Floyd-Hoare Style Program Verification

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23 October 2024

Outline of these lectures

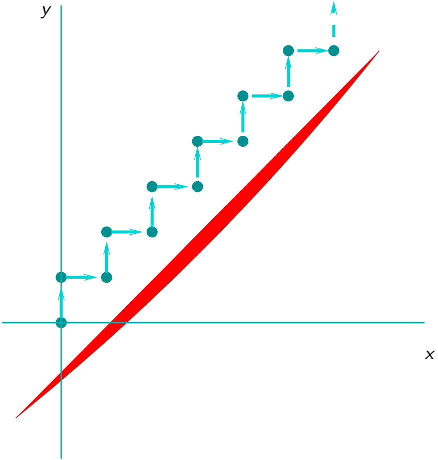
The Verification Problem

Given a system model M and a property P about the model, tell whether M satisfies P or not.

- Different kinds of system models. Here we are interested in (idealized) programs.
- Different kinds of properties: Safety, Temporal, Functionality based, Performance based, etc. Here we are interested in safety properties ("an unsafe/bad state is not reachable"). In particular, "pre-post" properties.

Example Program and Property

```
x := 0;
y := 0;
while (*) {
  if (x < y)
    x++;
  else
    y++;
}
// assert y != x - 1
```



How would one check that this program satisfies the given assertion?

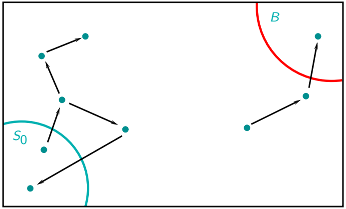
Transition System Model

A transition system $\mathcal{T}$ is specified by $(S, S_0, \rightarrow)$, where:

- S is a set of states
- $S_0 \subseteq S$ is a set of initial states
- $\rightarrow \subseteq S \times S$ is the transition relation.

Idea of Deductive Verification

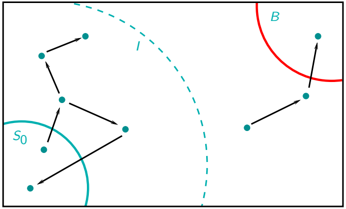
Problem: Given a transition system $\mathcal{T} = (S, S_0, \to)$ and an set of unsafe states $B \subseteq S$, does an execution of $\mathcal{T}$ reach a state in B?



Find a set of states I such that

1. $S_0 \subseteq I$ (initial states belong to I)
2. $s \in I$ and $s \to s'$, implies $s' \in I$ (I is inductive wrt trans)
3. $I \cap B = \emptyset$ (I disjoint from Bad states).

Such an I is called an adequate inductive invariant.



Idea of deductive verification

```
x := 0;
y := 0;
while (*) {
  if (x < y)
    x++;
  else
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}
// assert y != x - 1
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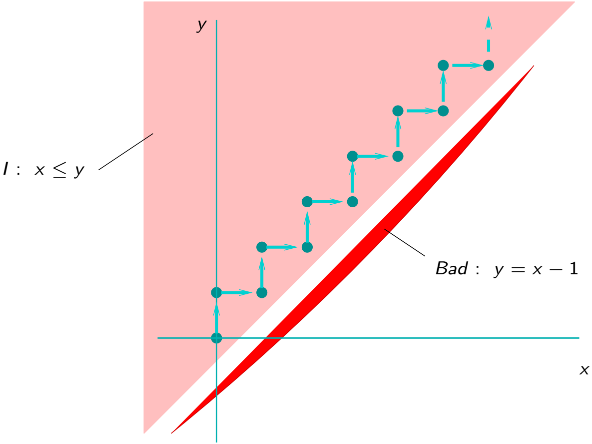

I is an adequate inductive invariant:

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3. $I \cap B = \emptyset$ (I disjoint from Bad states).

Floyd-Hoare Style of Program Verification

Robert W. Floyd: "Assigning meanings to programs" *Proceedings of the American Mathematical Society Symposia on Applied Mathematics* (1967)

C A R Hoare: "An axiomatic basis for computer programming", *Communications of the ACM* (1969).

Floyd-Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, if it terminates, it will terminate in a state satisfying condition B."
- Example assertion: $\{n \geq 0\}\ P\ \{a = n + m\}$, where P is the program:

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

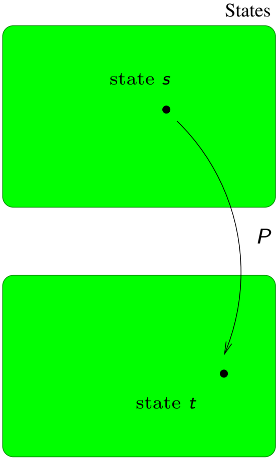
- Inductive Annotation ("consistent interpretation") (due to Floyd)
- A proof system (due to Hoare) for proving such assertions.
- A way of reasoning about such assertions using the notion of "Weakest Preconditions" (due to Dijkstra).

A Simple Programming Language

- `skip` (do nothing)
- `x := ` e (assignment)
- `if` b `then` S `else` T (if-then-else)
- `while` b `do` S (while loop)
- S ; T (sequencing)

Programs as State Transformers

- Program state is a valuation to variables of the program: $States = Var \to \mathbb{Z}$.
- View program P as a partial map $[\![P]\!] : States \to States$.

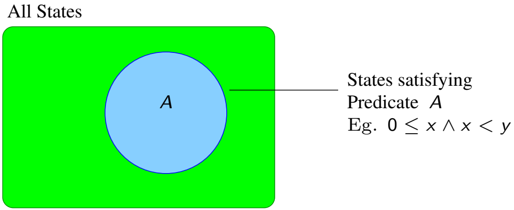


$s : \langle x \mapsto 2,\ y \mapsto 10,\ z \mapsto 3 \rangle$

```
y := y + 1;
z := x + y
```

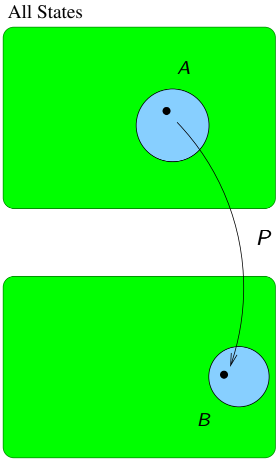
$t : \langle x \mapsto 2,\ y \mapsto 11,\ z \mapsto 13 \rangle$

Predicates on States



Assertion of "Partial Correctness" $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that "Whenever program P is started in a state satisfying condition A, either it will not terminate, or it will terminate in a state satisfying condition B."



$\{10 \leq y\}$

```
y := y + 1;
z := x + y
```

$\{x < z\}$

Mathematical meaning of a Hoare triple

- View program P as a relation on States (allows non-termination as well as non-determinism)

$$[\![P]\!] \subseteq \mathrm{States} \times \mathrm{States}.$$

  Here $(s, t) \in [\![P]\!]$ iff it is possible to start P in the state s and terminate in state t.
- $[\![P]\!]$ is possibly non-determinisitic, in case we also want to model non-deterministic assignment etc.
- Then the Hoare triple $\{A\}\ P\ \{B\}$ is true iff for all states s and t: whenever $s \models A$ and $(s, t) \in [\![P]\!]$, then $t \models B$.
- In other words $Post_{[\![P]\!]}([\![A]\!]) \subseteq [\![B]\!]$.

Example programs and pre/post conditions

```
// Pre: true

if (a <= b)
  min := a;
else
  min := b;

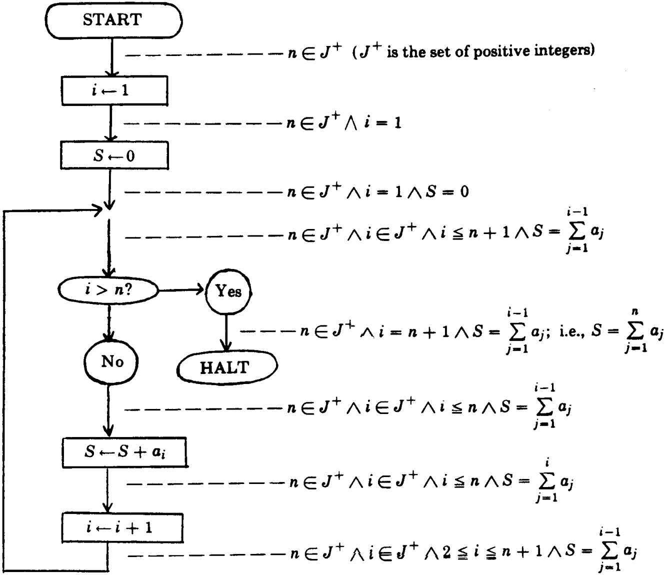
// Post: min <= a && min <= b
```

```
// Pre: 0 <= n

int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}

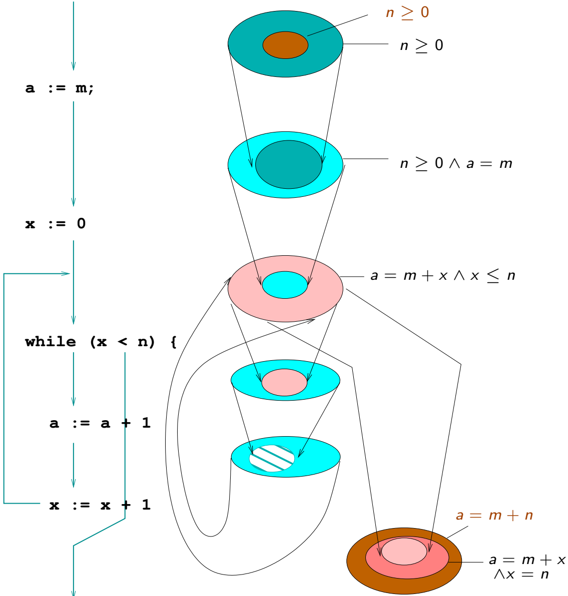
// Post: a = m + n
```

Floyd style proof: Inductive Annotation



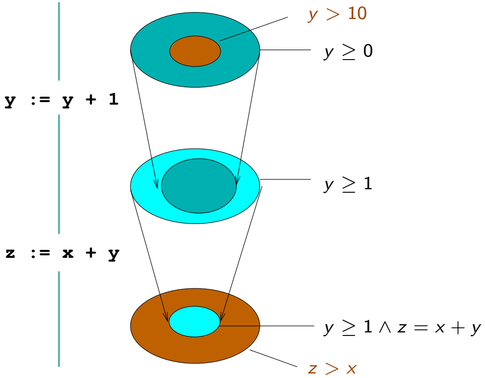
Inductive annotation based proof of `add` program

- Annotate each program point i with a predicate A_i
- Successive annotations must be inductive: $[\![S_i]\!]([\![A_i]\!]) \subseteq [\![A_{i+1}]\!]$, OR logically: $A_i \wedge [S_i] \implies A'_{i+1}$.
- Annotation must be adequate: $Pre \implies A_1$ and $A_n \implies Post$.
- Adequate inductive annotation constitutes a proof of $\{Pre\}$ Prog $\{Post\}$.



Example inductive annotation based proof

To prove: $\{y > 10\}$ `y := y+1; z := x+y` $\{z > x\}$



Example inductive annotation based proof

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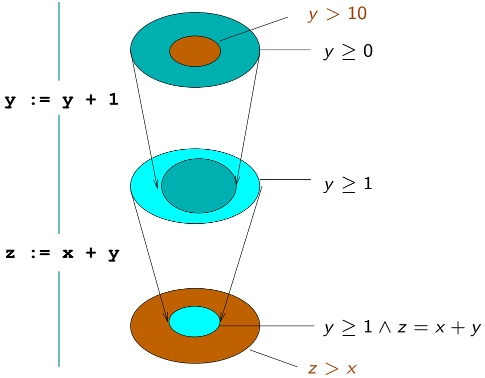


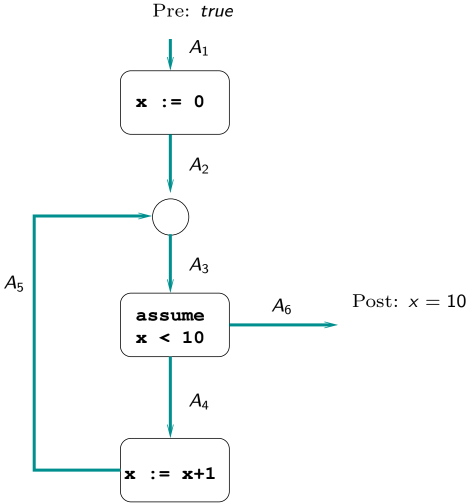
Logical proof obligations (Verification Conditions) check adequacy and inductiveness:
If VCs are logically valid then annotations are adequate and inductive.

$$(y > 10 \implies y \geq 0) \wedge ((y \geq 1 \wedge z = x + y) \implies z > x) \wedge$$
$$((y \geq 0 \wedge y' = y + 1 \wedge x' = x \wedge z' = z) \implies y' \geq 1) \wedge$$
$$((y \geq 1 \wedge z' = x + y \wedge x' = x \wedge y' = y) \implies y' \geq 1 \wedge z' = x' + y')$$

Exercise 1

Prove using Floyd-style annotation:

```
// Pre: true
int x := 0;
while (x < 10)
  x := x + 1;
// Post: x = 10
```



Also write out the proof obligations (verification conditions).

Exercise 2

Prove using Floyd's inductive annotation:

$$\{n \geq 1\}\ P\ \{a = n!\},$$

where P is the program:

```
x := n;
a := 1;
while (x ≥ 1) {
    a := a * x;
    x := x - 1
}
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n-1) \times \cdots \times 1 & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = 0 \\ -1 & \text{if} \quad n < 0 \end{cases}$$

Exercise 2

Prove using Floyd's inductive annotation:

$$\{n \geq 1\}\ P\ \{a = n!\},$$

where P is the program:

```
S1:  x := n;
S2:  a := 1;
S3:  while (x ≥ 1) {
S4:      a := a * x;
S5:      x := x - 1
     }
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n-1) \times \cdots \times 1 & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = 0 \\ -1 & \text{if} \quad n < 0 \end{cases}$$

Hoare's view: Program as a composition of statements

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

Hoare's view: Program as a composition of statements

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

```
S1: int a := m;
S2: int x := 0;
S3: while (x < n) {
      a := a + 1;
      x := x + 1;
    }
```

Program is `S1;S2;S3`

Proof rules of Hoare Logic

To be read as "If assertion above the line is true, the so is the assertion below the line".

Axiom of Valid formulas

$$\frac{}{A}$$

provided "$\models A$" (i.e. A is a valid logical formula, eg. $x > 10 \implies x > 0$).

Skip

$$\frac{}{\{A\}\ \texttt{skip}\ \{A\}}$$

Assignment

$$\frac{}{\{A[e/x]\}\ \texttt{x := e}\ \{A\}}$$

Proof rules of Hoare Logic

If-then-else

$$\frac{\{P \wedge b\}\ S\ \{Q\},\ \{P \wedge \neg b\}\ T\ \{Q\}}{\{P\}\ \texttt{if}\ b\ \texttt{then}\ S\ \texttt{else}\ T\ \{Q\}}$$

While (here P is called a *loop invariant*)

$$\frac{\{P \wedge b\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ b\ \texttt{do}\ S\ \{P \wedge \neg b\}}$$

Sequencing

$$\frac{\{P\}\ S\ \{Q\},\ \{Q\}\ T\ \{R\}}{\{P\}\ S\,;T\ \{R\}}$$

Weakening

$$\frac{P \implies Q,\ \{Q\}\ S\ \{R\},\ R \implies T}{\{P\}\ S\ \{T\}}$$

Loop invariants

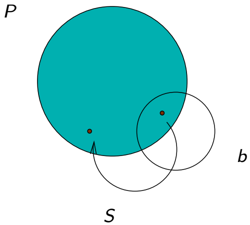
A predicate P is a loop invariant for the while loop:

```
while (b) {
  S
}
```

if $\{P \wedge b\}\ S\ \{P\}$ holds.

If P is a loop invariant then we can infer that:

$$\{P\}\ \texttt{while}\ b\ \texttt{do}\ S\ \{P \wedge \neg b\}$$



Proof of a Hoare triple in Hoare Logic

A proof of a Hoare triple $\{A\}\ P\ \{B\}$ in Hoare logic is a finite sequence of assertions

$$C_0, C_1, \ldots, C_n$$

such that:

- Each C_i is either an axiom of valid formulas or follows from earlier C_j's by one of the proof rules.
- C_n is $\{A\}\ P\ \{B\}$.

Can also be viewed as a "proof tree".

Some examples to work on

Use the rules of Hoare logic to prove the following assertions:

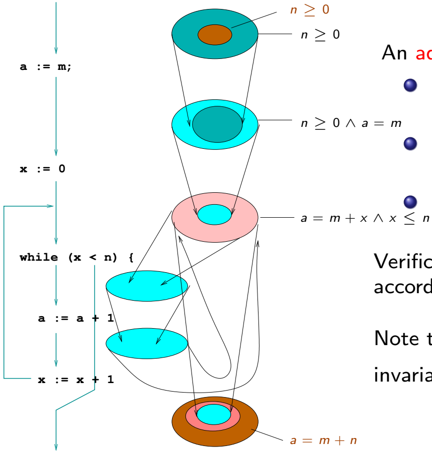
1. $\{x > 3\}$ `x := x + 2` $\{x \geq 5\}$
2. $\{(y \leq 0) \wedge (-1 < x)\}$ `if` $(y < 0)$ `then x:=x+1 else x:=y` $\{0 \leq x\}$
3. $\{x \leq 0\}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

Example proof using Hoare Logic

1. $\{x + 2 \geq 5\}\ x := x + 2\ \{x \geq 5\}$ [Assign. Rule]
2. $x > 3 \implies x + 2 \geq 5$ [Logical Axiom]
3. $x \geq 5 \implies x \geq 5$ [Logical Axiom]
4. $\{x > 3\}\ x := x + 2\ \{x \geq 5\}$ [Weak. on 1, 2, 3]

```
// pre: x > 3
   x := x + 2
// post: x >= 5
```

Adequate loop invariant



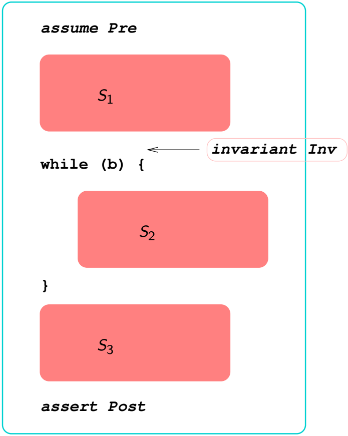
An adequate loop invariant needs to satisfy:

- $\{n \geq 0\}$ `a := m; x := 0` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x < n\}$ `a := a+1; x := x+1` $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x \geq n\}$ `skip` $\{a = m + n\}$.

Verification conditions are generated accordingly.

Note that $a = m + x$ is not an adequate loop invariant.

Generating Verification Conditions for a program



The following VCs are generated:

- $Pre \wedge [S_1] \implies Inv'$
  Or: $Pre \implies WP(S_1, Inv)$
- $Inv \wedge b \wedge [S_2] \implies Inv'$
  Or: $(Inv \wedge b) \implies WP(S_2, Inv)$
- $Inv \wedge \neg b \wedge [S_3] \implies Post'$
  Or: $Inv \wedge \neg b \implies WP(S_3, Post)$

Example proof using Hoare Logic

1. $\{n \geq 0\}$ S1 $\{n \geq 0 \wedge a = m\}$
2. $\{n \geq 0 \wedge a = m\}$ S2 $\{n \geq 0 \wedge a = m \wedge x = 0\}$
3. ...
4. $\{a = m + x \wedge 0 \leq x \leq n \wedge x < n\}$ S4;S5 $\{a = m + x \wedge 0 \leq x \leq n\}$ (From ...)
5. $\{a = m + x \wedge 0 \leq x \leq n\}$ S3 $\{a = m + x \wedge 0 \leq x \leq n \wedge x \geq n\}$ (From While rule, 4)
6. $\{n \geq 0\}$ S1;S2 $\{n \geq 0 \wedge a = m \wedge x = 0\}$ (From Seq rule, 1 and 2)
7. $(n \geq 0 \wedge a = m \wedge x = 0) \implies (a = m + x \wedge 0 \leq x \leq n)$ (From logical axiom)
8. $\{n \geq 0\}$ S1;S2 $\{a = m + x \wedge 0 \leq x \leq n\}$ (From Weakening rule, 6 and 7)
9. $\{n \geq 0\}$ (S1;S2);S3 $\{a = m + x \wedge 0 \leq x \leq n \wedge x \geq n\}$ (From Seq rule, 8, 5)
10. $(a = m + x \wedge 0 \leq x \leq n \wedge x \geq n) \implies (a = m + n)$
11. $\{n \geq 0\}$ (S1;S2);S3 $\{a = m + n\}$ (From Weakening rule, 9, 10).

```
// pre: n >= 0
S1: int a := m;
S2: int x := 0;
S3: while (x < n) {
S4:    a := a + 1;
S5:    x := x + 1;
    }
// post: a = m + n
```

Program is `S1;S2;S3`

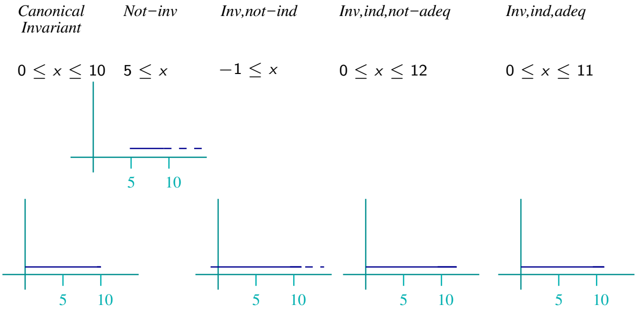
More on Adequate loop invariants

What is a "good" loop invariant for this program?

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```

Adequate loop invariant

```
x := 0;
while (x < 10) {
 if (x >= 0)
  x := x + 1;
 else
  x := x - 1;
}
assert(x <= 11);
```



Exercise

Prove using Hoare logic:

$$\{n \geq 1\}\ P\ \{a = n!\},$$

where P is the program:

```
x := n;
a := 1;
while (x ≥ 1) {
    a := a * x;
    x := x - 1
}
```

Assume that factorial is defined as follows:

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Assume that factorial is defined as follows:

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Soundness and Completeness

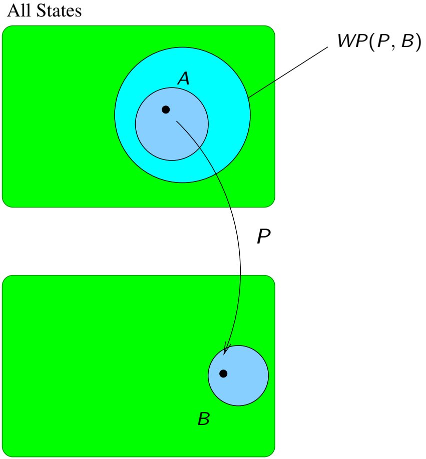
Soundness: If our proof system proves $\{A\}\ P\ \{B\}$ then $\{A\}\ P\ \{B\}$ indeed holds.

Completeness: If $\{A\}\ P\ \{B\}$ is true then our proof system can prove $\{A\}\ P\ \{B\}$.

- Floyd proof style is sound since any execution must stay within the annotations. Complete because the "collecting" set is an adequate inductive annotation for any program and any true pre/post condition. (Assumes collecting sets can be expressed logically).
- Hoare logic is sound, essentially because the individual rules can be seen to be sound.
- For completness of Hoare logic, we need weakest preconditions.

Weakest Precondition $WP(P, B)$

$WP(P, B)$ is "a predicate that describes the exact set of states s such that when program P is started in s, if it terminates it will terminate in a state satisfying condition B."



$\{10 < y\}$ (Valid Pre)

```
y := y + 1;
z := x + y;
```

$\{x < z\}$

Weakest Precondition $WP(P, B)$

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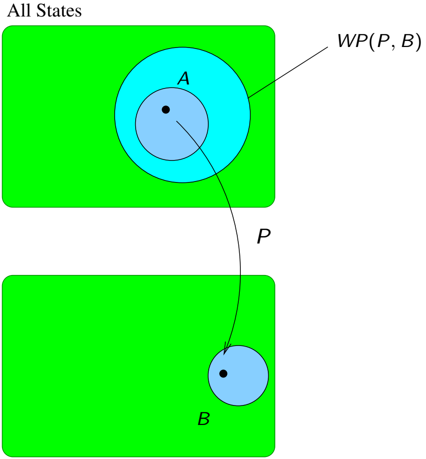


$\{-1 < y\}$ (Weakest Pre)

```
y := y + 1;
z := x + y;
```

$\{x < z\}$

Exercise: Give "weakest" preconditions

1. $\{? \quad\quad\}$ `x := x + 2` $\{x \geq 5\}$

Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$

2. $\{?\qquad\qquad\qquad\qquad\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$

Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$

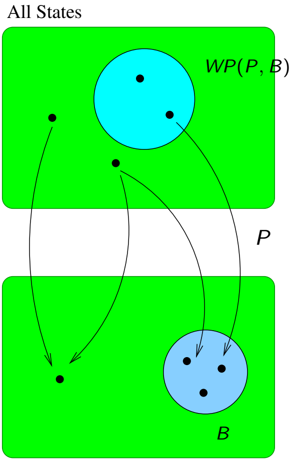
2. $\{\ (y < 0 \land x > -1) \lor (y > 0)\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$

3. $\{? \qquad\quad \}$ while $(x \leq 5)$ do x := x+1 $\{x = 6\}$

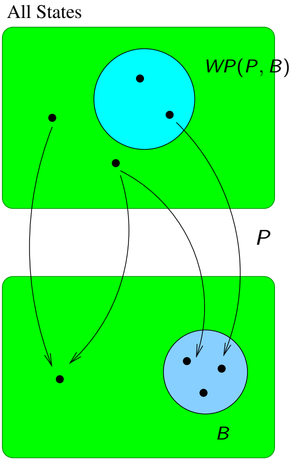
Exercise: Give "weakest" preconditions

1. $\{\ x \geq 3\}$ `x := x + 2` $\{x \geq 5\}$
2. $\{\ (y < 0 \wedge x > -1) \vee (y > 0)\}$
   `if (y < 0) then x := x+1 else x := y`
   $\{x > 0\}$
3. $\{\ x \leq 6\}$ while $(x \leq 5)$ do x $:=$ x$+1$ $\{x = 6\}$

Exercise: How will you define $WP(P, B)$?



Exercise: How will you define $WP(P, B)$?



$$WP(P,B) = \{s \mid \forall t[(s,t) \in [\![P]\!] \text{ implies } t \models B]\}$$

Using weakest preconditions to partially automate inductive proofs

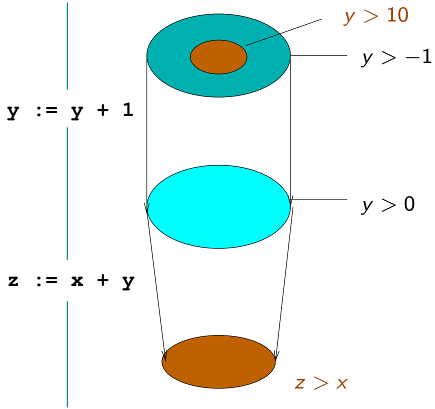
Weakest preconditions give us a way to:

- Check inductiveness of annotations

$$\{A_i\}\ S_i\ \{A_{i+1}\} \text{ iff } A_i \implies WP(S_i, A_{i+1})$$

- Reduce the amount of user-annotation needed
  - Programs without loops don't need any user-annotation
  - For programs with loops, user only needs to provide loop invariants

Checking $\{A\}\ P\ \{B\}$ using WP



Check that

$$(y > 10) \implies WP(P, z > x)$$

WP rules

- Hoare's rules for skip, assignment, and if-then-else are already WP rules.
- For Sequencing:

$$WP(S;T,\ B) = WP(S, WP(T, B)).$$

Weakest Precondition for `while` statements

- We can "approximate" $WP(\texttt{while}\ b\ \texttt{do}\ c)$.
- $WP_i(w, A) =$ the set of states from which the body c of the loop is either entered more than i times or we exit the loop in a state satisfying A.
- WP_i defined inductively as follows:

$$
\begin{array}{lcl}
WP_0 & = & b \vee A \\
WP_{i+1} & = & (\neg b \wedge A) \vee (b \wedge WP(c, WP_i))
\end{array}
$$

- Then $WP(w, A)$ can be shown to be the "limit" or least upper bound of the chain $WP_0(w, A)$, $WP_1(w, A), \ldots$ in a suitably defined lattice (here the join operation is "And" or intersection).

Illustration of WP_i through example

Consider the program w below:

```
while (x ≥ 10) do
    x := x - 1
```

- What is the weakest precondition of w with respect to the postcondition $(x \leq 0)$?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$, ...

Illustration of WP_i through example

Consider the program w below:

while $(x \geq 10)$ do
    x := x - 1

- What is the weakest precondition of w with respect to the postcondition $(x \leq 0)$?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$, ...

Automating checking of pre-post specifications for a program

To check:

$\{y > 10\}$

```
y := y + 1;
z := x + y;
```

$\{x < z\}$

Use the weakest precondition rules to generate the verification condition:

$$(y > 10) \implies (y > -1).$$

Check the verification condition by asking a theorem prover / SMT solver if the formula

$$(y > 10) \wedge \neg(y > -1).$$

is satisfiable.

Relative completeness of Hoare logic

Theorem (Cook 1974)

Hoare logic is complete provided the assertion language L can express the WP for any program P and post-condition B.

Proof uses WP predicates and proceeds by induction on the structure of the program P.

- Suppose $\{A\}$ `skip` $\{B\}$ holds. Then it must be the case that $A \implies B$ is true. By Skip rule we know that $\{B\}$ `skip` $\{B\}$. Hence by Weakening rule, we get that $\{A\}$ `skip` $\{B\}$ holds.
- Suppose $\{A\}$ `x := e` $\{B\}$ holds. Then it must be the case that $A \implies B[e/x]$. By Assignment rule we know that $\{B[e/x]\}$ `x := e` $\{B\}$ is true. Hence by Weakening rule, we get that $\{A\}$ `x := e` $\{B\}$ holds.
- Suppose $\{A\}$ `S;T` $\{B\}$ holds. Let $C = WP(\mathtt{T}, B)$. Then $\{A\}$ `S` $\{C\}$ and $\{C\}$ `T` $\{B\}$ must be valid assertions. By IH there must be Hoare logic proofs for them. We can now use the sequencing rule to conclude $\{A\}$ `S;T` $\{B\}$.

Relative completeness of Hoare logic

- Similarly for `if-then-else`.
- Suppose $\{A\}$ `while b do S` $\{B\}$ holds. Let $P = WP(\texttt{while b do S}, B)$.
  - Then it is not difficult to check that P is a loop invariant for the while statement. I.e $\{P \wedge b\}$ `S` $\{P\}$ is true. (Exercise!)
  - By induction hypothesis, this triple must be provable in Hoare logic. Hence we can conclude using the While rule, that $\{P\}$ `while b do S` $\{P \wedge \neg b\}$ is true.
  - But since P was a valid precondition, it follows that $(P \wedge \neg b) \implies B$. Since P was the WP, we should have $A \implies P$.
  - By the weakening rule, we have a proof of $\{A\}$ `while b do S` $\{B\}$.

Conclusion

- Features of this Floyd-Hoare style of verification:
  - Tries to find a proof in the form of an inductive annotation.
  - A Floyd-style proof can be used to obtain a Hoare-style proof; and vice-versa.
  - Reduces verification (given key annotations) to checking satisfiability of a logical formula (VCs).
  - Is flexible about predicates, logic used (for example can add quantifiers to reason about arrays).
- Main challenge is the need for user annotation (adequate loop invariants).
- Can be increasingly automated (using learning techniques).