Interprocedural analysis: Sharir-Pnueli's functional approach

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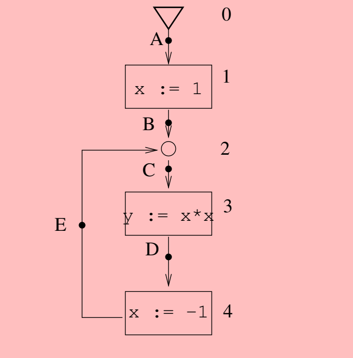
Outline

Equation solving approach

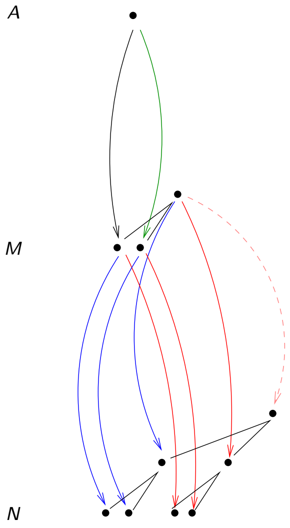
In non-procedural case, we set up equations to capture JOP assuming distributivity. Least solution to these equations gave us exact/over-approx JOP depending on distributive/monotonic framework.

$$
\begin{aligned}
x_A &= \emptyset \\
x_B &= f_1(x_A) \\
x_C &= x_B \sqcup x_E \\
x_D &= f_3(x_C) \\
x_E &= f_4(x_D)
\end{aligned}
$$



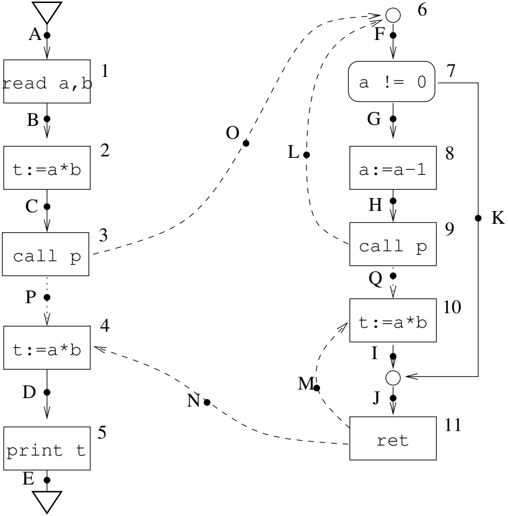
Equations to capture JOP: why it works

- We want JOP at N.
- Suppose M is an intermediate point such that all paths to N pass through M.
- If transfer functions are distributive, then we can take join over paths at point M, and then join over paths from M to N.



Equation solving: Problems with naive approach

- Try to set up similar equations for x_N (JVP at program point N).
- How do we describe x_N in terms of x_J?

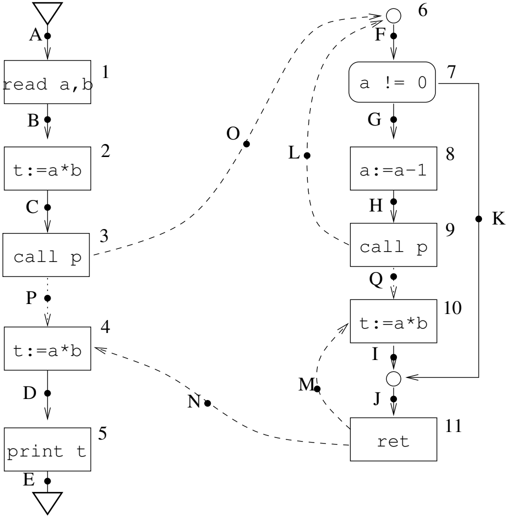


Instead try to capture join over complete paths first

- Set up equations to capture join over complete paths.
- Now set up equations to capture JVP using join over complete path values.

Notation

- Root of procedure p is denoted r_p.
- Exit (return) of procedure p is denoted e_p.
- Sometimes use r_1 for r_{main}.
- Assume WLOG that `main` is not called.

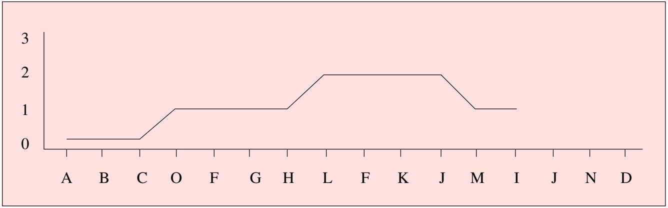


Valid and complete paths

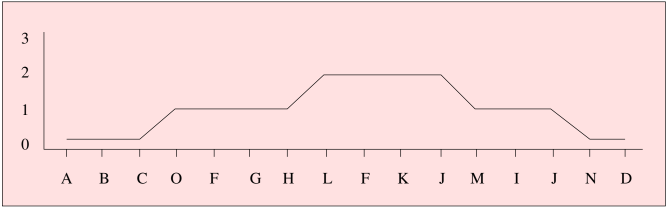
- A path ρ in G' is valid and complete if it is an interprocedurally valid path in G' and the associate call-string is empty.
- Denote the set of such valid and complete paths by $IVP_0(G')$
- That is $\rho \in IVP_0(G')$ iff $\rho \in IVP(G')$ and $cs(\rho) = \epsilon$.

Example paths

An example valid path in $IVP(r_1, I)$:



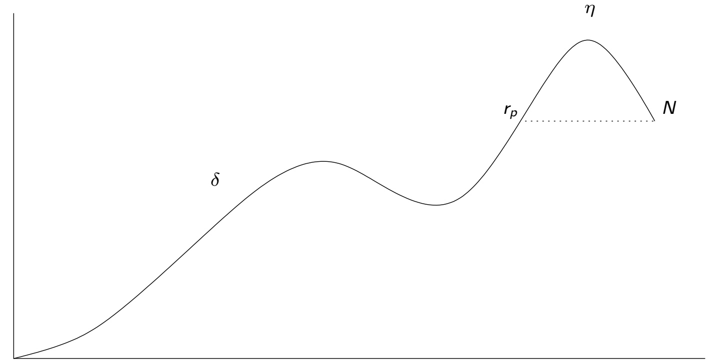
An example valid and complete path in $IVP_0(r_1, D)$:



Path "FGHLFKJMIJ" is valid and complete and is in $IVP_0(r_p, J)$.

Basic idea: Why join over complete paths help

An IVP path ρ from r_1 to N in procedure p can be written as $\delta \cdot \eta$ where δ is in $\mathrm{IVP}(r_1, r_p)$, and η is in $\mathrm{IVP}_0(r_p, N)$.



Path η is suffix after last pending call to procedure p was made.

Join over valid and complete paths from r_p to N

For a procedure p and node N in p, define:

$$\phi_{r_p,N} : D \to D$$

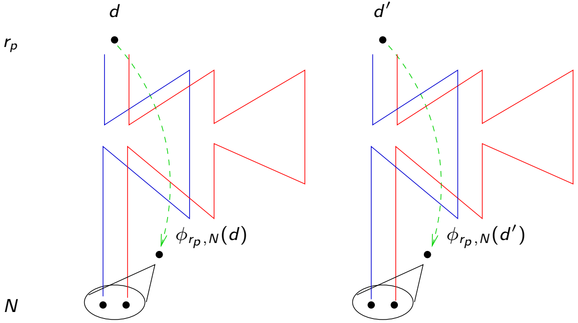
given by

$$\phi_{r_p,N}(d) = \bigsqcup_{\text{paths } \rho \in \mathrm{IVP}_0(r_p,N)} f_\rho(d).$$

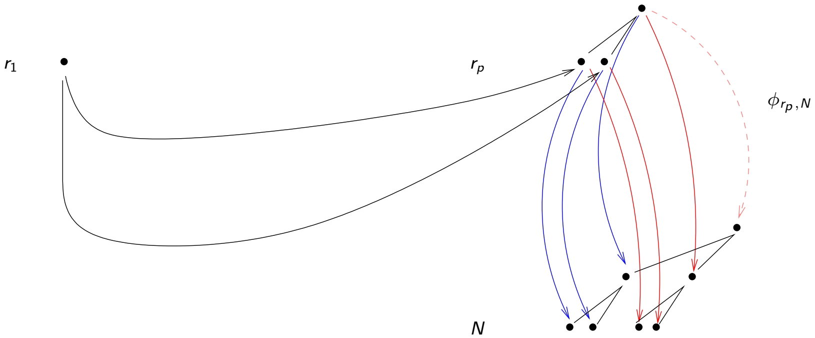
$\phi_{r_p,N}$ is thus the join of all functions f_ρ where ρ is an interprocedurally valid and complete path from r_p to N.

We call $\phi_{r_p,N}$ the Join over Valid and Complete Paths (JVCP) from r_p to N.

Visualizing $\phi_{r_p,N}$



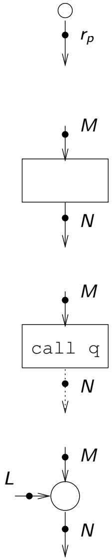
Using $\phi_{r_p,N}$'s to get JVP values



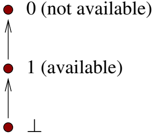
Assuming distributivity of underlying transfer functions, JVP value at N equals $\phi_{r_p,N}$ applied to JVP value at r_p.

Equations (1) to capture $\phi_{r_p,N}$

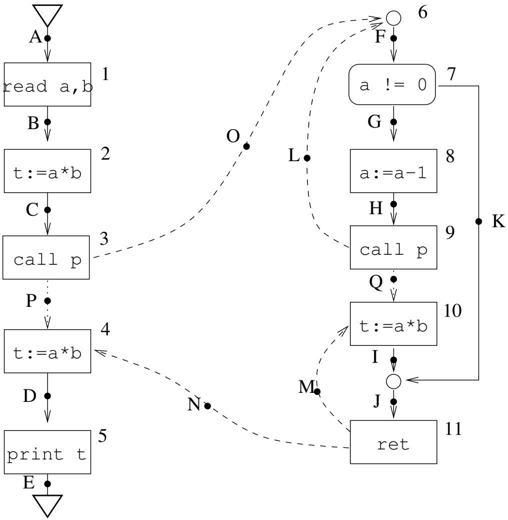
$$
\begin{aligned}
y_{r_p,r_p} &= id_D && (root)\\
y_{r_p,N} &= f_{MN} \circ y_{r_p,M} && (stmt)\\
y_{r_p,N} &= y_{r_q,e_q} \circ y_{r_p,M} && (call)\\
y_{r_p,N} &= y_{r_p,L} \sqcup y_{r_p,M} && (join)
\end{aligned}
$$



Example: Available expressions analysis



Lattice for Av-Exp analysis.



Functions we will use for example analysis

- $D = \{\perp, 1, 0\}$.
- $\mathbf{0} : D \to D$ given by

$$\begin{array}{ccc} \perp & \mapsto & \perp \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 0 \end{array}$$

- $\mathbf{1} : D \to D$ given by

$$\begin{array}{ccc} \perp & \mapsto & \perp \\ 0 & \mapsto & 1 \\ 1 & \mapsto & 1 \end{array}$$

- $id : D \to D$ given by

$$\begin{array}{ccc} \perp & \mapsto & \perp \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \end{array}$$

- Ordering: $\mathbf{1} \leq id \leq \mathbf{0}$.

Example: Equations (1) for ϕ's

$$
\begin{aligned}
y_{A,A} &= id \\
y_{A,B} &= \mathbf{0} \circ y_{A,A} \\
y_{A,C} &= \mathbf{1} \circ y_{A,B} \\
y_{A,P} &= y_{F,J} \circ y_{A,C} \\
y_{A,D} &= \mathbf{1} \circ y_{A,P} \\
y_{A,E} &= id \circ y_{A,D}
\end{aligned}
$$

$$
\begin{aligned}
y_{F,F} &= id \\
y_{F,G} &= id \circ y_{F,F} \\
y_{F,K} &= id \circ y_{F,F} \\
y_{F,H} &= \mathbf{0} \circ y_{F,G} \\
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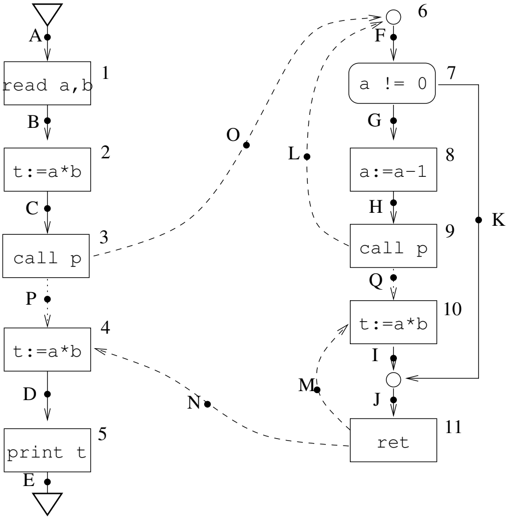
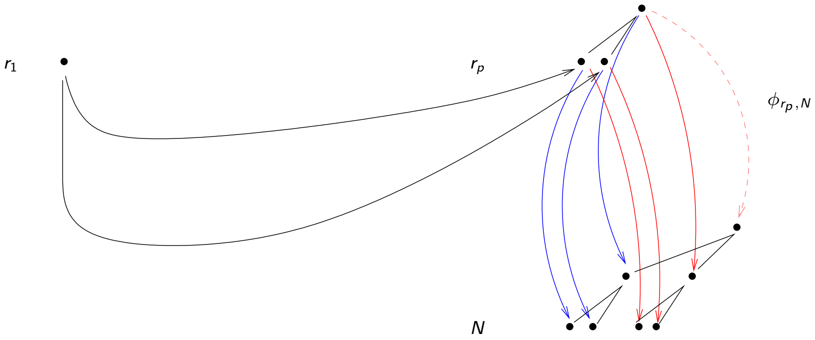


Using $\phi_{r_p,N}$'s to get JVP values



Assuming distributivity of underlying transfer functions, JVP value at N equals $\phi_{r_p,N}$ applied to JVP value at r_p.

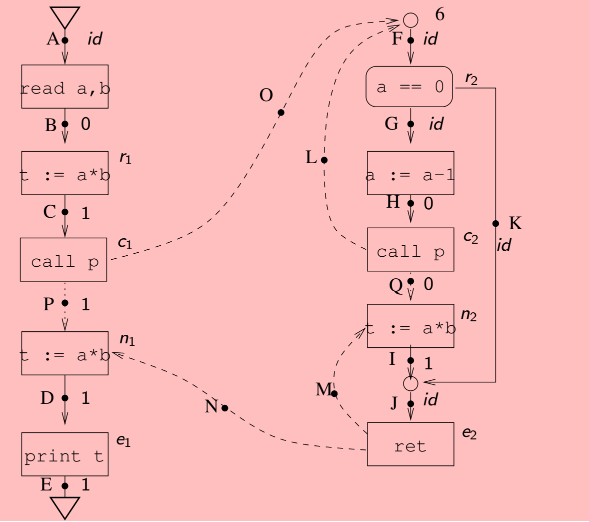
Equations (2) to capture JVP

$$
\begin{array}{rcll}
x_1 & = & d_0 & \\
x_{r_p} & = & \bigsqcup_{\text{calls } C \text{ to } p} x_C & \\
x_N & = & \phi_{r_p,N}(x_{r_p}) & \text{for } N \in \text{ProgPts}(p) - \{r_p\}.
\end{array}
$$

Example: Equations for x_N's (JVP)

$$
\begin{aligned}
x_A &= 0\\
x_B &= \phi_{AB}(x_A)\\
x_C &= \phi_{AC}(x_A)\\
x_P &= \phi_{AP}(x_A)\\
x_D &= \phi_{AD}(x_A)\\
x_E &= \phi_{AE}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= x_C \sqcup x_H\\
x_G &= \phi_{FG}(x_F)\\
x_K &= \phi_{FK}(x_F)\\
x_H &= \phi_{FH}(x_F)\\
x_Q &= \phi_{FQ}(x_F)\\
x_I &= \phi_{FI}(x_F)\\
x_J &= \phi_{FJ}(x_F).
\end{aligned}
$$



Example: Equations for x_N's (JVP)

$$
\begin{aligned}
x_A &= 0 \\
x_B &= \mathbf{0}(x_A) \\
x_C &= \mathbf{1}(x_A) \\
x_P &= \mathbf{1}(x_A) \\
x_D &= \mathbf{1}(x_A) \\
x_E &= \mathbf{1}(x_A)
\end{aligned}
$$

$$
\begin{aligned}
x_F &= x_C \sqcup x_H \\
x_G &= id(x_F) \\
x_K &= id(x_F) \\
x_H &= \mathbf{0}(x_F) \\
x_Q &= \mathbf{0}(x_F) \\
x_I &= \mathbf{1}(x_F) \\
x_J &= id(x_F).
\end{aligned}
$$

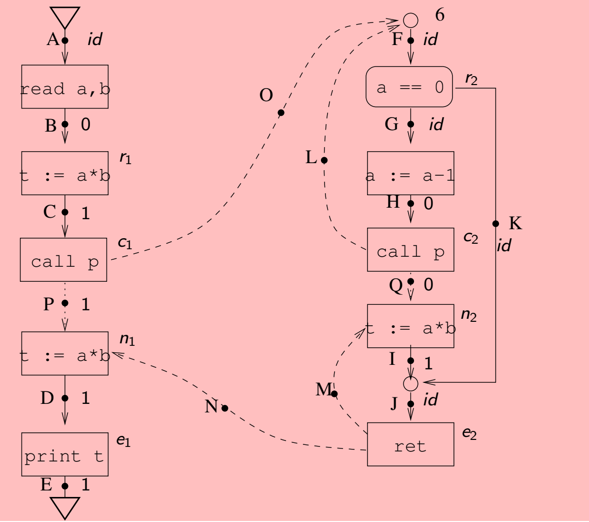


Fig. shows values of $\phi_{r_p,N}$'s in bold.

Solving a system of equations using Knaster-Tarski Theorem

Given equations (E_1)

$$
\begin{array}{rcl}
y_1 & = & f_1(y_1, \ldots, y_n) \\
& \ldots & \\
y_n & = & f_n(y_1, \ldots, y_n)
\end{array}
$$

Consider a complete lattice $(D, \leq)$ such that:

- D is closed under each f_i.
- Each f_i is a monotonic function on this lattice: if $\langle d_1, \ldots, d_n \rangle \leq \langle e_1, \ldots, e_n \rangle$ then $f_i(d_1, \ldots, d_n) \leq f_i(e_1, \ldots, e_n)$.
- Equivalently, the function $\overline{F}$ on $(D^n, \leq)$ given by

  $$\overline{F}(\langle d_1, \ldots, d_n \rangle) = \langle f_1(d_1, \ldots, d_n), \ldots, f_n(d_1, \ldots, d_n) \rangle,$$

  is monotonic.

Then, by Knaster-Tarski, the function $\overline{F}$ on $(D^n, \leq)$ has a LFP, which coincides with the least solution (in D^n) to equations (E_1).

Solving Eq (1) using Knaster-Tarski

- Consider lattice $(F, \leq)$ of functions from D to D, obtained by closing the transfer functions, identity, and $f_{\perp} : d \mapsto \perp$ under composition and join. (Alternatively we can take F to be all monotone functions on D.)
- Ordering is $f \leq g$ iff $f(d) \leq g(d)$ for each $d \in D$.
- $(F, \leq)$ is also a complete lattice.
- $\overline{f}$ induced by Eq (1) is monotone on complete lattice $(\overline{F}, \overline{\leq})$.
    - Sufficient to argue that function composition $\circ$ is monotone when applied to monotone functions.
    - Join operation $\bigsqcup$ is monotone.
- LFP / least solution (say $y^*_{r_p,N}$'s) exists by Knaster-Tarski.
- Each $y^*_{r_p,N}$ is necessarily monotonic.

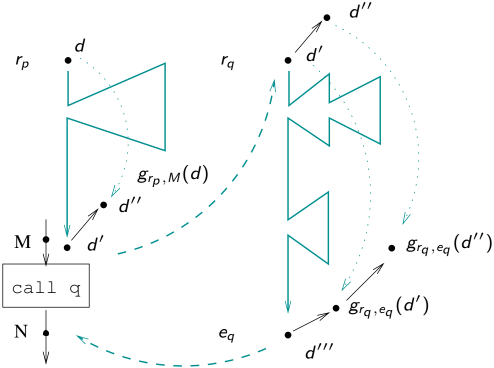
Correctness claim I

Claim

1. The least solution to Eq (1) dominates $\phi_{r_p,N}$'s (i.e. $\phi_{r_p,N} \leq y^*_{r_p,N}$ for each N).
2. $\phi_{r_p,N}$'s are the least solution to Eq (1) (i.e. $\phi_{r_p,N} = y^*_{r_p,N}$ for each N) when f_{MN}'s are distributive.

Proof

1. For part 1:
   - Let $g_{r_p,N}$ be any monotone solution to Eq (1).
   - Sufficient to prove: For each $d \in D$, and ρ an IVP_0 path from r_p to N, $f_\rho(d) \leq g_{r_p,N}(d)$.
   - Proof by induction on length of path ρ.
2. For part 2: Prove that $\phi_{r_p,N}$'s are a solution to Eq (1), and hence they will dominate the least solution.



Using Kildall to compute LFP

- We can use Kildall's algo to compute the LFP of these equations as follows.
  - Initialize the value at all program points with RHS of the constant equations (in this case *id* at entry of procedures), and the bottom value (in this case $f_\perp$).
  - Mark all values
  - Pick a marked value at point say N, and "propagate" it (i.e. for any node M in the LHS of an equation in which N occurs in the RHS, evaluate M and join it with the existing value at M). Mark as before in Kildall's algo.
  - Stop when no more marked values to propagate.
- Kildall's algo will compute $y^*_{r_p,N}$ if D is finite. Note that finite height of $(D, \leq)$ is not sufficient for termination.

Correctness and Algo II

Consider Eq (2)':

$$\begin{array}{lcll} x_1 & = & d_0 & \\ x_{r_p} & = & \bigsqcup_{\text{calls } C \text{ to } p} x_C & \\ x_N & = & y^*_{r_p,N}(x_{r_p}) & \quad \text{for } N \in N_p - \{r_p\}. \end{array}$$

(Recall that $y^*_{r_p,N}$'s are the least solution of Eq (1).)

- $\overline{f}$ induced by Eq (2)' is a monotone function on the complete lattice $(\overline{D}, \overline{\leq})$.
- LFP / least solution (say x^*_N's) exists by Knaster-Tarski.

Claim

JVP values are the least solution to Eq (2)' (i.e. $\mathrm{JVP}_N = x^*_N$) when f_{MN}'s are distributive. Otherwise $\mathrm{JVP}_N \leq x^*_N$ for each N.

Kleene/Kildall's algo will compute x^*_N's (assuming D finite).

Example: Computing $\phi_{r_p,N}$'s ($y^*_{r_p,N}$ to be precise) using Kildall's algo

$$
\begin{aligned}
y_{A,A} &= id \\
y_{A,B} &= \mathbf{0} \circ y_{A,A} \\
y_{A,C} &= \mathbf{1} \circ y_{A,B} \\
y_{A,P} &= y_{F,J} \circ y_{A,C} \\
y_{A,D} &= \mathbf{1} \circ y_{A,P} \\
y_{A,E} &= id \circ y_{A,D}
\end{aligned}
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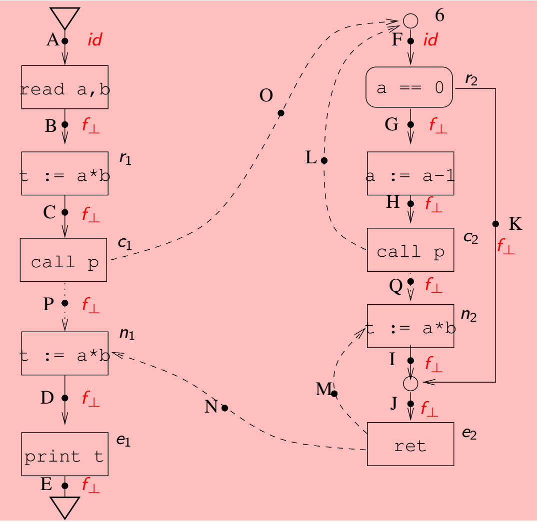

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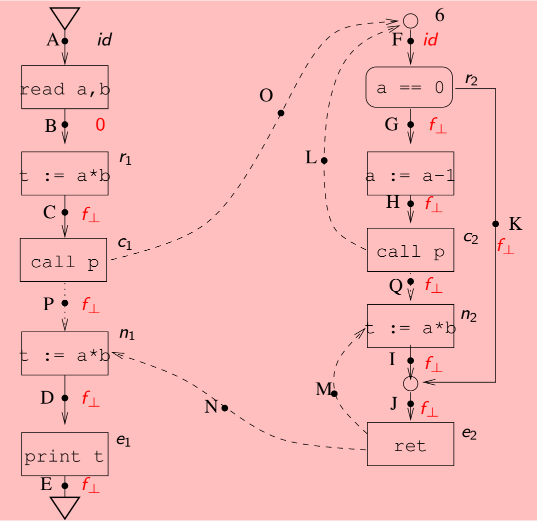

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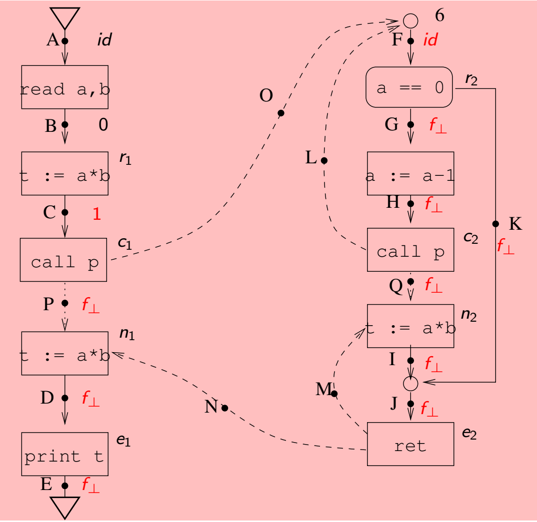

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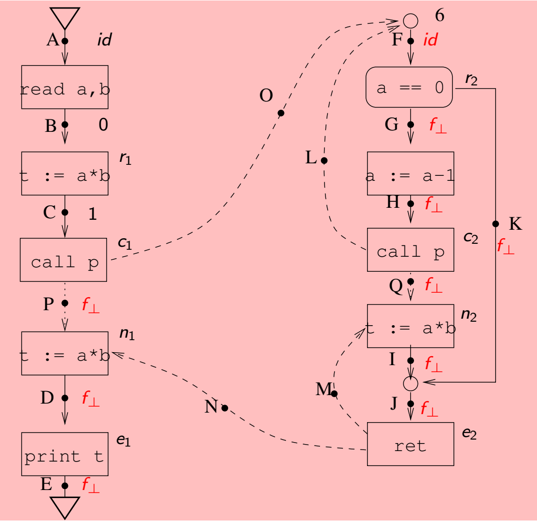

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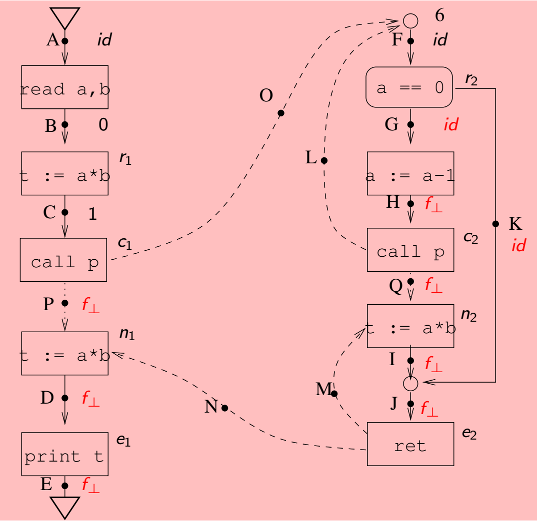

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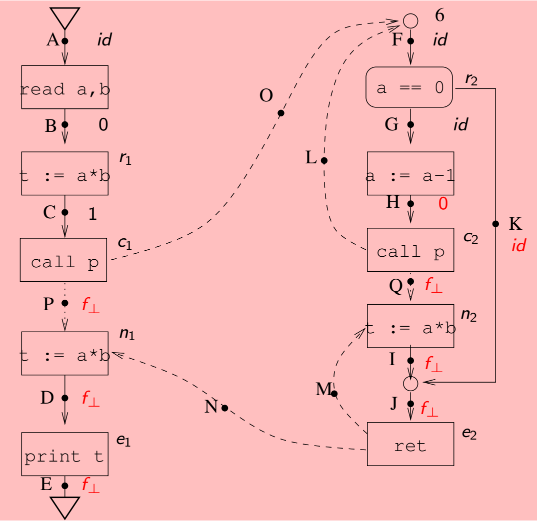

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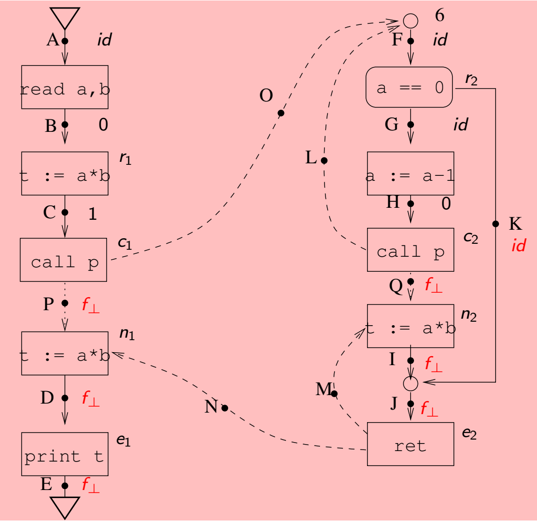

Example: Computing $\phi_{r_p,N}$'s ($y^*_{r_p,N}$ to be precise) using Kildall's algo

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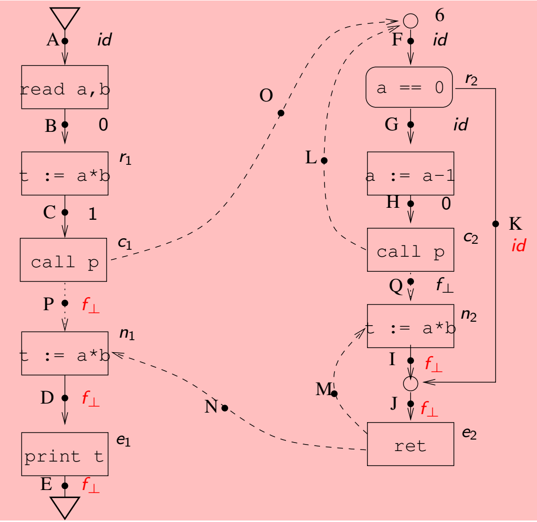

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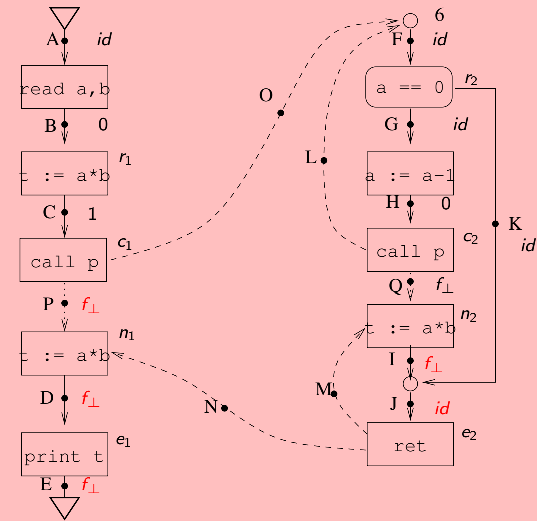

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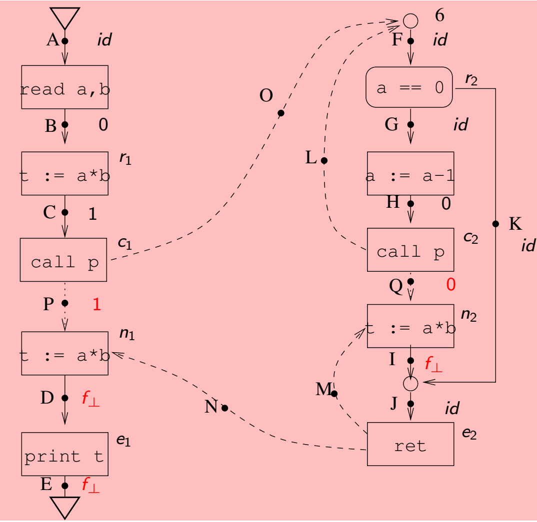

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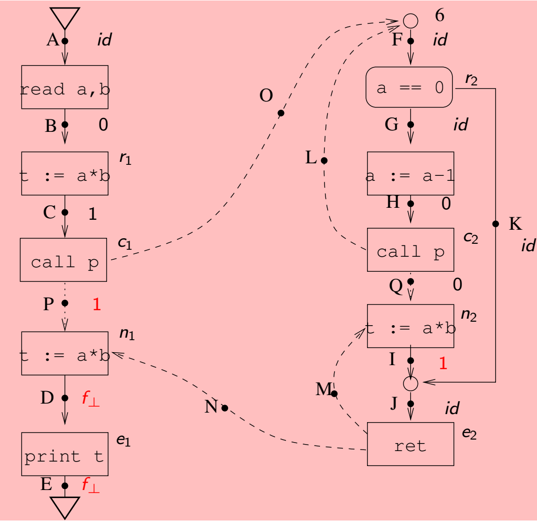

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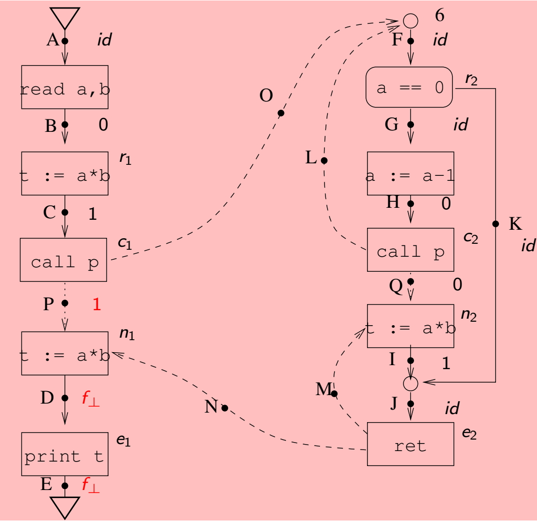

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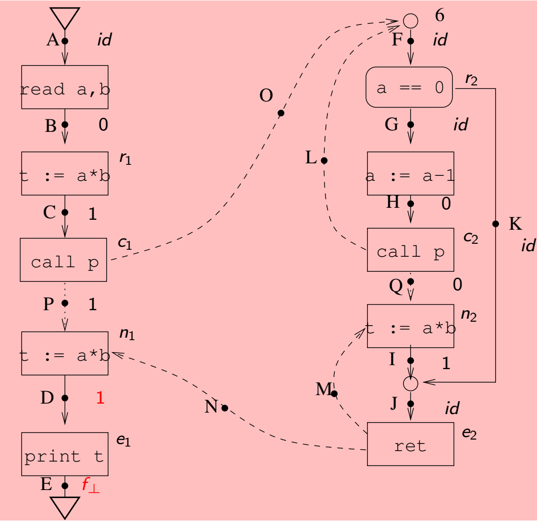

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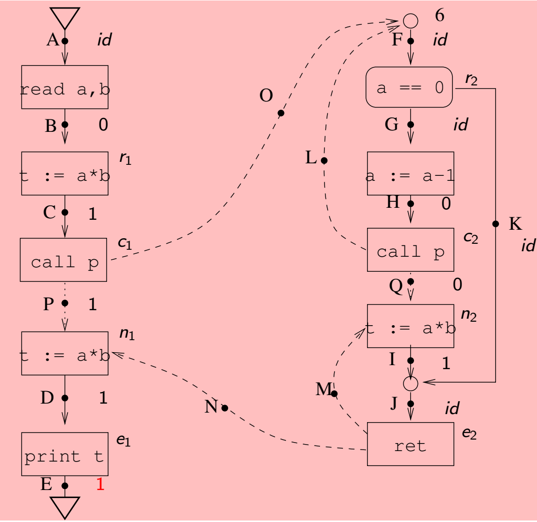

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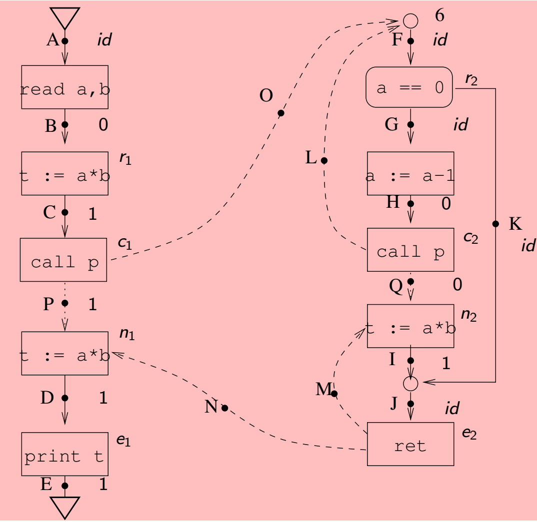

Example: Computing JVP values (x_N^*'s to be precise)

$$
\begin{aligned}
x_A &= 0\\
x_B &= \mathbf{0}(x_A)\\
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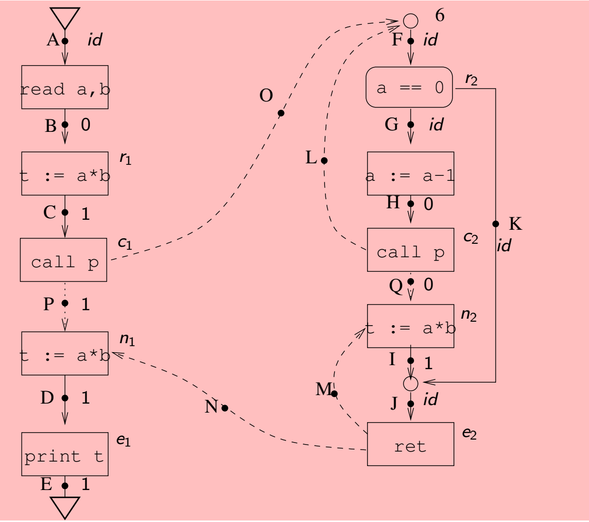

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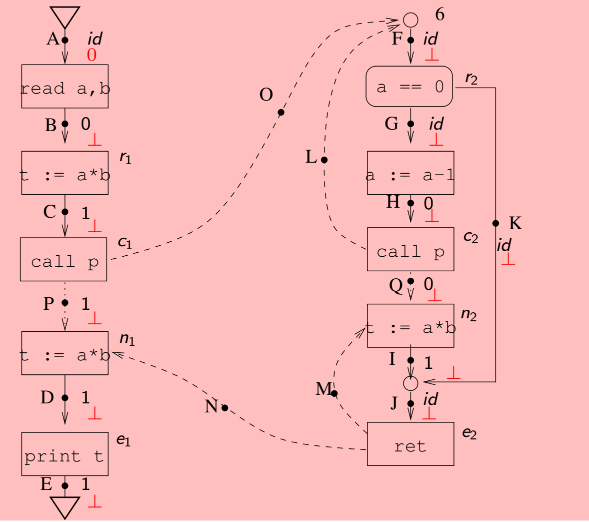

Fig shows initial (red) and final (blue) values.

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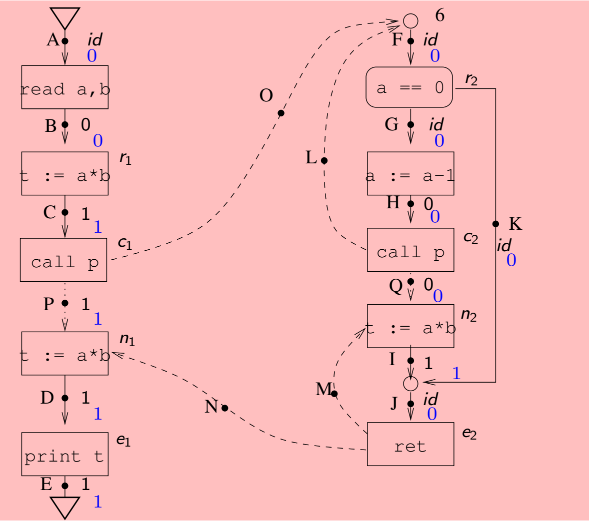


Fig shows initial (red) and final (blue) values.

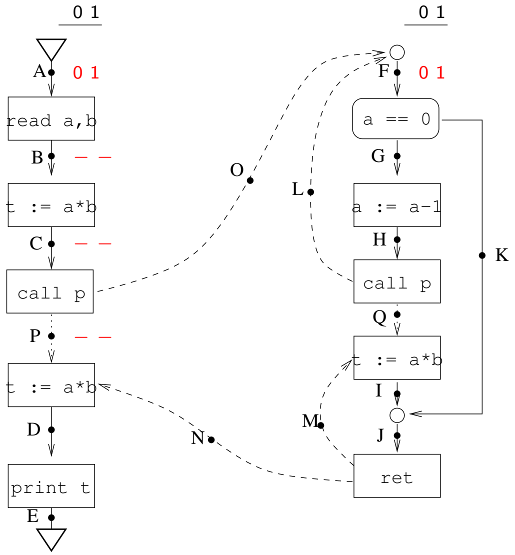
Summary of functional approach

- Uses a two step approach
  1. Compute $\phi_{r_p,N}$'s (JVCPs for each function).
  2. Compute x_n's (JVPs) at each point.

Summary of conditions: For each property (column heading), the conjunction of the ticked conditions (row headings) are sufficient to ensure the property.

	Termination	Least Sol of Eq(2) $\geq$ JVP	Least Sol of Eq(2)$=$ JVP
f_{MN}'s monotonic	$\surd$	$\surd$	
Finite underlying lattice	$\surd$		
f_{MN}'s distributive			$\surd$

Viewing ϕ computation as a table



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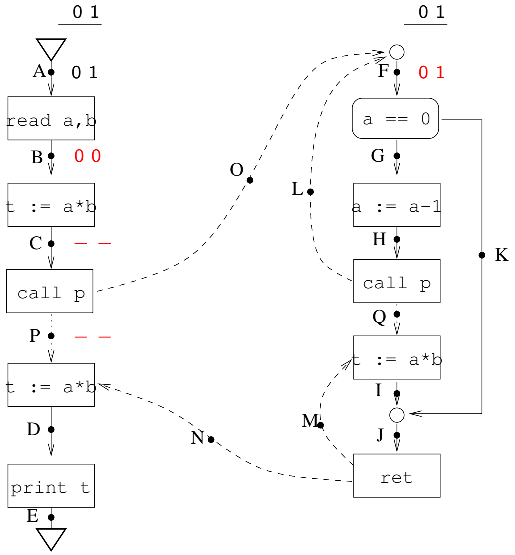

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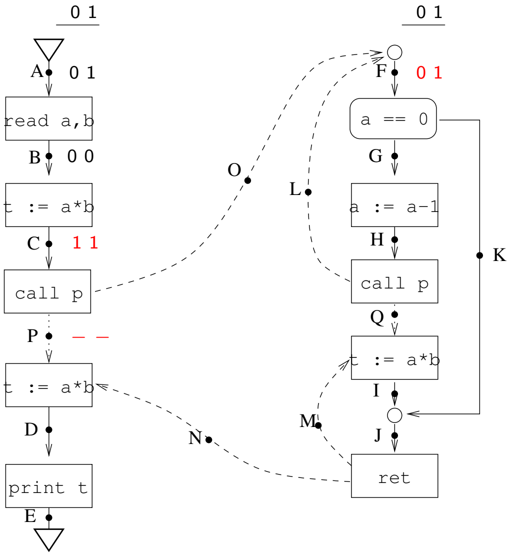

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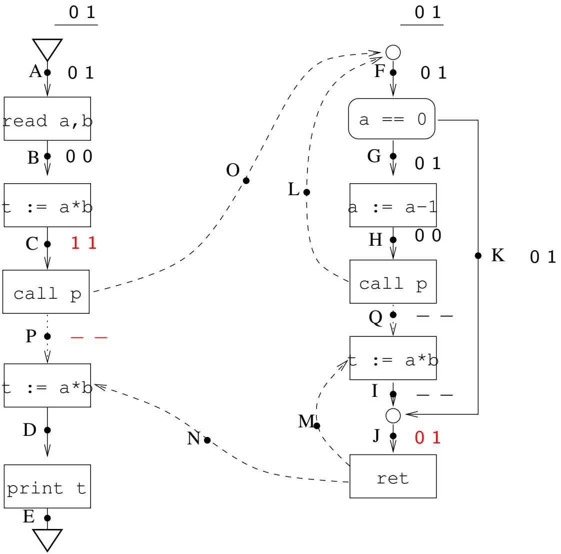

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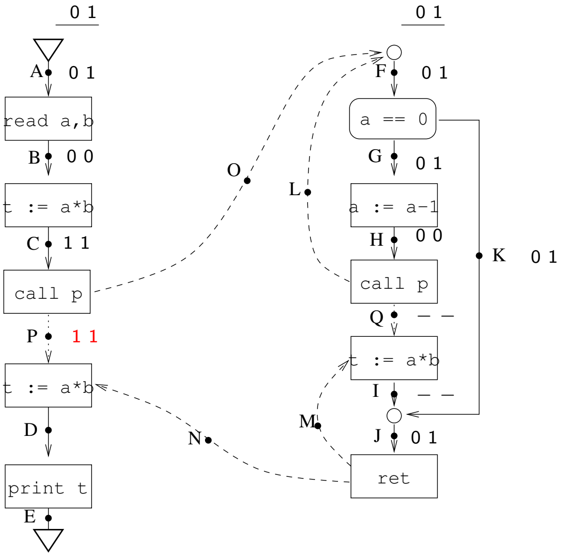


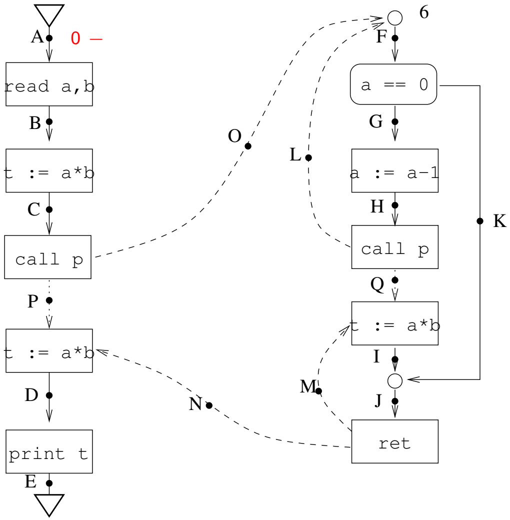
Iterative/Tabulation Approach

- Main idea: de-couple the propagation of function rows.
- Maintain a table of values representing the current value of $\phi_{r_p,N}$ for each program point N in procedure p.
- Expand column for data value d in procedure p only if d is reachable at r_p.
- Informally, at N in procedure p, the table has an entry $d \mapsto d'$ if we have seen
  1. valid paths ρ from r_1 to r_p with $\bigsqcup_\rho f_\rho(d_0) = d$, and
  2. valid and complete paths δ from r_p to N with $\bigsqcup_\delta f_\delta(d) = d'$.

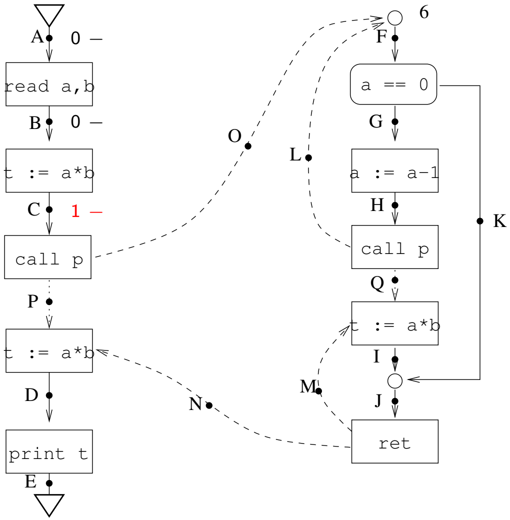
Iterative/Tabulation Approach

- Apply Kildall's algo with initial value of $d_0 \mapsto d_0$ at r_1.
- Propagating value d across a call to procedure p: (a) begin a column for d at root of p if not already there; Also (b) if d is mapped to d' at the end of p, then propogate d' to the return site of the call.
- Propagating across return nodes from procedure p: value d' in column for d is propagated to each column at a return site of a call to procedure p that has the value d in the preceding entry.

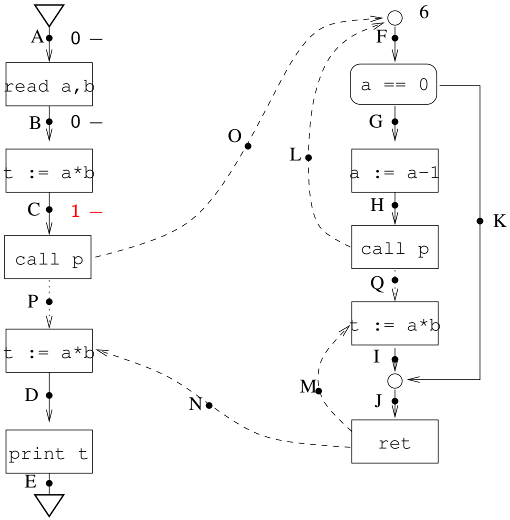
Example: Computing ϕ's iteratively: 1



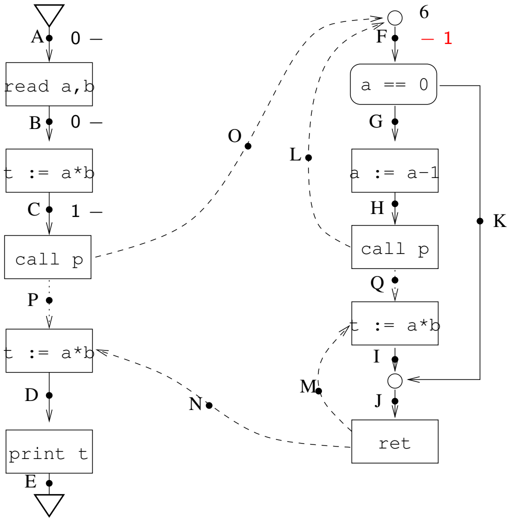
Example: Computing ϕ's iteratively: 2



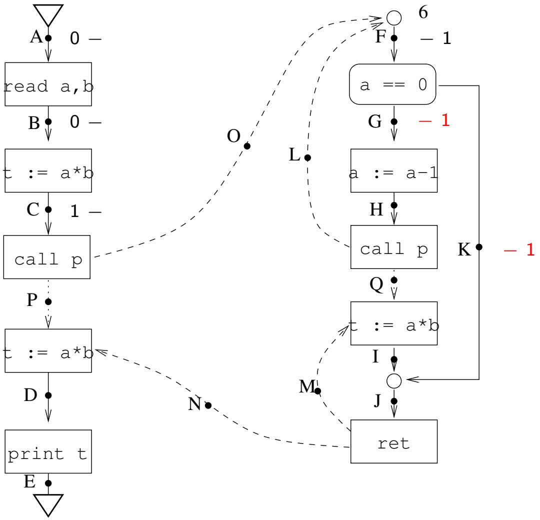
Example: Computing ϕ's iteratively: 3



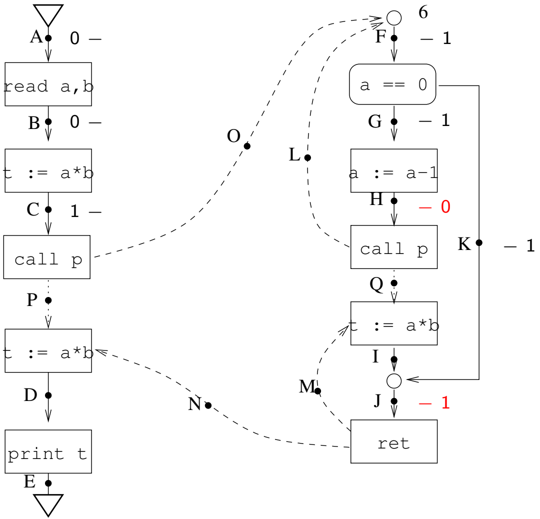
Example: Computing ϕ's iteratively: 4



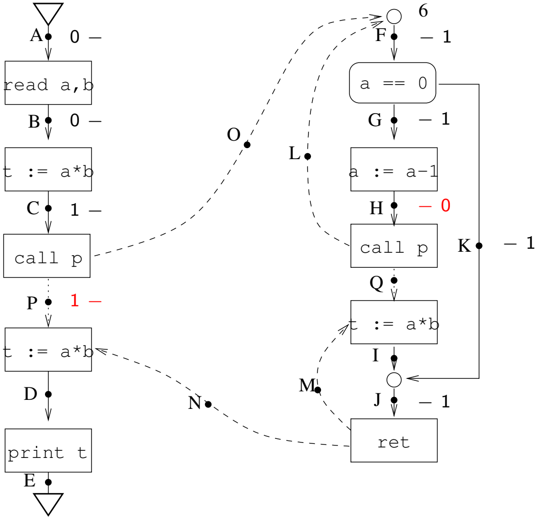
Example: Computing ϕ's iteratively: 5



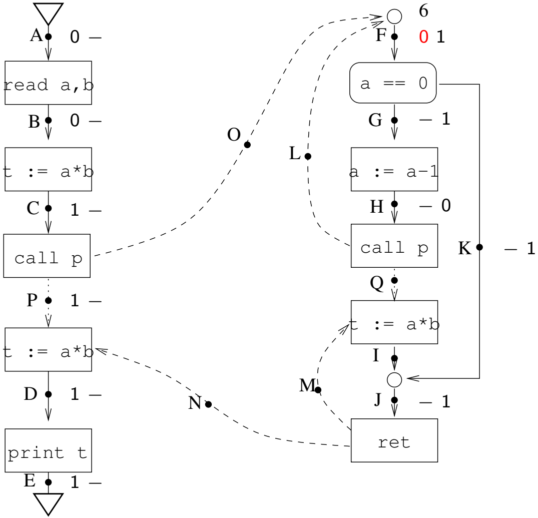
Example: Computing ϕ's iteratively: 6



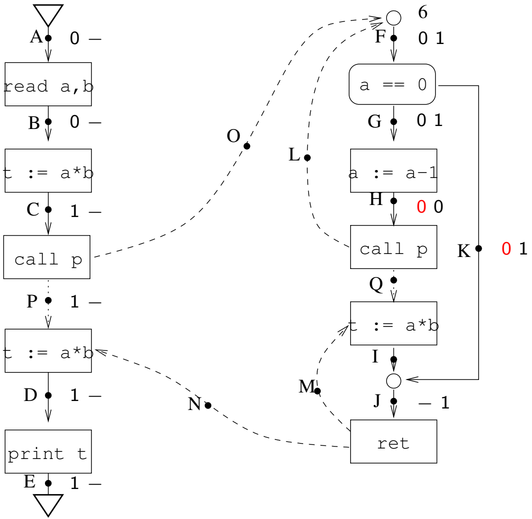
Example: Computing ϕ's iteratively: 7



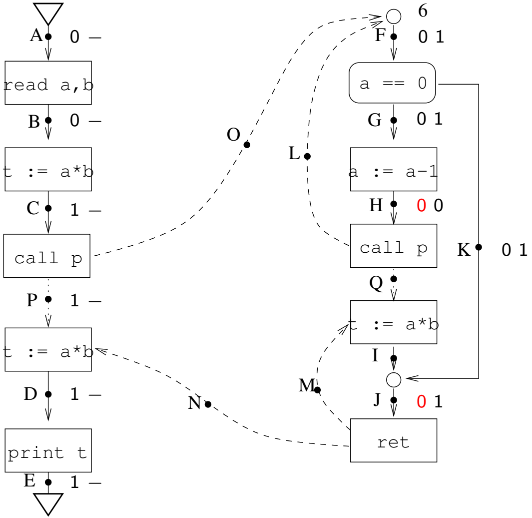
Example: Computing ϕ's iteratively: 8



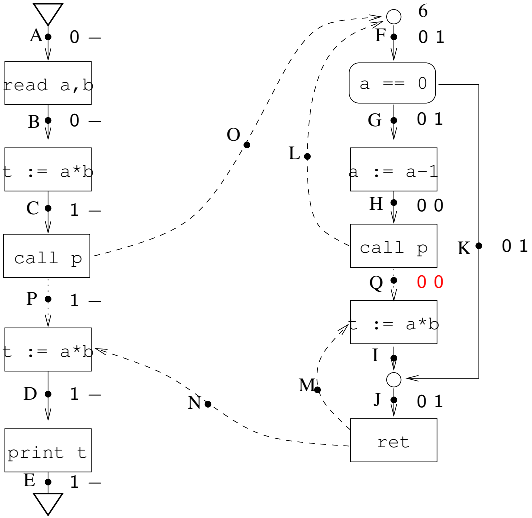
Example: Computing ϕ's iteratively: 9



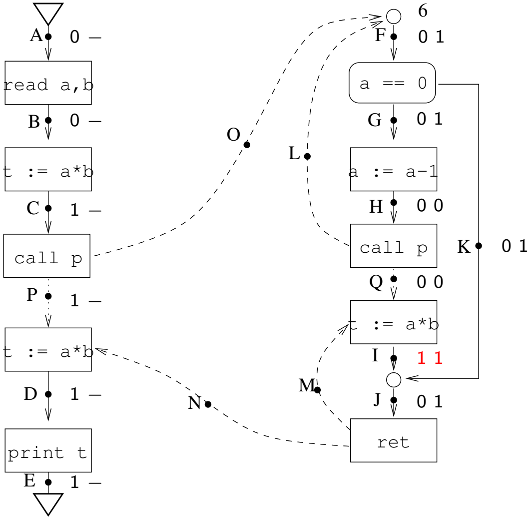
Example: Computing ϕ's iteratively: 10



Example: Computing ϕ's iteratively: 11



Example: Computing ϕ's iteratively: 12



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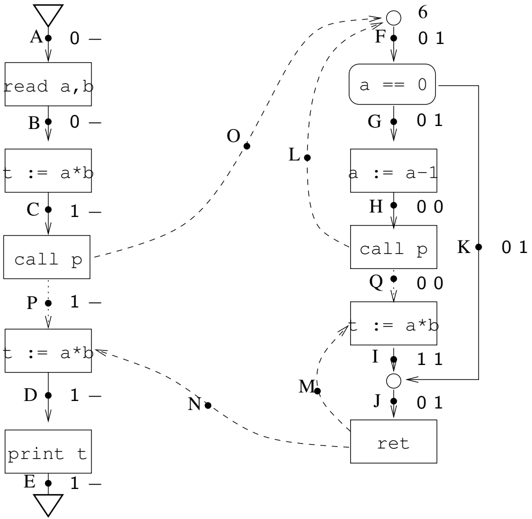
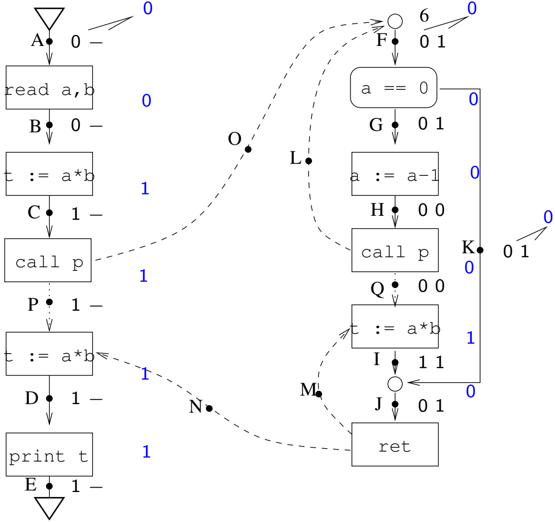


Example: Finally compute x_N's from ϕ values

At each point N take join of reachable $\phi_{r_p,N}$ values.



Correctness of iterative algo

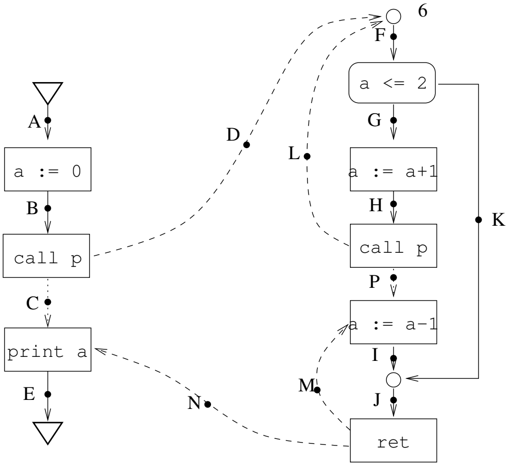
- Iterative algo terminates provided underlying lattice is finite.
- It computes the $y^*_{r_p,N}$'s (where $y^*_{r_p,N}$'s are the least solution to Eq (1)) "partially": If it maps d to $d' \neq \bot$ then $y^*_{r_p,N}(d) = d'$.
- The JVP values it gives (say z_N's) are such that

$$\mathrm{JVP}_N \leq z_N \leq x^*_N$$

  (where x^*_N's are the solution to Eq (2')).
- If underlying transfer functions are distributive it computes $\phi_{r_p,N}$'s correctly (though partially), and the JVP values correctly.
- It thus computes an overapproximation of JVP for monotonic transfer functions, and exact JVP when transfer functions are distributive.

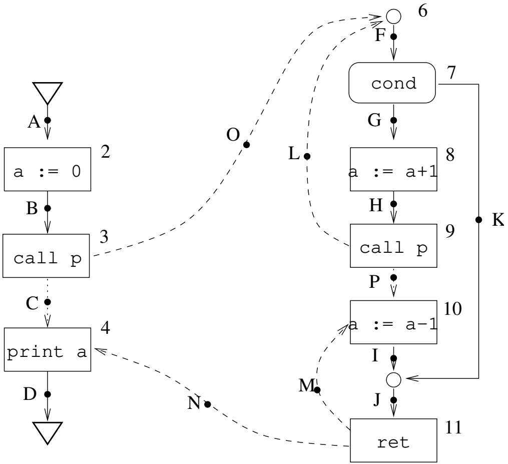
Exercise 1: Iterative algo

Run the iterative algo to do constant propagation analysis for the program below with initial value $\emptyset$.



Exercise 2: Functional vs Iterative algo

Run the functional and iterative algos to do constant propagation analysis for the program below with initial value $\emptyset$:



Comparing functional vs iterative approach

- Functional algo can terminate even when underlying lattice is infinite, provided we can represent and compose/join functions "symbolically".
- Iterative is typically more efficient than functional since it only computes $\phi_{r_p,N}$'s for values reachable at start of procedure.