Lattices and the Knaster-Tarski Theorem

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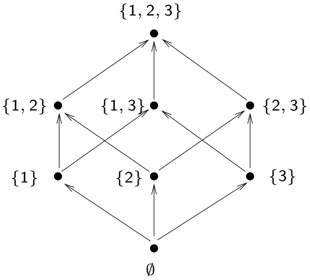
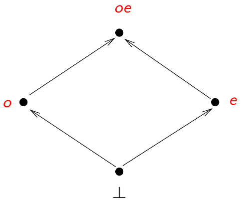
Outline

What a lattice looks like



Subsets of $\{1, 2, 3\}$, "subset"



Odd/even, "contained in"

Why study lattices in program analysis?

Why lattices?

- Natural way to obtain the "collecting state" at a point is to take union of states reached along each path leading to the point.
- With abstract states also we want a "union" or "join" over all paths (JOP).

Why fixpoints?

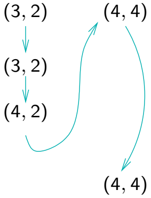
- Guaranteed to safely approximate JOP (* Conditions apply).
- Easier to compute than JOP.
- Knaster-Tarski theorem tells us about the existence of fixpoints and their structure in a lattice.

Motivation: Interpreting a program with even/odd abstract values

What values can this program print when input values are odd p and even q?

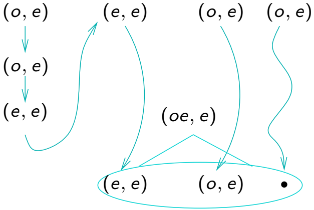
```
1.  while (p > q) {
2.    p := p+1;
3.    q := q+2;
    }
4.  print p,q;
```

Motivation: Interpreting a program with even/odd abstract values



A concrete execution
with p=3 and q=2.

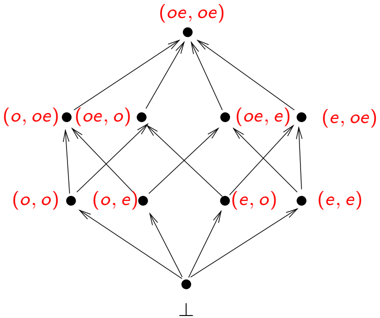
```
1. while  (p > q) {
2.     p := p+1;
3.     q := q+2;
   }
4. print p,q;
```



Join over all paths
reaching line 4.

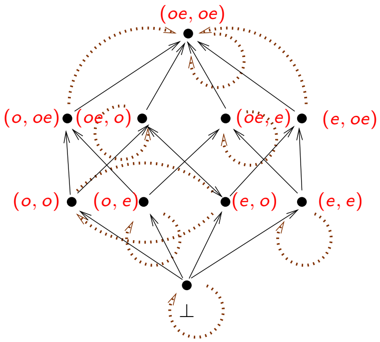
Motivation: Interpreting a program with even/odd abstract values

```
1.  while (p > q) {
2.    p := p+1;
3.    q := q+2;
    }
4.  print p;
```



Why Fixed Points?

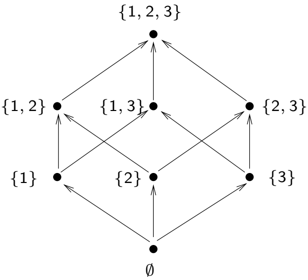
- JOP not always possible to compute
- LFP guaranteed to conservatively approximate JOP
- More efficient to compute LFP



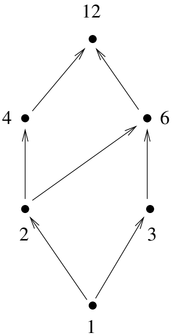
Transfer function for p:=p+q

Partial Orders

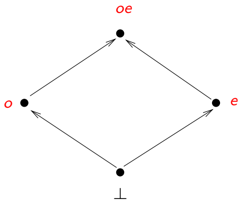
- Usual order (or total order) on numbers: $1 \leq 2 \leq 3$.
- Some domains are naturally "partially" ordered:



Subsets of $\{1, 2, 3\}$, "subset"



Divisors of 12, "divides"



Odd/even, "contained in"

Partial orders: definition

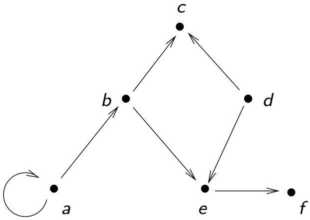
- A partially ordered set is a non-empty set D along with a partial order $\leq$ on D. Thus $\leq$ is a binary relation on D satisfying:
  - $\leq$ is reflexive ($d \leq d$ for each $d \in D$)
  - $\leq$ is transitive ($d \leq d'$ and $d' \leq d''$ implies $d \leq d''$)
  - $\leq$ is anti-symmetric ($d \leq d'$ and $d' \leq d$ implies $d = d'$).

Binary relations as Graphs

We can view a binary relation on a set as a directed graph.
For example, the binary relation

$$\{(a,a),(a,b),(b,c),(b,e),(d,e),(d,c),(e,f)\}$$

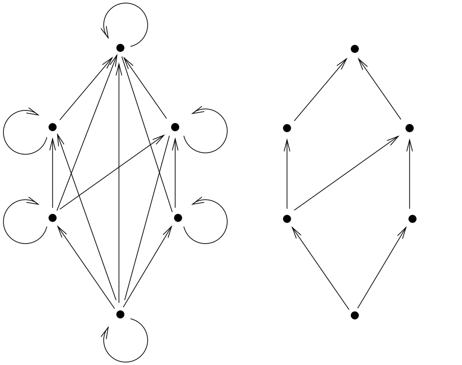
can be represented as the graph:



Partial Order as a graph

A partial order is then a special kind of directed graph:

- Reflexive = self-loop on each node
- Antisymmetric = no 2-length cycles
- Transitive = "transitivity" of edges.



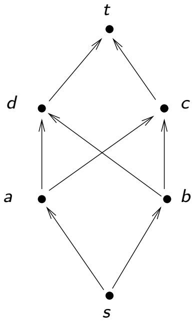
Graph representation

Hasse-diagram representation

Upper bounds etc.

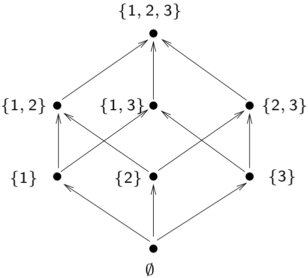
In a partially ordered set $(D, \leq)$:

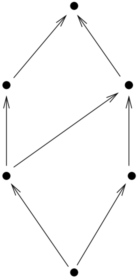
- An element $u \in D$ is an upper bound of a set of elements $X \subseteq D$, if $x \leq u$ for all $x \in X$.
- u is the least upper bound (or lub or join) of X if u is an upper bound for X, and for every upper bound y of X, we have $u \leq y$. We write $u = \bigsqcup X$.
- Similarly, $v = \bigsqcap X$ (v is the greatest lower bound or glb or meet of X).

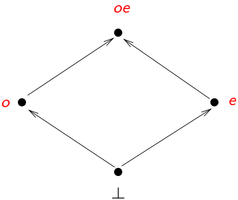


Lattices

- A lattice is a partially order set in which every pair of elements has an lub and a glb.
- A complete lattice is a lattice in which every subset of elements has a lub and glb.

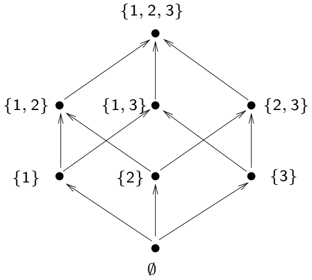


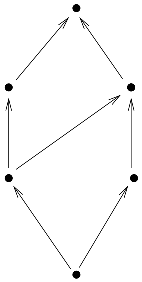


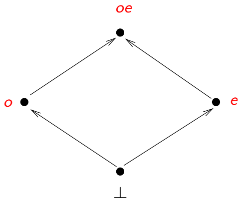


Lattices

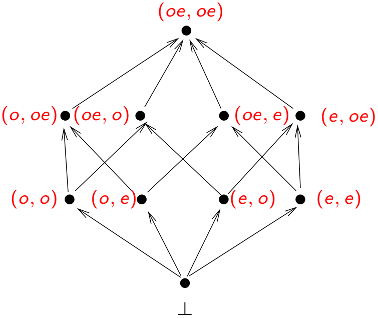
- A lattice is a partially order set in which every pair of elements has an lub and a glb.
- A complete lattice is a lattice in which every subset of elements has a lub and glb.
- Examples below are all complete lattices.







More lattices



Exercise

1. Example of a partial order that is not a lattice?

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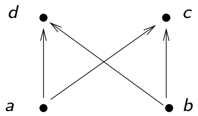
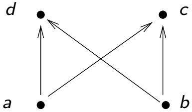


Exercise

1. Example of a partial order that is not a lattice?



2. "Simplest" example of a partial order that is not a lattice?

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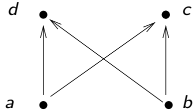
1. Example of a partial order that is not a lattice?



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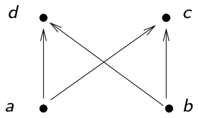

2. "Simplest" example of a partial order that is not a lattice?

3. Example of a lattice which is not complete?

Exercise

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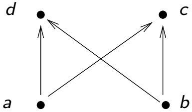
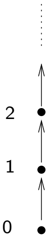


2. "Simplest" example of a partial order that is not a lattice?

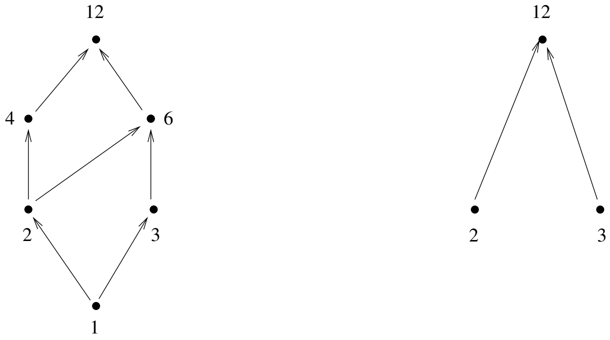
3. Example of a lattice which is not complete? $(\mathbb{N}, \leq)$



Partial order induced by a subset of elements

Let $(D, \leq)$ be a partially ordered set, and X be a non-empty subset of D. Then X induces a partial order, which we call the partial order *induced by* X in $(D, \leq)$, and defined to be $(X, \leq \cap (X \times X))$.

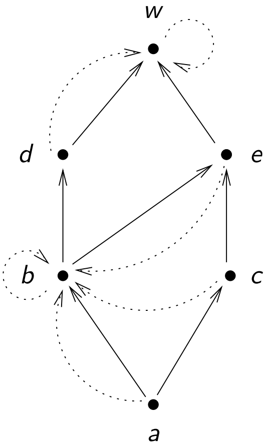
Example: the partial order induced by the set of elements $X = \{2, 3, 12\}$.



Monotonic functions

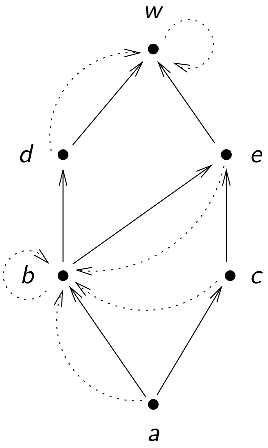
Let $(D, \leq)$ be a partially ordered set.

- A function $f : D \to D$ is monotonic or order-preserving if whenever $x \leq y$ we have $f(x) \leq f(y)$.



Fixpoints

- A fixpoint of a function $f : D \to D$ is an element $x \in D$ such that $f(x) = x$.
- A pre-fixpoint of f is an element x such that $x \leq f(x)$.
- A post-fixpoint of f is an element x such that $f(x) \leq x$.



Knaster-Tarski Fixpoint Theorem

Theorem (Knaster-Tarski)

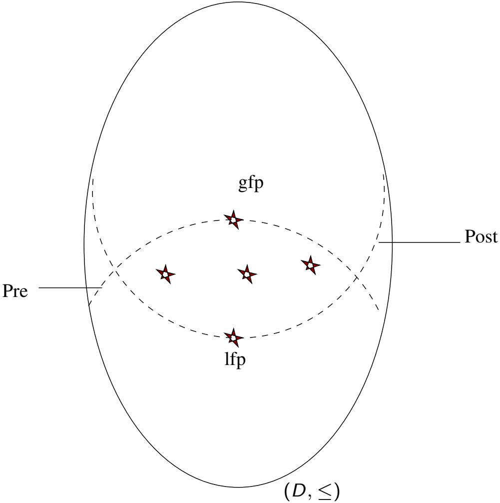
Let $(D, \leq)$ be a complete lattice, and $f : D \to D$ a monotonic function on $(D, \leq)$. Then:

(a) *f has at least one fixpoint.*

(b) *f has a least fixpoint which coincides with the glb of the set of postfixpoints of f, and a greatest fixpoint which coincides with the lub of the prefixpoints of f.*

(c) *The set of fixpoints P of f itself forms a complete lattice under $\leq$.*

Fixpoints of f



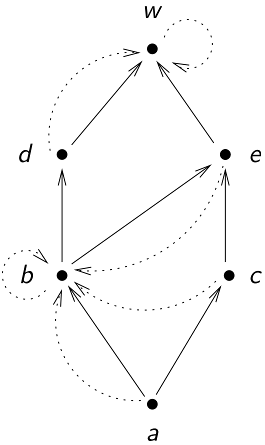
Stars denote fixpoints.

Exercise

Consider the complete lattice and monotone function f below.

1. Mark the pre-fixpoints with up-triangles ($\triangle$).
2. What is the lub of the pre-fixpoints?
3. Mark post-fixpoints with down-triangles ($\triangledown$).
4. Fixpoints are the stars (✡).

Check that claims of K-T theorem hold here.



Exercise

If you drop one of the conditions of the K-T theorem

- Monotonicity of the function f
- Completeness of the lattice

does the conclusion of the theorem still hold?

Proof of Knaster-Tarski theorem

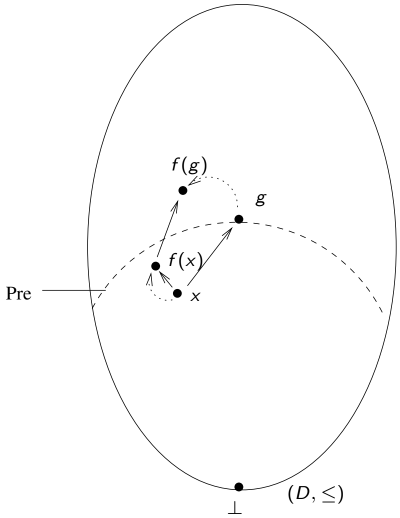
(a) $g = \bigsqcup Pre$ is a fixpoint of f.

(b) g is the greatest fixpoint of f.

(c) Similarly $l = \bigsqcap Post$ is the least fixpoint of f.

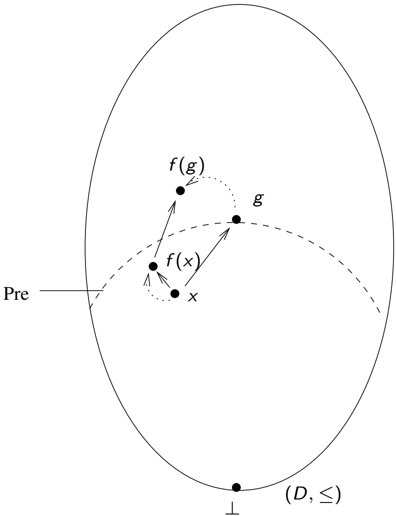
(d) Let P be the set of fixpoints of f. Then $(P, \leq)$ is a *complete* lattice.

Proof of K-T theorem: (a)



Proof of K-T theorem: (a)

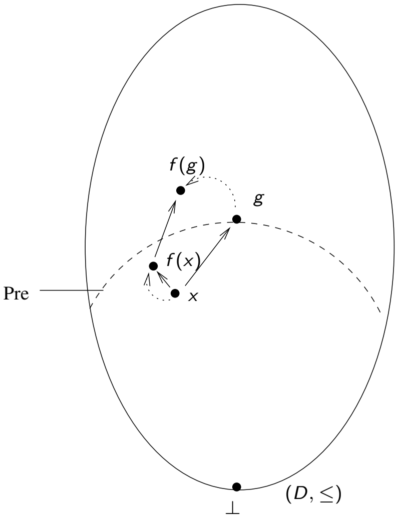
To show $g = f(g)$:



Proof of K-T theorem: (a)

To show $g = f(g)$:

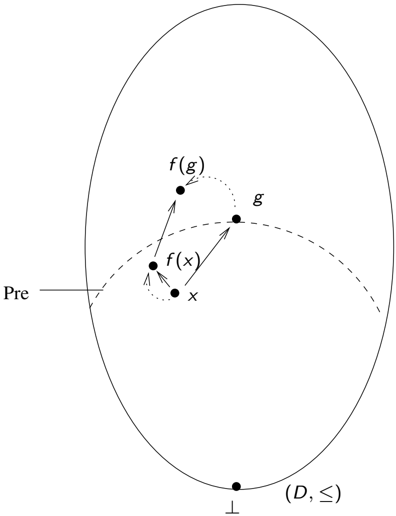
- $g \leq f(g)$



Proof of K-T theorem: (a)

To show $g = f(g)$:

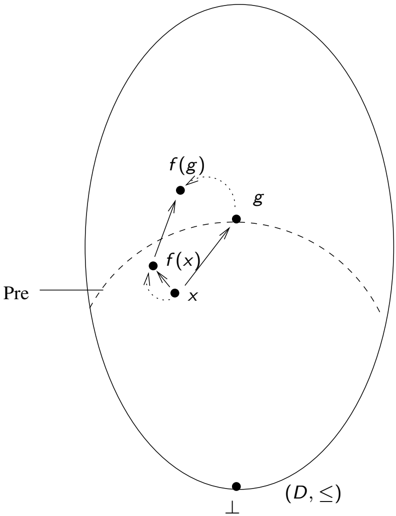
- $g \leq f(g)$
  - Since $f(g)$ can be seen to be u.b. of *Pre*.



Proof of K-T theorem: (a)

To show $g = f(g)$:

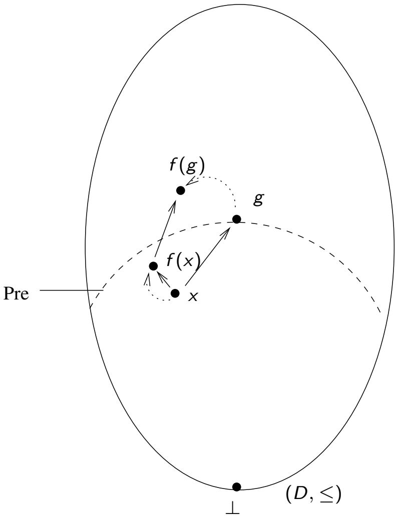
- $g \leq f(g)$
  - Since $f(g)$ can be seen to be u.b. of *Pre*.
- $f(g) \leq g$



Proof of K-T theorem: (a)

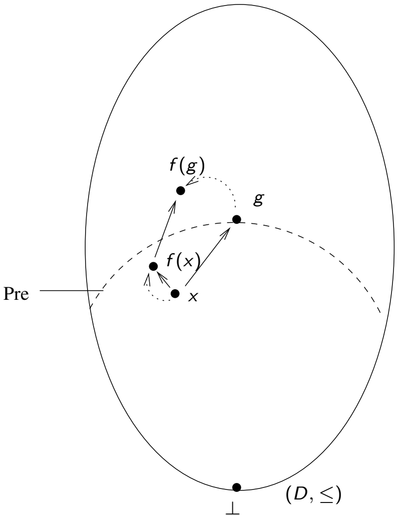
To show $g = f(g)$:

- $g \leq f(g)$
  - Since $f(g)$ can be seen to be u.b. of *Pre*.
- $f(g) \leq g$
  - Since $f(g)$ can be seen to be prefixpoint of f.



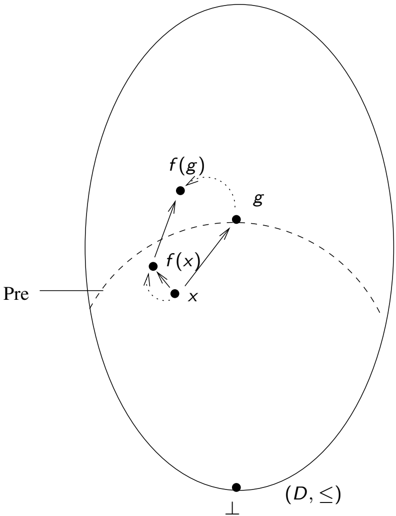
Proof of K-T theorem: (b)

g is the greatest fixpoint of f:



Proof of K-T theorem: (b)

g is the greatest fixpoint of f: Any other fixpoint is also a pre-fixpoint of f, and hence g must dominate it.



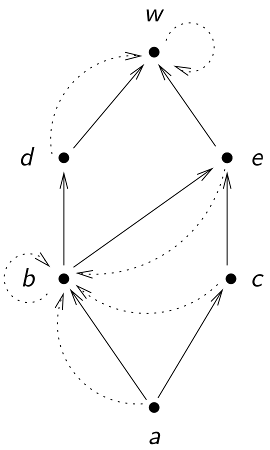
Exercise: intervals and closure

Let $(D, \leq)$ be a partial order, and let $f : D \to D$.

- Let $a, b \in D$. The interval from a to b, written $[a, b]$, is the set $\{d \mid a \leq d \leq b\}$.
- A subset $X \subseteq D$ is said to be closed wrt to f, if $f(x) \in X$ for each $x \in X$.

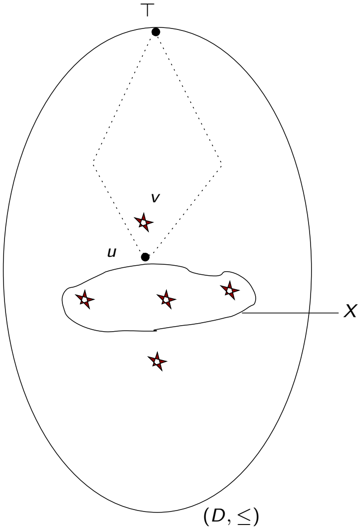
Exercise: Let $(D, \leq)$ be a partial order with a $\top$ element, and let $f : D \to D$ be a monotone function on D.

1. Show that an interval in D need *not* be closed wrt f.
2. Let $u \in D$ be the lub of a set X of fixpoints of f. Prove that the interval $[u, \top]$ is closed wrt f.



Proof of K-T theorem: (d)

- $(P, \leq)$ is also a partial order.
- $(P, \leq)$ is a complete lattice
  - Let $X \subseteq P$. We show there is an lub of X in $(P, \leq)$.
    - Let u be lub of X in $(D, \leq)$.
    - Consider "interval" $I = [u, \top] = \{x \in D \mid u \leq x\}$. $(I, \leq)$ is also a complete lattice.
    - $f : I \to I$ as well, and monotonic on $(I, \leq)$.
    - Hence by part (a) f has a least fixpoint in I, say v.
    - Argue that v is the lub of X in $(P, \leq)$.



Chains in partial orders

- A chain in a partial order $(D, \leq)$ is a totally ordered subset of D.
- An ascending chain is an infinite sequence of elements of D of the form:

$$d_0 \leq d_1 \leq d_2 \leq \cdots .$$

- An ascending chain $\langle d_i \rangle$ is eventually stable if there exists n_0 such that $d_i = d_{n_0}$ for each $i \geq n_0$.
- $(D, \leq)$ has finite height if each chain in it is finite.
- $(D, \leq)$ has bounded height if there exists k such that each chain in D has height at most k (i.e. number of elements in each chain is at most $k + 1$.)

Monotonicity, distributivity, and continuity

- f is monotone:

  $$x \leq y \implies f(x) \leq f(y).$$

- f is distributive:

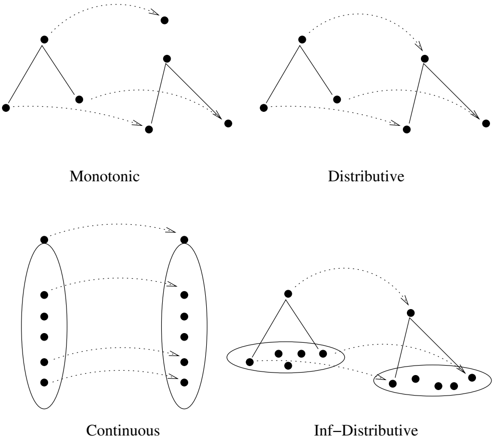
  $$f(x \sqcup y) = f(x) \sqcup f(y).$$

- f is continuous: For any asc chain X:

  $$f(\bigsqcup X) = \bigsqcup (f(X)).$$

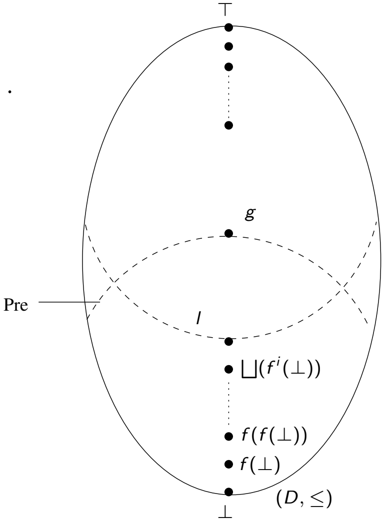
- f is inf distributive: For any $X \subseteq D$:

  $$f(\bigsqcup X) = \bigsqcup (f(X)).$$



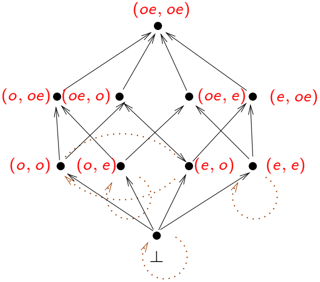
Characterising LFP of a function f in a complete lattice $(D, \leq)$

- If f is monotonic
  - Then
    $$\bot \leq f(\bot) \leq f^2(\bot) \leq f^3(\bot) \leq \cdots$$
    is an ascending chain.
  - If this chain stabilizes the stable value will be $lfp(f)$.
  - If $(D, \leq)$ has finite height then we can compute $lfp(f)$ by finding the stable value of this chain.
- If f is continuous then
  $$lfp(f) = \bigsqcup_{i \geq 0} (f^i(\bot)).$$



Exercise

Consider the statement "`p := p + q`". Show the transfer function of this statement in the parity lattice below.



Is it monotonic/distributive/continuous?