, interpreted over timed words. We then show that the two semantics coincide for this first-order logic, by translating every continuous FO formula to an equivalent pointwise one.

1 Introduction

The Timed Temporal Logic (TPTL) of Alur and Henzinger [1] is a well-known temporal logic for specifying real-time behaviors. The logic is interpreted over timed words and extends classical LTL with the "freeze" quantifier $x.\theta$ which binds x to the value of the current time point, along with the ability to constrain these time points using linear constraints of the form $x \sim y + c$. For example the formula $x.(\Diamond y.(a \wedge y = x + 2))$ says that with respect to the current time point, an action a occurs exactly two time units later. Another popular timed temporal logic is Metric Temporal Logic (MTL) [2–4], which extends the U operator of classical LTL with an interval index, to allow formulas of the form $\theta U_I \eta$ which says that w.r.t. the current time point, there is a future time point where η is satisfied which lies at a distance that falls within the interval I, and at all time points in between θ is satisfied.

For both TPTL and MTL (as well as for other timed logics) there are two natural ways to interpret these logics over timed words, namely the "pointwise" and the "continuous" interpretations. In the pointwise semantics, formulas may be asserted only at points where an action occurs (the so called "action points"), while in the continuous semantics formulas may be asserted at arbitrary time points. As an example, consider the timed word σ in which the first action is an a at time 2, followed subsequently by only b's. Then the TPTL formula $\Diamond x.\Diamond y.(a \wedge y = x+1)$ is satisfied in σ in the continuous semantics, but not in the pointwise semantics, since there is no action point at time 1. It is not difficult to see that for a typical timed temporal logic the continuous semantics is at least as expressive as the pointwise one, since one can ask for a time point to be an action point by asserting $\bigvee_{a \in \Sigma} a$ at each quantified time point.

There have been several results in the literature which show that for the logic MTL and its variants the continuous semantics is in fact strictly more expressive than the pointwise one [5–7]. Thus the logics MTL, MTL_S (MTL with the "since" operator S), MTL_{S_I} (MTL with the S_I operator), and MITL (MTL restricted to non-singular intervals), are all strictly more expressive in the continuous semantics than their pointwise counterparts.

In this paper we consider this issue for the logic TPTL and show that for the logic extended with the S modality (which we denote by TPTL_S), the expressiveness in the pointwise and continuous semantics coincide. In fact, we show that the logic TPTL_S corresponds to a natural first-order logic $\mathrm{FO}(<_+)$ which is interpreted over timed words and allows constraints of the form $x \sim y + c$, and that the pointwise and continuous semantics for this logic coincide. To the best of our knowledge, this is the first instance of a real-time logic for which the pointwise and continuous semantics have been shown to be equally expressive.

The main proof idea is to show that we can go from an arbitrary sentence in $\mathrm{FO}(<_+)$ to a sentence in $\mathrm{FO}(<_+)$ which uses only "active" quantifiers. We say a subformula of the form $\exists x \varphi$ is an actively quantified formula if φ is of the form $a(x) \wedge \psi$ for some action a and formula ψ; and say it is "passively" quantified otherwise. A sentence in which all quantifiers are active, is clearly equivalent to a pointwise formula. Thus, we show how to eliminate passively quantified variables using only actively quantified ones. As an example, the formula $\exists x(1 \leq x \wedge \exists y(b(y) \wedge x + 1 \leq y))$ has x passively quantified. We can eliminate x from it, by giving the equivalent actively quantified formula $\exists y(b(y) \wedge 2 \leq y)$.

Along the way we also exhibit a useful normal form for $\mathrm{FO}(<_+)$ (and hence also for TPTL_S) sentences, in which all quantified subformulas are essentially of the form $\exists x(a(x) \wedge \pi \wedge \nu)$, where a is an action in Σ, π is a conjunction of linear constraints, and ν is a conjunction of formulas satisfying a similar restriction on quantified subformulas.

2 Preliminaries

We begin with some preliminary definitions. Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers, $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{N}$ denote the set of non-negative integers. We use the standard notation to represent intervals, which are convex subsets of $\mathbb{R}$. For example $[1, \infty)$ denotes the set $\{t \in \mathbb{R} \mid 1 \leq t\}$.

For an alphabet A we denote by A^ω the set of infinite words over A. Let Σ be a finite alphabet of actions, which we fix for the rest of this paper. An (infinite) timed word σ over Σ is an element of $(\Sigma \times \mathbb{R}_{\geq 0})^\omega$ of the form $(a_0, t_0)(a_1, t_1) \cdots$, satisfying the conditions that: for each $i \in \mathbb{N}$, $t_i < t_{i+1}$ (*monotonicity*), and for each $t \in \mathbb{R}_{\geq 0}$ there exists an $i \in \mathbb{N}$ such that $t < t_i$ (*progressiveness*). For convenience, we will also assume in this paper that $t_0 = 0$. We will sometimes represent the timed word σ above as a pair (α, τ), where $\alpha = a_0 a_1 \cdots$ and $\tau = t_0 t_1 \cdots$. Thus $\alpha(i)$ and $\tau(i)$ denote the the action and the time stamp respectively, in σ at position i. We write $T\Sigma^\omega$ to denote the set of all timed words over Σ.

We now introduce the linear constraints we use in this paper, and some notations for manipulating them. We assume a supply of variables $Var = \{x, y, \ldots\}$ which we will use in constraints as well as later in our logics. We use restricted linear constraints of the form $x \sim y + c$ or $x \sim c$, where x and y are variables in Var, $\sim$ is one of the relations $\{<, \leq, =, \geq, >\}$, and c is a rational constant. We call these constraints *simple constraints*. In general, we will allow constraints to be boolean combinations of simple constraints given by the syntax $\delta ::= g \mid \neg\delta \mid \delta \wedge \delta \mid \delta \vee \delta$, where g is a simple constraint.

An *interpretation* for variables is a map $\mathbb{I} : Var \to \mathbb{R}$. For $t \in \mathbb{R}$ and $x \in Var$ we will use $\mathbb{I}[t/x]$ to represent the interpretation which sends x to t, and agrees with $\mathbb{I}$ on all other variables. When we are interested in a finite set of variables $\{x_1, \ldots, x_n\}$ we will write $[t_1/x_1, \ldots, t_n/x_n]$ to represent an interpretation that maps each x_i to t_i.

For an interpretation $\mathbb{I}$ and a constraint δ, we write $\mathbb{I} \models \delta$ to mean that the constraint δ is satisfied in the interpretation $\mathbb{I}$, and defined in the expected way.

Consider a conjunction of simple constraints π over the variables $x, y_1, \ldots, y_n$, such that x occurs in each of the constraints in π. In such a case we will sometimes write π as $\pi(x)$ to emphasize this fact. For an interpretation $\mathbb{I}$ for the variables $y_1, \ldots, y_n$, the set of values of x which satisfy π under the interpretation $\mathbb{I}$, forms an interval which we denote by $I(\pi, x, \mathbb{I})$. Thus, $I(\pi, x, \mathbb{I})$ is the set of all $t \in \mathbb{R}$ such that $\mathbb{I}[t/x] \models \pi$. Let $Lt_x(\pi)$ be the conjunction of constraints which define the "left boundary" of x, that is all constraints in π of the form $e \prec x$ with $\prec \in \{<, \leq\}$. Similarly let $Rt_x(\pi)$ denote the conjunction of constraints which define the "right boundary" of x, namely all constraints of the form $x \prec e$ in π. Then, the set of values for x which don't satisfy the constraint $\pi(x)$ under the interpretation $\mathbb{I}$, are the set of values t such that $\mathbb{I}[t/x] \models \neg Lt_x(\pi) \vee \neg Rt_x(\pi)$. Similarly, for conjunctions of constraints $\pi_1(x)$ and $\pi_2(x)$, we have $\mathbb{I} \models Lt_x(\pi_1) \wedge \neg Lt_x(\pi_2)$ iff the left boundary of $I(\pi, x, \mathbb{I})$ begins before the left boundary of $I(\pi, x, \mathbb{I})$. Similarly, $\neg Lt_x(\pi_1) \wedge \neg Rt_x(\pi_2)$ says that the left boundary of π_1 begins after the right boundary of π_2.

As a final piece of notation, we recall the well-known Fourier-Motzkin method (see [8]) for eliminating variables from constraints. Consider a conjunction of simple constraints π. The constraints in π can be written as:

$$\begin{array}{lll} e_i & \leq x & (i = 1, \ldots, m_1) \\ x & \leq e_j & (j = m_1 + 1, \ldots, m_2) \\ e_k & \leq f_k & (k = m_2 + 1, \ldots, m_3), \end{array}$$

for some $m_1, m_2, m_3 \geq 0$ (we consider the case when π has strict constraints later).

The first two rows of constraints above are equivalent to:

$$\max_{1 \leq i \leq m_1} e_i \leq x \leq \min_{m_1+1 \leq j \leq m_2} e_j. \tag{1}$$

The variable x can be eliminated using $(m_1 \times (m_2 - m_1)) + (m_3 - m_2)$ simple constraints which do not contain x:

$$e_i \leq e_j \;\; (i = 1, \ldots, m_1; j = m_1 + 1, \ldots, m_2) \tag{2}$$
$$e_k \leq f_k \; (k = m_2 + 1, \ldots, m_3). \tag{3}$$

We denote the conjunction of the constraints above by $\mathrm{FME}_x(\pi)$. The constraint $\mathrm{FME}_x(\pi)$ preserves the solution set of π in that it is the projection of the solution set of π to the variables in π other than x. More precisely, if $x, y_1, \ldots, y_n$ were the variables in π, then $[t_1/y_1, \ldots, t_n/y_n]$ is a solution for $\mathrm{FME}_x(\pi)$ iff there exists $t \in \mathbb{R}$ such that $[t/x, t_1/y_1, \ldots, t_n/y_n]$ is a solution to π.

The method can be extended to the case when some of the constraints are strict, by writing $e_i < e_j$ in (2) above whenever at least one of the i-th or j-th constraints is strict, and $e_i \leq e_j$ otherwise.

3 The logics $\mathbf{TPTL}_S$ and FO

We recall the definition of Timed Propositional Temporal Logic from [1]. We use the version of the logic with the "since" operator S, and denote this logic by TPTL_S. The formulas of TPTL_S over the alphabet Σ, are defined as follows:

$$\theta ::= a \mid g \mid \neg\theta \mid (\theta \wedge \theta) \mid (\theta \vee \theta) \mid (\theta U \theta) \mid (\theta S \theta) \mid x.\theta$$

where $a \in \Sigma$, x is a variable in Var, and g is a simple constraint.

We first define the *pointwise* semantics for TPTL_S. Let θ be a TPTL_S formula. Let $\sigma = (\alpha, \tau)$ be a timed word over Σ, let $i \in \mathbb{N}$, and let $\mathbb{I}$ be an interpretation for variables. Then the satisfaction relation $\sigma, i, \mathbb{I} \models_{pw} \theta$, (read "$\sigma$ satisfies the formula θ at position i in the interpretation $\mathbb{I}$ in the pointwise semantics") is inductively defined as:

$$
\begin{array}{lll}
\sigma, i, \mathbb{I} \models_{pw} a & \text{iff} & \alpha(i) = a \\
\sigma, i, \mathbb{I} \models_{pw} g & \text{iff} & \mathbb{I} \models g \\
\sigma, i, \mathbb{I} \models_{pw} \neg\theta & \text{iff} & \sigma, i, \mathbb{I} \not\models_{pw} \theta \\
\sigma, i, \mathbb{I} \models_{pw} \theta \wedge \eta & \text{iff} & \sigma, i, \mathbb{I} \models_{pw} \theta \text{ and } \sigma, i, \mathbb{I} \models_{pw} \eta \\
\sigma, i, \mathbb{I} \models_{pw} \theta \vee \eta & \text{iff} & \sigma, i, \mathbb{I} \models_{pw} \theta \text{ or } \sigma, i, \mathbb{I} \models_{pw} \eta \\
\sigma, i, \mathbb{I} \models_{pw} \theta U \eta & \text{iff} & \exists k : \; k > i \text{ such that } \sigma, k, \mathbb{I} \models_{pw} \eta \text{ and} \\
 & & \forall j : \; i < j < k : \sigma, j, \mathbb{I} \models_{pw} \theta \\
\sigma, i, \mathbb{I} \models_{pw} \theta S \eta & \text{iff} & \exists k : \; 0 \leq k < i \text{ such that } \sigma, k, \mathbb{I} \models_{pw} \eta \text{ and} \\
 & & \forall j : \; k < j < i, \sigma, j, \mathbb{I} \models_{pw} \theta \\
\sigma, i, \mathbb{I} \models_{pw} x.\theta & \text{iff} & \sigma, i, \mathbb{I}[\tau(i)/x] \models_{pw} \theta
\end{array}
$$

A TPTL_S formula is "closed" if every occurrence of a variable x is within the scope of a freeze quantifier "$x.$". If a formula θ is closed, the interpretation for variables plays no role in the satisfaction relation, and we say $\sigma \models_{pw} \theta$ if $\sigma, 0 \models_{pw} \theta$. We define the timed language defined by a closed formula θ in the pointwise semantics to be $L^{pw}(\theta) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_{pw} \theta\}$. If a TPTL_S formula is interpreted in the pointwise semantics, we call it a TPTL_S^{pw} formula.

We now define the *continuous* semantics for TPTL_S. Let $\sigma = (\alpha, \tau)$ be a timed word over Σ, $t \in \mathbb{R}_{\geq 0}$ and $\mathbb{I}$ be an interpretation for variables. The satisfaction relation $\sigma, t, \mathbb{I} \models_c \theta$ (read "σ satisfies the formula θ at time t in the interpretation $\mathbb{I}$ in the continuous semantics") is inductively defined as:

$$
\begin{array}{lll}
\sigma, t, \mathbb{I} \models_c a & \text{iff} & \exists i : \alpha(i) = a \text{ and } \tau(i) = t \\
\sigma, t, \mathbb{I} \models_c g & \text{iff} & \mathbb{I} \models g \\
\sigma, t, \mathbb{I} \models_c \neg\eta & \text{iff} & \sigma, t, \mathbb{I} \not\models_c \eta \\
\sigma, t, \mathbb{I} \models_c \theta \wedge \eta & \text{iff} & \sigma, t, \mathbb{I} \models_c \theta \text{ and } \sigma, t, \mathbb{I} \models_c \eta \\
\sigma, t, \mathbb{I} \models_c \theta \vee \eta & \text{iff} & \sigma, t, \mathbb{I} \models_c \theta \text{ or } \sigma, t, \mathbb{I} \models_c \eta \\
\sigma, t, \mathbb{I} \models_c \theta U \eta & \text{iff} & \exists t' : t' > t \text{ such that } \sigma, t', \mathbb{I} \models_c \eta \text{ and} \\
 & & \forall t'' : t < t'' < t', \sigma, t'', \mathbb{I} \models_c \theta \\
\sigma, t, \mathbb{I} \models_c \theta S \eta & \text{iff} & \exists t' : 0 \leq t' < t \text{ such that } \sigma, t', \mathbb{I} \models_c \eta \text{ and} \\
 & & \forall t'' : t' < t'' < t, \sigma, t'', \mathbb{I} \models_c \theta \\
\sigma, t, \mathbb{I} \models_c x.\eta & \text{iff} & \sigma, t, \mathbb{I}[t/x] \models_c \eta
\end{array}
$$

The timed language defined by a closed TPTL_S formula θ in the continuous semantics is given by $L^c(\theta) = \{\sigma \in T\Sigma^\omega \mid \sigma, 0 \models_c \theta\}$. When a TPTL_S formula is interpreted in the continuous semantics, we call it a TPTL_S^c formula.

We use the standard syntactic abbreviations of $\Diamond, \diamondminus, \Box$ and $\boxminus$ for temporal logic, defined in a way that they are "reflexive". Thus, we define $\Diamond\theta = \theta \vee (\top U \theta)$, $\diamondminus\theta = \theta \vee (\top S \theta)$, $\Box\theta = \neg\Diamond\neg\theta$, $\boxminus\theta = \neg\diamondminus\neg\theta$.

We note that in the logic TPTL_S, it is possible to express both U and S operators using $\Diamond$ and $\diamondminus$ operators in both the continuous and pointwise semantics.

$$
\begin{array}{ll}
\theta U \eta & \equiv x.\Diamond y.(\eta \wedge x < y \wedge \boxminus z.(x < z \wedge z < y \Rightarrow \theta)) \\
\theta S \eta & \equiv x.\diamondminus y.(\eta \wedge y < x \wedge \Box z.(z < x \wedge y < z \Rightarrow \theta)).
\end{array}
$$

We now introduce our first order logic with simple constraints $\mathrm{FO}(<_+)$, which is interpreted over timed words over the alphabet Σ. The formulas of $\mathrm{FO}(<_+)$ are given by:

$$\varphi ::= a(x) \mid g \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x \varphi,$$

where $a \in \Sigma$, $x \in \mathit{Var}$, and g is a simple constraint.

We now define the continuous semantics for $\mathrm{FO}(<_+)$. Let φ be a formula in $\mathrm{FO}(<_+)$. Let $\sigma = (\alpha, \tau)$ be the timed word over Σ, and let $\mathbb{I}$ be an interpretation for variables. Then the satisfaction relation $\sigma, \mathbb{I} \models_c \varphi$ (read "σ satisfies φ in the interpretation $\mathbb{I}$ in the continuous semantics") is inductively defined as:

$$
\begin{array}{lll}
\sigma, \mathbb{I} \models_c a(x) & \text{iff} & \exists i : \alpha(i) = a \text{ and } \tau(i) = \mathbb{I}(x) \\
\sigma, \mathbb{I} \models_c g & \text{iff} & \mathbb{I} \models g \\
\sigma, \mathbb{I} \models_c \neg\nu & \text{iff} & \sigma, \mathbb{I} \not\models_c \nu \\
\sigma, \mathbb{I} \models_c \nu \wedge \psi & \text{iff} & \sigma, \mathbb{I} \models_c \nu \text{ and } \sigma, \mathbb{I} \models_c \psi \\
\sigma, \mathbb{I} \models_c \nu \vee \psi & \text{iff} & \sigma, \mathbb{I} \models_c \nu \text{ or } \sigma, \mathbb{I} \models_c \psi \\
\sigma, \mathbb{I} \models_c \exists x \nu & \text{iff} & \exists t : t \in \mathbb{R}_{\geq 0} \text{ such that } \sigma, \mathbb{I}[t/x] \models_c \nu.
\end{array}
$$

A variable x is said to occur *free* in a formula φ if it is occurs outside the scope of a quantifier $\exists x$. A *sentence* is a formula in which there are no free occurrences of variables. Again, the satisfaction of a sentence is independent of an interpretation for variables.

The timed language defined by an FO($<_+$) sentence φ in the continuous semantics is given by $L^c(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_c \varphi\}$. We denote this continuous version of the logic by FO^c($<_+$).

We can similarly define the pointwise version of the logic FO($<_+$), where the quantification is over action points in the timed word. The satisfaction relation $\sigma, \mathbb{I} \models_{pw} \varphi$, read "$\sigma$ satisfies φ in the interpretation $\mathbb{I}$ in the pointwise semantics", is defined as above, except for the $\exists$ clause which is interpreted as follows:

$$\sigma, \mathbb{I} \models_{pw} \exists x \nu \ \text{ iff } \ \exists i \in \mathbb{N} : \sigma, \mathbb{I}[\tau(i)/x] \models_{pw} \nu.$$

The timed language defined by a sentence φ in the pointwise semantics is given by $L^{pw}(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_{pw} \varphi\}$. We denote the pointwise version of the logic by FO^{pw}($<_+$).

4 A normal form for FO sentences

In this section we exhibit a normal form for FO($<_+$) sentences which will be useful in our proofs. We begin with a normal form for formulas of the form $\exists x \varphi$. An FO($<_+$) formula is said to be in $\exists$-*normal form* if it is of the form simple constraints each containing x, and ν is a conjunction of formulas of the form ψ or $\neg\psi$, where each ψ

$$\exists x(a(x) \wedge \pi(x) \wedge \nu),$$

where $a \in \Sigma$, $\pi(x)$ is a conjunction of is again in $\exists$-normal form. In addition, we allow any of the components $a(x)$ and ν to be absent.

Theorem 1. *Any* FO($<_+$) *sentence can be written as a boolean combination of sentences which are in* $\exists$*-normal form.*

Proof. Let φ be an FO($<_+$) sentence. We transform φ into an equivalent sentence in normal form (i.e. a boolean combination of sentences in $\exists$-normal form), by repeatedly transforming the formula tree of φ.

Since φ is a sentence it must be a boolean combination of sentences of the form $\exists x \varphi'$. We now carry out the following steps to translate φ into normal form.

1. In every subtree rooted at an $\exists$ node, in the formula tree of φ, push every $\neg$ operator downwards over $\vee$, $\wedge$, and all the way through g nodes, till it reaches an $\exists$ node or an a (action) node. After this step, the subtree below every $\exists$ node contains only conjunctions and disjunctions of a, $\neg a$, $\exists$, $\neg\exists$, and g nodes.
2. For convenience, in this step we will consider $\neg\exists$ as a single composite node in the formula tree.
   Pull all the $\vee$'s upwards in the resulting formula tree for φ. using the following identities: $\nu_1 \wedge (\nu_2 \vee \nu_3) \equiv (\nu_1 \wedge \nu_2) \vee (\nu_1 \wedge \nu_3), \exists x(\nu_1 \vee \nu_2) \equiv (\exists x \nu_1) \vee (\exists x \nu_2)$ and $\neg\exists x(\nu_1 \vee \nu_2) \equiv (\neg\exists x(\nu_1)) \wedge (\neg\exists x(\nu_2))$. After this step, the subtree rooted at each $\exists$ node or $\neg\exists$ node contains only conjunctions of a, $\neg a$, $\exists$, $\neg\exists$, and g nodes.

3. In this step we push out from a subtree rooted at an $\exists x$ node, all nodes which are independent of x, namely nodes of the form $b(y)$, $\neg b(y)$ (with $y \neq x$), and g where g does not contain x. Those can be pushed up in the tree using following relations (we again treat $\neg\exists$ nodes as a single composite node): $\exists x(b(y) \wedge \nu) \equiv b(y) \wedge \exists x(\nu)$ and $\neg\exists x(b(y) \wedge \nu) \equiv \neg b(y) \vee \neg\exists x(\nu)$. We can use similar relations for $\neg b(y)$ and g to move them up the tree. Move all newly generated $\vee$'s up the tree using step (2).
   After this step, the subtrees rooted at each $\exists x$ node is a conjunction of $a(x)$, $\neg a(x)$, $\exists$, $\neg\exists$ and $g(x)$ nodes.
4. We now update the formula tree with the following relations: $a(x) \wedge b(x) \equiv \perp$ and $a(x) \wedge \neg b(x) \equiv a(x)$. After this step, the only action-related nodes in a subtree rooted at a $\exists x$ node are a single action node $a(x)$ or a conjunction of negation of actions $\bigwedge_{a \in A} \neg a(x)$.
5. We can now replace formulas of the form $\bigwedge_{a \in A} \neg a(x)$ by a disjunction of formulas which contain at most one action, as described below. We then push the newly generated $\vee$ nodes up the tree using step (2). After this step, the subtree rooted at every $\exists x$ node contains only conjunctions of $a(x)$, $\exists$, $\neg\exists$ and $g(x)$ nodes. We can collect the $g(x)$ nodes together to get a single conjunction of constraints $\pi(x)$. Thus finally each $\exists$ node is in $\exists$-normal form.

To see how we can replace formulas of the form $\bigwedge_{a \in A} \neg a(x)$ by a disjunction of formulas in $\exists$-normal form, consider a formula φ of the form $\exists x(\bigwedge_{a \in A} \neg a(x) \wedge \pi(x) \wedge \nu)$. Let $A(x)$ be shorthand for the formula $\bigvee_{a \in A} a(x)$.

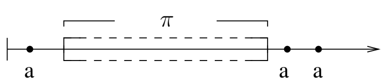

Fig. 1.

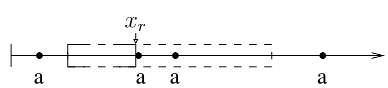

Fig. 2.

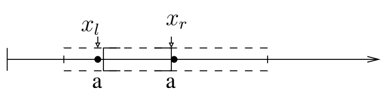

Fig. 3.

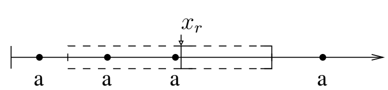

Fig. 4.

We replace φ by disjunction of $\psi_1, \psi_2, \psi_3, \psi_4$ which are defined as follows:

1. If an action $A(x)$ did not occur anywhere in the interval $I(\pi, x, \mathbb{I})$, then φ is satisfied if ν is satisfied for any $x \in I(\pi, x, \mathbb{I})$. (See Fig(1)).
   $\psi_1 \equiv \neg(\exists x(A(x) \ \wedge \pi(x)) \wedge \exists x(\pi(x) \ \wedge \ \nu)$.
2. If an action $A(x)$ first occurred in $I(\pi, x, \mathbb{I})$ at x_r, then φ is satisfied, if ν is satisfied for any $x \in I(\pi \wedge x < x_r, x, \mathbb{I})$. (See Fig(2)).
   $\psi_2 \equiv \exists_{\mathit{first}} x_r(A(x_r) \ \wedge \exists x(\pi(x) \wedge x < x_r \ \wedge \ \nu))$,
   where $\exists_{\mathit{first}} x(\varphi) \equiv \exists x(\varphi \wedge \neg(\exists x'(\varphi \wedge x' < x)))$.
3. If an action $A(x)$ occurred consecutively at x_l and at x_r in the interval $I(\pi, x, \mathbb{I})$, then φ is satisfied if ν is satisfied for any $x \in I(\pi \wedge x_l < x \wedge x <$

$x_r, x, \mathbb{I}$).(See Fig(3)).
$\psi_3 \equiv \exists x_l(A(x_l) \ \wedge \exists_{next} x_r(A(x_r) \ \wedge \exists x(\pi(x) \wedge x < x_r \wedge x_l < x \ \wedge \ \nu)))$,
where $\exists_{next} x(\varphi) \equiv \exists x(\varphi \wedge x_l < x \wedge \neg(\exists x'(\varphi \wedge x_l < x' \wedge x' < x)))$.

4. If an action $A(x)$ last occurred last in $I(\pi, x, \mathbb{I})$ at x_l, then φ is satisfied, if ν is satisfied for any $x \in I(\pi \wedge x_l < x, x, \mathbb{I})$. (See Fig(4)).
$\psi_4 \equiv \exists_{last} x_l(A(x_l) \ \wedge \exists x(\pi(x) \wedge x_l < x \ \wedge \ \nu))$,
where $\exists_{last} x(\varphi) \equiv \exists x(\varphi \wedge \ \neg(\exists x'(\varphi \wedge x < x')))$.

It is easy to show that, for a timed word σ, $\sigma \models_c \varphi$ iff for one of the ψ_i, $\sigma \models_c \psi_i$ □

5 Equivalence of FO and TPTL$_S$

We can now argue that the logics TPTL$_S$ and FO($<_+$) are expressively equivalent in the continuous as well as pointwise semantics.

For a closed formula θ in TPTL$_S^c$ we show how to give a formula *tptl-fo*(θ) in FO^c($<_+$), which has a single free variable z, such that for any timed word σ, $\sigma, t \models_c \theta$ if and only if $\sigma, [t/z] \models_c$ *tptl-fo*(θ). The translation *tptl-fo* is defined inductively on the structure of θ as follows:

$$\begin{aligned}
\mathit{tptl\text{-}fo}(a) &= a(z)\\
\mathit{tptl\text{-}fo}(\pi_1 \leq \pi_2) &= \pi_1 \leq \pi_2\\
\mathit{tptl\text{-}fo}(\neg\theta') &= \neg \mathit{tptl\text{-}fo}(\theta')\\
\mathit{tptl\text{-}fo}(\theta_1 \wedge \theta_2)) &= \mathit{tptl\text{-}fo}(\theta_1) \wedge \mathit{tptl\text{-}fo}(\theta_2)\\
\mathit{tptl\text{-}fo}(\Diamond y.(\theta')) &= \exists y(y > x \wedge \mathit{tptl\text{-}fo}(\theta'[y/x]))\\
\mathit{tptl\text{-}fo}(\diamondminus y.(\theta')) &= \exists y(y < x \wedge \mathit{tptl\text{-}fo}(\theta'[y/x]))
\end{aligned}$$

Now we can translate a closed formula θ in TPTL$_S^c$ to the FO^c($<_+$) sentence $\varphi = \exists z(z = 0 \wedge \mathit{tptl\text{-}fo}(\theta))$, so that we have $L^c(\theta) = L^c(\varphi)$.

In the other direction, we translate an FO^c($<_+$) sentence φ, to an equivalent closed TPTL$_S^c$ formula *fo-tptl*(φ) as follows. We first transform φ into its normal form as given in Theorem(1). The translation *fo-tptl* is defined inductively in a similar manner to *tptl-fo* above, with the $\exists$ subformulas being translated via the rule:

$$\begin{aligned}
\mathit{fo\text{-}tptl}(\exists x(a(x) \wedge \pi(x) \wedge \nu)) = {} & \Diamond x.(a \wedge \pi(x) \wedge \mathit{fo\text{-}tptl}(\nu)) \vee\\
& \diamondminus x.(a \wedge \pi(x) \wedge \mathit{fo\text{-}tptl}(\nu)).
\end{aligned}$$

It is easy to see that $L^c(\varphi) = L^c(\mathit{fo\text{-}tptl}(\varphi))$.

Thus we can say that the logics TPTL$_S^c$ and FO^c($<_+$) are expressively equivalent.

The translations above are also applicable for the pointwise semantics, and hence we can conclude similarly that the logics TPTL$_S^{pw}$ and FO^{pw}($<_+$) are expressively equivalent.

6 Equivalence of FO^c and FO^{pw} semantics

In this section our aim is to show that the logics FO$^{pw}(<_+)$ and FO$^c(<_+)$ are expressively equivalent. It is easy to translate an FO$^{pw}(<_+)$ sentence φ to an equivalent FO$^c(<_+)$ sentence by simply replacing every $\exists x\varphi'$ subformula, by $\exists x(\bigvee_{a\in\Sigma} a(x) \wedge \varphi'')$, where φ'' is obtained by similarly replacing $\exists$ subformulas in φ'.

In the converse direction, let us call an FO$^c(<_+)$ formula φ *actively quantified* (or simply *active*) if every $\exists$ subformula is of the form $\exists x(a(x) \wedge \varphi')$, for some action $a \in \Sigma$ and formula φ'. Then, an active FO$^c(<_+)$ formula clearly defines the same language of timed words, regardless of the semantics being pointwise or continuous. Hence, our aim in the rest of this section is to show how we can go from an arbitrary formula in FO$^c(<_+)$ to an equivalent active formula.

An arbitrary formula in the continuous semantics has the obvious advantage of being able to associate any value in $\mathbb{R}_{\geq 0}$ to its variables, whereas an actively quantified variable can be asserted only at the action points in a timed word. For e.g., consider the language of all timed strings over a and b, where for every b in the interval [1,2], there is an a in [0,1] which is exactly at a distance of one time unit from that of the b. This can be written easily in FO^c, as shown below:

$$\neg\exists x((\neg a(x) \wedge 0 \leq x \wedge x \leq 1 \wedge \exists y(b(y) \wedge y = x + 1)) \tag{4}$$

But if we restrict x to be actively quantified, then the above formula does not recognize the same language. This formula is not in the $\exists$-normal form and the normalization of this formula yields a disjunction of four formulas $\psi_1, \psi_2, \psi_3, \psi_4$, where x is the only variable which is passively quantified. If we can eliminate x from $\psi_1, \psi_2, \psi_3, \psi_4$, without introducing any new passively quantified variables, the disjunction of these actively quantified formulas recognizes the required language. The subformula involving x in each of ψ_i-s looks like $\exists x(\pi(x) \;\wedge\; \exists y(b(y) \wedge y = x + 1))$. In the latter part of the current section, we prove that, it is possible to remove x from such formulas using the Fourier-Motzkin variable elimination method.

Consider another example where x is passively quantified.

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \exists y(a(y) \wedge x + 1 \leq y \wedge y \leq x + 1.2)). \tag{5}$$

The above formula is true, iff there is a point in [0,1], from which there is an action a at a distance which lies in [1,1.2]. Equivalently, (5) is true iff there is an action a in the interval [1,2.2]. So the equivalent active formula is:

$$\exists y(a(y) \wedge 1 \leq y \wedge y \leq 2.2). \tag{6}$$

As an another example, consider the following modified formula:

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \neg\exists y(a(y) \wedge x + 1 \leq y \wedge y \leq x + 1.2)) \tag{7}$$

To eliminate the passively quantified variable x from this formula, first consider "all" intervals of x in [0,1] which satisfy (6) in the given model (the bracketed region in the Fig. 5.). (7) is true iff there are some "gaps" where formula (5) is not satisfied. There can be the following four types of gaps.

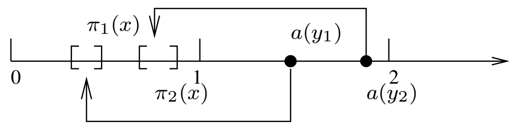

Fig. 5.

1. No x satisfies (6), so whole of [0,1] is a gap.
2. Gap from 0 to the beginning of the first interval of x satisfying (6).
3. Gap between two consecutive intervals of x satisfying formula (6).
4. Gap from the end of the last interval of x satisfying (6) to 1.

The formula (7) can be satisfied iff any of the above gaps exist. In the next section, we prove that it is possible to identify those gaps using the syntax of $\mathrm{FO}^c(<_+)$.

Before going to the proof, we shall see some of the definitions here. *Quantifier depth* of any FO formula is the maximum depth of quantifier operators in that formula. A quantified variable x in a FO^c formula is said to be *positively quantified* if it is not under the scope of any of the $\neg$ operators. For example, in the formula $\exists x(\pi(x) \wedge \neg\exists y(\exists z(a(z))))$, only x is positively quantified. If φ is satisfied for timed word σ with the interpretation $\mathbb{I}$, then the *witness* of the satisfied formula φ is the set of values of all positively quantified variables of φ in $\sigma, \mathbb{I} \models \varphi$.

Theorem 2. *Let $\widehat{\nu}$ be an active formula which is the conjunction of formulas in the $\exists$-normal form or in negated $\exists$-normal form. It is possible to eliminate x from the formula $\psi = \exists x(\pi \wedge \widehat{\nu})$ to get an equivalent formula $\bigvee_i(\widehat{\mu}_i)$, where each $\widehat{\mu}_i$ is active and independent of x. Further, we can give a constraint $\pi_i(x)$ for each of the $\widehat{\mu}_i$'s containing free variables and positively quantified variables of $\widehat{\mu}_i$ such that:*

(a) If $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}$, then $\exists i : \ \sigma, \mathbb{I} \models \widehat{\mu}_i$ and further, if $t_1, \ldots, t_k$ are the witnesses for the positively quantified variables $y_1, \ldots, y_k$ in $\sigma, \mathbb{I} \models \widehat{\mu}_i$, then $t \in I(\pi_i, x, \mathbb{I}[t_1/y_1, \ldots, t_k/y_k])$. i.e., $\mathbb{I}[t/x, t_1/y_1, \ldots, t_k/y_k] \models \pi_i$.
(b) If $\sigma, \mathbb{I} \models \widehat{\mu}_i$ for some i and $t_1, \ldots, t_k$ are the witnesses of the positively quantified variables $y_1, \ldots, y_k$ in $\sigma, \mathbb{I} \models \widehat{\mu}_i$, then for any $t \in I(\pi_i, x, \mathbb{I}[t_1/y_1, \ldots, t_k/y_k])$ i.e., $\mathbb{I}[t/x, t_1/y_1, \ldots, t_k/y_k] \models \pi_i$, we have $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}$.

Proof. Intuitively, first part of the theorem statement, claim the existence of equivalent active formulas and (a) and (b) says that the union of all the intervals corresponding to all π_is exactly characterize all the possible values of x satisfying ψ.

We prove this by induction on the quantifier depth of the formula $\widehat{\nu}$. Let d be the quantifier depth of $\widehat{\nu}$.

For the base case of the induction, consider $d = 0$. ψ is of the form $\exists x(\pi)$. x can be eliminated from this formula using Fourier-Motzkin Elimination to get an equivalent formula $\widehat{\mu}$ which is a boolean combination of simple constraints and does not contain x. The constraint will be π itself, (a) if $\sigma, \mathbb{I}[t/x] \models \pi$ then $t \in I(\pi, x, \mathbb{I})$ and (b) for every $t \in I(\pi, x, \mathbb{I})$, $\sigma, \mathbb{I}[t/x] \models \pi$.

To prove for the formula $\widehat{\nu}$ having depth d, consider $\psi = \exists x(\pi \wedge \widehat{\nu})$, where $\widehat{\nu}$ is of the form $\widehat{\nu}_1 \wedge \ldots \wedge \widehat{\nu}_n$ where each $\widehat{\nu}_i$ is in $\exists$-normal form or in the negated $\exists$-normal form. By induction hypothesis, Each $\widehat{\nu}_i$ has quantifier depth of at most d and are active.

First we consider the case when $n = 1$. ψ is either in the form $\exists x(\pi \wedge \widehat{\psi})$ or in the form $\exists x(\pi \wedge \neg\widehat{\psi})$ where $\widehat{\psi} = \exists y(a(y) \wedge \pi' \wedge \widehat{\nu})$.

Case $\psi = \exists x(\pi \wedge \widehat{\psi}) = \exists x(\pi \wedge \exists y(a(y) \wedge \pi' \wedge \widehat{\nu}))$: This formula is equivalent to $\exists y(a(y) \wedge \exists x(\pi \wedge \pi' \wedge \widehat{\nu}))$. Since, $\widehat{\nu}$ has quantifier depth strictly less than d, by induction hypothesis, $\exists x(\pi \wedge \pi' \wedge \widehat{\nu})$ can be translated to $\vee_i(\widehat{\mu}'_i)$ with corresponding constraints π'_i. The final formula is, $\exists y(a(y) \wedge (\vee_i(\widehat{\mu}'_i)))$, which is equivalent to $\vee_i(\exists y(a(y) \wedge \widehat{\mu}'_i))$. Let $\widehat{\mu}_i = \exists y(a(y) \wedge \widehat{\mu}'_i)$. Now, we prove that, π'_i is the constraint for $\widehat{\mu}_i$ as well.

Suppose (a) $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\psi}$ and the value of positively quantified variable y be t_1. This implies that $\sigma, \mathbb{I}[t/x, t_1/y] \models \pi \wedge \pi' \wedge \widehat{\nu}$. By induction hypothesis, for some i, $\sigma, \mathbb{I}[t_1/y] \models \widehat{\mu}'_i$ and $t \in I(\pi'_i, x, \mathbb{I}[t_1/y])$. At $y = t_1,\ a(t_1)$ is satisfied, which implies, $\widehat{\mu}_i$ is also satisfied at $y = t_1$ and $t \in I(\pi'_i, x, \mathbb{I})$.

Suppose (b) $\sigma, \mathbb{I} \models \widehat{\mu}_i$ with witness y takes value t'_1, then, $\sigma, \mathbb{I}[t'_1/y] \models \widehat{\mu}'_i$. By induction hypothesis, for every $t \in I(\pi'_i, x, \mathbb{I}[t'_1/y])$, $\sigma, \mathbb{I}[t/x, t'_1/y] \models \exists x(\pi \wedge \pi' \wedge \widehat{\nu})$. Since at $y = t_1,\ a(t_1)$ is satisfied, $\sigma, \mathbb{I} \models \psi$.

Case $\psi = \exists x(\pi \wedge \neg\widehat{\psi}) = \exists x(\pi \wedge \neg\exists y(a(y) \wedge \pi' \wedge \widehat{\nu}))$: Let $\psi' = \exists x(\pi \wedge \widehat{\psi}))$ which falls into the previous case and x can be eliminated from ψ' to give an equivalent formula $\vee_i(\widehat{\mu}_i)$ and respective constraint π_i for each i. Let $\widehat{\mu}'_i$ be the formula obtained by replicating formula $\widehat{\mu}_i$, but renaming every positively quantified variable y of $\widehat{\mu}_i$ by y'. If $y_1, \ldots y_n$ are the positively quantified variables of $\widehat{\mu}_i$, $\widehat{\mu}'_i = \widehat{\mu}_i[y'_1/y_1, \ldots y'_n/y_n]$. If π_i is the constraint of $\widehat{\mu}_i$, then π'_i is the constraint of $\widehat{\mu}'_i$, obtained by renaming positively quantified variables in π_i. Finally, if $\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $t_1, \ldots, t_n$ are the witnesses for the positively quantified variables $y_1, \ldots, y_n$ in $\sigma, \mathbb{I} \models \widehat{\mu}_i$, we abbreviate $I(\pi_i, x, \mathbb{I}[t_1/y_1, \ldots, t_n/y_n])$ with $I(\pi_i, x, \mathbb{I}_i)$.

Before continuing with the proof, we introduce a new operator $\overline{\wedge}$. $\varphi_1 \mathbin{\overline{\wedge}} \varphi_2$ will rearrange the brackets of φ_1 such that, every positively quantified variable of φ_1 is in the scope of φ_2. For example, $\exists y(a(y) \wedge y = x + 1) \mathbin{\overline{\wedge}} \exists z(a(z) \wedge z = y + 1)) \equiv \exists y(a(y) \wedge y = x + 1 \wedge \exists z(a(z) \wedge z = y + 1)))$.

We prove that, ψ is equivalent to the disjunction of following formulas:

1.
$$\mathrm{FME}_x(\pi) \bigwedge_i \neg\widehat{\mu}_i. \tag{8}$$

No where in $I(\pi, x, \mathbb{I})$, ψ' is satisfied. So, ψ is satisfied for any $t \in I(\pi, x, \mathbb{I})$. Constraint is : π itself.

2.

$$\bigvee_i \Big(\widehat{\mu}_i \,\overline{\wedge}\, \mathrm{FME}_x(Lt_x(\pi) \wedge \neg Lt_x(\pi_i)) \,\overline{\wedge} \bigwedge_j \neg \Big(\widehat{\mu}'_j \,\overline{\wedge}\, \mathrm{FME}_x(Lt_x(\pi'_j) \wedge \neg Lt_x(\pi_i)) \Big) \Big). \tag{9}$$

$\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $I(\pi_i, x, \mathbb{I}_i)$ begins after the beginning of $I(\pi, x, \mathbb{I})$. Further, it is the first interval, i.e., for none of the j', $\sigma, \mathbb{I} \models \widehat{\mu}'_j$ and $I(\pi'_j, x, \mathbb{I}_{j'})$ begins before $I(\pi_i, x, \mathbb{I}_i)$. Constraint for each formula(i) is : $Lt_x(\pi) \wedge \neg Lt_x(\pi_i)$.

3.

$$\bigvee_i \bigvee_j \Big(\widehat{\mu}_i \,\overline{\wedge}\, \widehat{\mu}_j \,\overline{\wedge}\, \mathrm{FME}_x(\neg Rt_x(\pi_i) \wedge \neg Lt_x(\pi_j)) \,\overline{\wedge}$$
$$\bigwedge_k \neg \Big(\widehat{\mu}'_k \,\overline{\wedge}\, \mathrm{FME}_x(Lt_x(\pi'_k) \wedge Rt_x(\pi_i)) \,\overline{\wedge}\, \mathrm{FME}_x(\neg Rt_x(\pi_i) \wedge Rt_x(\pi'_k)) \Big) \,\overline{\wedge}$$
$$\bigwedge_l \neg \Big(\widehat{\mu}'_l \,\overline{\wedge}\, \mathrm{FME}_x(Lt_x(\pi'_l) \wedge \neg Lt_x(\pi_i)) \,\overline{\wedge}\, \mathrm{FME}_x(\neg Lt_x(\pi_j) \wedge Lt_x(\pi'_l)) \Big) \Big). \tag{10}$$

$\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $\sigma, \mathbb{I} \models \widehat{\mu}_j$ for some i, j and $I(\pi_j, x, \mathbb{I}_j)$ begins after the ending of $I(\pi_i, x, \mathbb{I}_i)$. Further, for none of the k', $\sigma, \mathbb{I} \models \widehat{\mu}'_k$ and $I(\pi'_k, x, \mathbb{I}'_k)$ begins before and ends after the ending of $I(\pi_i, x, \mathbb{I}_i)$ and for none of the l', $\sigma, \mathbb{I} \models \widehat{\mu}'_l$ and $I(\pi'_l, x, \mathbb{I}'_l)$ begins after the ending of $I(\pi_i, x, \mathbb{I}_i)$ and before the beginning of $I(\pi_j, x, \mathbb{I}_j)$. Constraint for each formula(i, j) is : $\neg Rt_x(\pi_i) \wedge \neg Lt_x(\pi_j)$.

4.

$$\bigvee_i \Big(\widehat{\mu}_i \,\overline{\wedge}\, \mathrm{FME}_x(\neg Rt_x(\pi_i) \wedge Rt_x(\pi)) \,\overline{\wedge} \bigwedge_j \neg \Big(\widehat{\mu}'_j \,\overline{\wedge}\, \mathrm{FME}_x(Rt_x(\pi'_j) \wedge \neg Rt_x(\pi_i)) \Big) \Big). \tag{11}$$

$\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $I(\pi_i, x, \mathbb{I}_i)$ ends before the ending of $I(\pi, x, \mathbb{I})$. Further, it is the last interval, i.e., for none of the j', $\sigma, \mathbb{I} \models \widehat{\mu}'_j$ and $I(\pi'_j, x, \mathbb{I}_{j'})$ ends after $I(\pi_i, x, \mathbb{I}_i)$. Constraint for each formula(i) is : $Rt_x(\pi) \wedge \neg Rt_x(\pi_i)$.

To prove the equivalence, we show that, if ψ is satisfied at $x = t$, then, one of the equations (8), (9), (10), (11) will be satisfied and t belongs to the corresponding interval. Conversely, if any of the four equations are satisfied, then, every t' in the corresponding interval, will satisfy ψ.

To prove implication of forward direction, suppose $\sigma, \mathbb{I}[t/x] \models \psi$. Find the point t_l which is immediately to the left of t and t_r which is immediately to the right of t, such that, $\sigma, \mathbb{I}[t_l/x] \models \psi'$ and $\sigma, \mathbb{I}[t_r/x] \models \psi'$. If we can not find both t_l and t_r (see Fig.(6)), then $\sigma, \mathbb{I}[t'/x] \models \neg\psi'$ for every $t' \in I(\pi, x, \mathbb{I})$. Since $t \in I(\pi, x, \mathbb{I})$, $I(\pi, x, \mathbb{I})$ is not empty, $\sigma, \mathbb{I} \models \mathrm{FME}_x(\pi)$. This implies (8) is true.

If we can find only t_r, then for some i, $\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $t_r \in I(\pi_i, x, \mathbb{I}_i)$. Further, t_r must be its left most boundary as there is no $t'_r \in I(\pi, x, \mathbb{I})$ and $t'_r < t_r$ and $\sigma, \mathbb{I}[t'_r/x] \models \psi'$ (if not, that must be the immediate right point of t. See Fig.(7)). This implies, (9) must be true, as there is no such interval to the left of t such

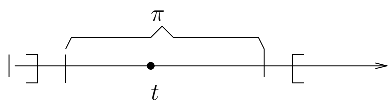

Fig. 6.

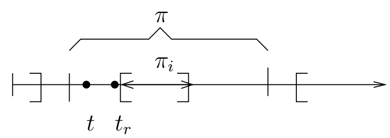

Fig. 7.

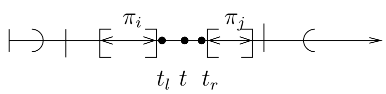

Fig. 8.

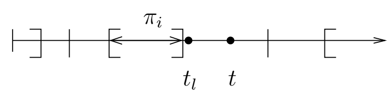

Fig. 9.

that, $\sigma, \mathbb{I} \models \widehat{\mu}'_j$. t is in the interval spanning from the left boundary of π to x_r. This implies $t \in I(Lt_x(\pi) \wedge \neg Lt_x(\pi_i), x, \mathbb{I}_i)$.

If we can find both t_l and t_r, then for some i, $\sigma, \mathbb{I} \models \widehat{\mu}_i$ and for some j, $\sigma, \mathbb{I} \models \widehat{\mu}_j$. Further, $t_l \in I(\pi_i, x, \mathbb{I}_i)$ and t_l is the right most boundary of this interval, $t_r \in I(\pi_j, x, \mathbb{I}_j)$ and t_r is the left most boundary of this interval (See Fig.(8)). This implies, (10) must be true, as there is no such interval to the left of t such that, $\sigma, \mathbb{I} \models \widehat{\mu}'_k$ and no interval to the right of t such that, $\sigma, \mathbb{I} \models \widehat{\mu}'_l$. t is in the interval (t_l, t_r). This implies $t \in I(\neg Rt_x(\pi_i) \wedge \neg Lt_x(\pi_j), x, \mathbb{I}_{i,j})$.

If we can find only t_l, then for some i, $\sigma, \mathbb{I} \models \widehat{\mu}_i$ and $t_l \in I(\pi_i, x, \mathbb{I}_i)$ and t_l must be its left most boundary. (See Fig.(7)). This implies, (9) must be true, as there is no such interval to the right of t such that, $\sigma, \mathbb{I} \models \widehat{\mu}'_j$. t is in the interval spanning from x_l to the right boundary of π. This implies $t \in I(\neg Rt_x(\pi_i) \wedge Rt_x(\pi), x, \mathbb{I}_i)$.

To prove the implication of reverse direction, suppose (8) is true, then for every $t' \in I(\pi, x, \mathbb{I})$, $\sigma, \mathbb{I}[t'/x] \models \pi \wedge \neg\widehat{\psi}$.

If (9) is true, consider any $t' \in I(Lt_x(\pi) \wedge \neg Lt_x(\pi_i), x, \mathbb{I}_i)$ (interval is non empty since $\mathrm{FME}_x(Lt_x(\pi) \wedge \neg Lt_x(\pi_i))$ is true). $\widehat{\psi}$ is false at $x = t'$, or else $\widehat{\mu}'_j$ will be true at t', which is a contradiction. This implies, $\sigma, \mathbb{I}[t'/x] \models \pi \wedge \neg\widehat{\psi}$.

If (10) is true, consider any $t' \in I(\neg Rt_x(\pi_i) \wedge \neg Lt_x(\pi_j), x, \mathbb{I}_{i,j})$ (interval is non empty since $\mathrm{FME}_x(\neg Rt_x(\pi_i) \wedge \neg Lt_x(\pi_j))$ is true). $\widehat{\psi}$ is false at $x = t'$, or else $\widehat{\mu}'_k$ or $\widehat{\mu}'_l$ will be true at t', which is a contradiction. This implies, $\sigma, \mathbb{I}[t'/x] \models \pi \wedge \neg\widehat{\psi}$.

If (11) is true, consider any $t' \in I(\neg Rt_x(\pi_i) \wedge Rt_x(\pi), x, \mathbb{I}_i)$, which is non empty. $\widehat{\psi}$ is false at $x = t'$, or else $\widehat{\mu}'_j$ will be true at t', which is a contradiction. This implies, $\sigma, \mathbb{I}[t'/x] \models \pi \wedge \neg\widehat{\psi}$.

Now consider formulas of the form $\psi \equiv \exists x(\pi \wedge \widehat{\nu}_1 \wedge \widehat{\nu}_2)$. We prove that, $\exists x(\pi \wedge \widehat{\nu}_1 \wedge \widehat{\nu}_2) \equiv \bigvee_i \left(\widehat{\mu}_i \barwedge \exists x(\pi_i \wedge \widehat{\nu}_2)\right)$ where, $\widehat{\mu}_i$ and π_i are obtained by eliminating x from the formula $\exists x(\pi \wedge \widehat{\nu}_1)$.

If $x = t$ satisfies L.H.S., then it should satisfy $\pi \wedge \widehat{\nu}_1$, which implies $\widehat{\mu}_i$ is satisfied and $t \in I(\pi_i, x, \mathbb{I}_i)$ for some i. Since t must also satisfy $\widehat{\nu}_2$ as well, t will satisfy R.H.S.

If $x = t$ satisfies R.H.S., then $t \in I(\pi_i, x, \mathbb{I}_i)$, it will satisfy $\pi \wedge \widehat{\nu}_1$. But t also satisfies $\widehat{\nu}_2$ which implies it will satisfy L.H.S.

We need to eliminate x from $\bigvee_i \left(\widehat{\mu}_i \,\overline{\wedge}\, \exists x(\pi_i \wedge \widehat{\nu}_2)\right)$. (we assume each disjunct to be in $\exists$-normal form. If it is not in that form, we need to normalize it). Consider $\exists x(\pi_i \wedge \widehat{\nu}_2)$, which can be translated to disjunction of active formulas $\widehat{\mu}'_j$ with corresponding π'_j. So

$$\psi \equiv \bigvee_i \bigvee_j (\widehat{\mu}_i \,\overline{\wedge}\, \widehat{\mu}'_j)$$

We claim that, π'_j is the constraint for each of $\widehat{\mu}_i \overline{\wedge} \widehat{\mu}'_j$. Suppose $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}_1 \wedge \widehat{\nu}_2$. This implies, $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}_1$. Implies $\sigma, \mathbb{I} \models \widehat{\mu}_i$ for some i and $t \in I(\pi_i, x, \mathbb{I}_i)$. We also know that, $\sigma, \mathbb{I}[t/x] \models \widehat{\nu}_2$. Combining these two, $\sigma, \mathbb{I}_i[t/x] \models \pi_i \wedge \widehat{\nu}_2$. This implies, $\sigma, \mathbb{I}_{j'} \models \widehat{\mu}'_j$ for some j' and $t \in I(\pi'_j, x, \mathbb{I}_{j'})$. This implies, $\widehat{\mu}_i \wedge \widehat{\mu}'_j$ is satisfied and t is in the interval corresponding to π'_j.

To show in the converse direction, if $\sigma, \mathbb{I} \models \widehat{\mu}_i \,\overline{\wedge}\, \widehat{\mu}'_j$, then we need to show that, for every $t \in I(\pi'_j, x, \mathbb{I}_{i,j'})$, $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}_1 \wedge \widehat{\nu}_2$.
We know that, $\sigma, \mathbb{I}_{j'} \models \widehat{\mu}'_j$
$\implies$, for every t such that $\mathbb{I}[t/x, t_1/y_1, \ldots, t_p/y_p, t'_1/y'_1, \ldots, t'_q/y'_q] \models \pi'_j$ then $\sigma, \mathbb{I}[t/x, t_1/y_1, \ldots, t_p/y_p] \models \pi_i \wedge \widehat{\nu}_2$.
$\implies$ $\sigma, \mathbb{I}[t/x] \models \widehat{\nu}_2$ and $\mathbb{I}[t/x, t_1/x_1, \ldots t_n/x_n] \models \pi_i$.
$\implies$ $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}_1$. Combining these two, $\sigma, \mathbb{I}[t/x] \models \pi \wedge \widehat{\nu}_1 \wedge \widehat{\nu}_2$.

It is possible to extend this to any number of disjuncts $\widehat{\nu}_i$ in ψ. □

Theorem 3. *It is possible to translate any* FO^c *formula to an equivalent formula which when evaluated in pointwise semantics, will accept the same language.*

Proof. If we can express every FO^c formula by an equivalent active FO^c formula, the theorem is proved. Any FO^c formula φ, can be written as a boolean combination of $\exists$-normal form formulas. Since translation is intact across the boolean operations, it is sufficient if we can eliminate all the passive variables from formulas in $\exists$-normal form. Consider the formula tree of a formula in $\exists$-normal form. Identify those nodes which are passive and having only active nodes as ancestors. It is easy to see that, if we can eliminate passively quantified variables from each of these trees rooted at these nodes, translation is complete. Now consider any specific node(N), which is of the form $\psi \equiv \exists x(\pi(x) \wedge \bigwedge_{i \le t} \nu_i)$.

We argue that it is possible to convert any passively quantified formula of the form ψ, into an equivalent formula with finite disjunction of active formulas. We prove this based on induction on number of passive variables n in the tree rooted under N.

As a base case, if $n = 1$, the passive variable must be x itself and by Theorem(2), we can convert this formula into an equivalent formula which is a finite disjunction of active formulas. By induction, it is possible to convert any formula with $n < r$ passive variables into an equivalent formula with finite disjunction of active formulas. Consider ψ with $n = r$ number of passive variables. Since x itself is a passive variable, each of the ν_i's will have utmost $r - 1$ passive variables. But by induction each ν_i can be expressed as finite disjunction of $\widehat{\mu}_j$'s $(1 \le j \le u)$, which are active formulas. Since there are finitely many ν_i's, ψ will be of the form,

$$\bigvee_{i \leq t, j \leq u} \exists x \Big(\pi(x) \wedge \widehat{\mu}_{i,j} \Big) \tag{12}$$

Since each of the disjunct is in $\exists$-normal form, and only x is the passive variable, using Theorem(2), we can translate (12), to disjunction of active formulas. □

We give an example to demonstrate the translation of a passively quantified formula to actively quantified form. Consider the language consisting of all timed words such that, there exists an a at one unit from some point in [0,1].

$$\begin{aligned} &\exists x(0 \leq x \wedge x \leq 1 \wedge (\exists y(a(y) \wedge y = x + 1) \\ \equiv &\exists y(a(y) \wedge \mathrm{FME}_x(0 \leq x \wedge y \leq x + 1 \wedge x \leq 1 \wedge x \leq y + 1)) \\ \equiv &\exists y(a(y) \wedge 1 \leq y \wedge y \leq 2) \end{aligned}$$

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