

Lattices and the Knaster-Tarski Theorem

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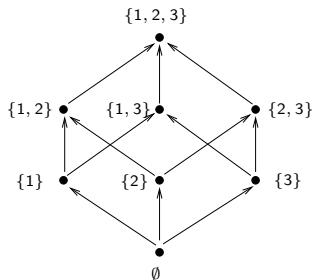
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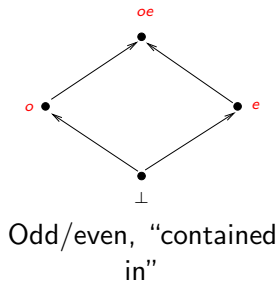
Outline

- 1 Why study lattices
- 2 Partial Orders
- 3 Lattices
- 4 Knaster-Tarski Theorem
- 5 Computing LFP

What a lattice looks like



Subsets of $\{1, 2, 3\}$,
"subset"



Odd/even, "contained
in"

Why study lattices in program analysis?

Why **lattices**?

- Natural way to obtain the “collecting state” at a point is to take union of states reached along each path leading to the point.
- With abstract states also we want a “union” or “join” over all paths (JOP).

Why **fixpoints**?

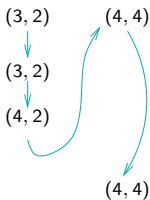
- Guaranteed to safely approximate JOP (* Conditions apply).
- Easier to compute than JOP.
- Knaster-Tarski theorem tells us about the existence of fixpoints and their structure in a lattice.

Motivation: Interpreting a program with even/odd abstract values

What values can this program print when input values are odd p and even q ?

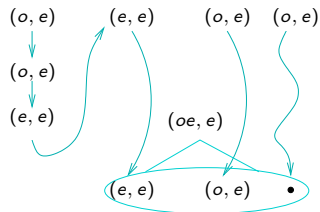
```
1. while (p > q) {  
2.   p := p+1;  
3.   q := q+2;  
   }  
4. print p,q;
```

Motivation: Interpreting a program with even/odd abstract values



A concrete execution
with $p=3$ and $q=2$.

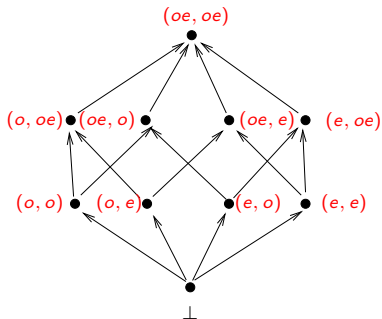
```
1. while (p > q) {
2.     p := p+1;
3.     q := q+2;
4. }
5. print p,q;
```



Join over all paths
reaching line 4.

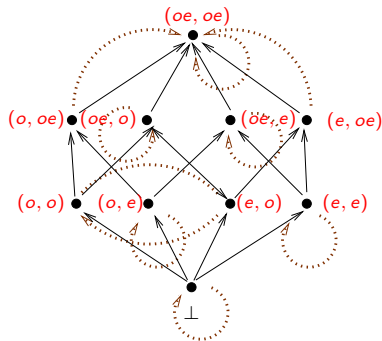
Motivation: Interpreting a program with even/odd abstract values

```
1. while (p > q) {  
2.   p := p+1;  
3.   q := q+2;  
4. }  
5. print p;
```



Why Fixed Points?

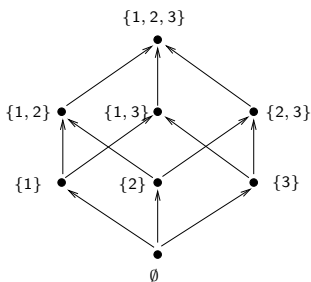
- JOP not always possible to compute
- LFP guaranteed to conservatively approximate JOP
- More efficient to compute LFP



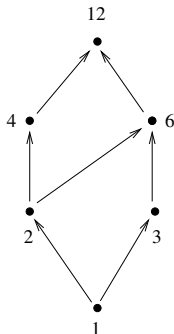
Transfer function for $p := p + q$

Partial Orders

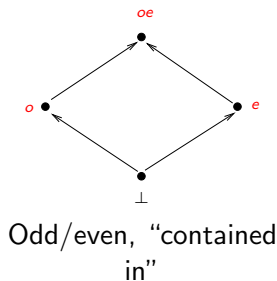
- Usual order (or **total** order) on numbers: $1 \leq 2 \leq 3$.
- Some domains are naturally “partially” ordered:



Subsets of $\{1, 2, 3\}$,
“subset”



Divisors of 12, “divides”



Odd/even, “contained
in”

Partial orders: definition

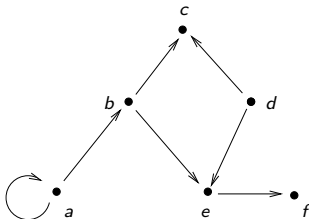
- A **partially ordered set** is a non-empty set D along with a partial order $\leq$ on D . Thus $\leq$ is a binary relation on D satisfying:
 - $\leq$ is reflexive ($d \leq d$ for each $d \in D$)
 - $\leq$ is transitive ($d \leq d'$ and $d' \leq d''$ implies $d \leq d''$)
 - $\leq$ is anti-symmetric ($d \leq d'$ and $d' \leq d$ implies $d = d'$).

Binary relations as Graphs

We can view a binary relation on a set as a **directed graph**.
For example, the binary relation

$$\{(a, a), (a, b), (b, c), (b, e), (d, e), (d, c), (e, f)\}$$

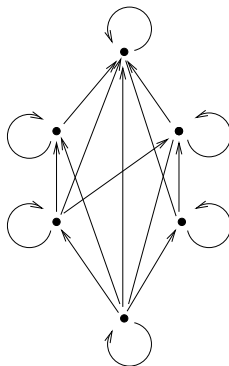
can be represented as the graph:



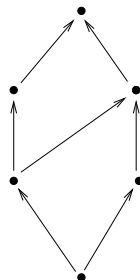
Partial Order as a graph

A **partial order** is then a special kind of directed graph:

- Reflexive = self-loop on each node
- Antisymmetric = no 2-length cycles
- Transitive = “transitivity” of edges.



Graph
representation

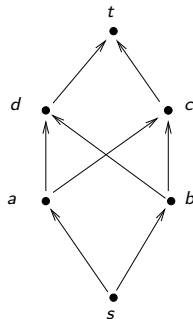


Hasse-diagram
representation

Upper bounds etc.

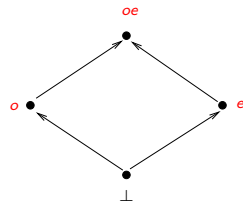
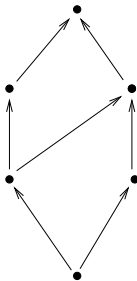
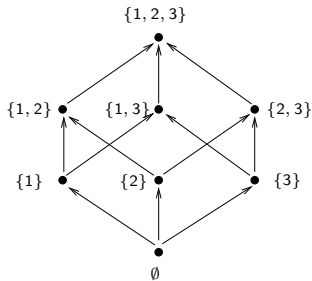
In a partially ordered set $(D, \leq)$:

- An element $u \in D$ is an **upper bound** of a set of elements $X \subseteq D$, if $x \leq u$ for all $x \in X$.
- u is the **least upper bound** (or **lub** or **join**) of X if u is an upper bound for X , and for every upper bound y of X , we have $u \leq y$. We write $u = \bigsqcup X$.
- Similarly, $v = \bigsqcap X$ (v is the **greatest lower bound** or **glb** or **meet** of X).



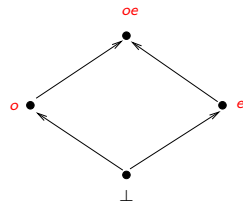
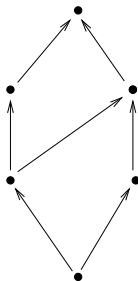
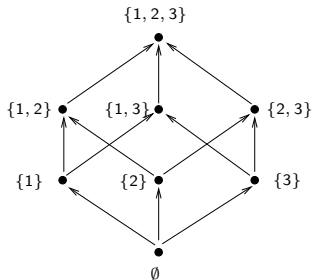
Lattices

- A **lattice** is a partially order set in which every pair of elements has an lub and a glb.
- A **complete** lattice is a lattice in which every **subset** of elements has a lub and glb.

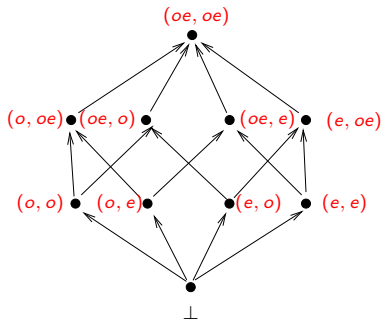


Lattices

- A **lattice** is a partially order set in which every pair of elements has an lub and a glb.
- A **complete** lattice is a lattice in which every **subset** of elements has a lub and glb.
- Examples below are all complete lattices.



More lattices

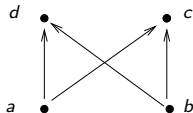


Exercise

- 1 Example of a partial order that is not a lattice?

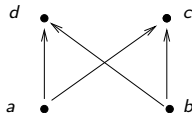
Exercise

- ① Example of a partial order that is not a lattice?



Exercise

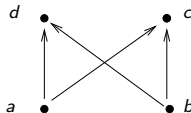
- ① Example of a partial order that is not a lattice?



- ② “Simplest” example of a partial order that is not a lattice?

Exercise

- ① Example of a partial order that is not a lattice?

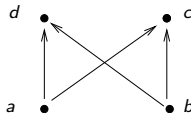


- ② “Simplest” example of a partial order that is not a lattice?



Exercise

- ① Example of a partial order that is not a lattice?



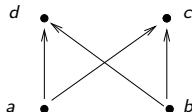
- ② “Simplest” example of a partial order that is not a lattice?



- ③ Example of a lattice which is **not** complete?

Exercise

- ① Example of a partial order that is not a lattice?



- ② “Simplest” example of a partial order that is not a lattice?



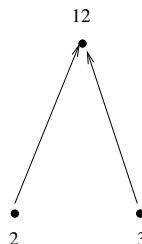
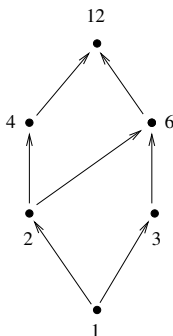
- ③ Example of a lattice which is **not** complete? $(\mathbb{N}, \leq)$



Partial order induced by a subset of elements

Let $(D, \leq)$ be a partially ordered set, and X be a non-empty subset of D . Then X induces a partial order, which we call the partial order *induced by X* in $(D, \leq)$, and defined to be $(X, \leq \cap (X \times X))$.

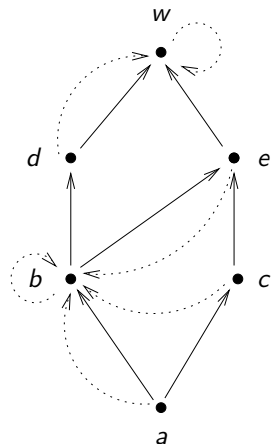
Example: the partial order induced by the set of elements $X = \{2, 3, 12\}$.



Monotonic functions

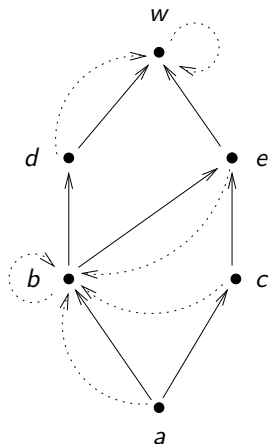
Let $(D, \leq)$ be a partially ordered set.

- A function $f : D \rightarrow D$ is **monotonic** or **order-preserving** if whenever $x \leq y$ we have $f(x) \leq f(y)$.



Fixpoints

- A **fixpoint** of a function $f : D \rightarrow D$ is an element $x \in D$ such that $f(x) = x$.
- A **pre-fixpoint** of f is an element x such that $x \leq f(x)$.
- A **post-fixpoint** of f is an element x such that $f(x) \leq x$.



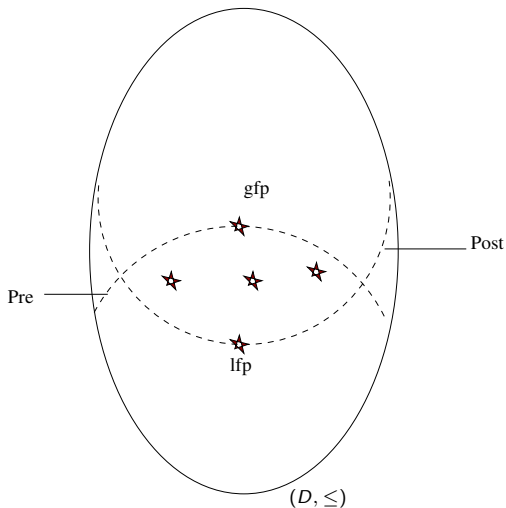
Knaster-Tarski Fixpoint Theorem

Theorem (Knaster-Tarski)

Let $(D, \leq)$ be a complete lattice, and $f : D \rightarrow D$ a monotonic function on $(D, \leq)$. Then:

- (a) f has at least one fixpoint.
- (b) f has a **least fixpoint** which coincides with the glb of the set of postfixpoints of f , and a **greatest fixpoint** which coincides with the lub of the prefixpoints of f .
- (c) The set of fixpoints P of f itself forms a complete lattice under $\leq$.

Fixpoints of f



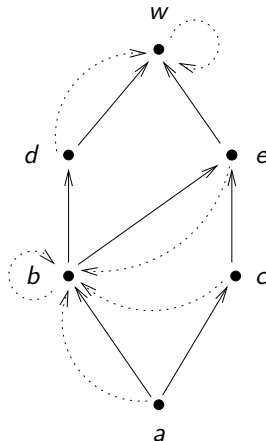
Stars denote fixpoints.

Exercise

Consider the complete lattice and monotone function f below.

- 1 Mark the pre-fixpoints with up-triangles ($\triangle$).
- 2 What is the lub of the pre-fixpoints?
- 3 Mark post-fixpoints with down-triangles (∇).
- 4 Fixpoints are the stars ($\star$).

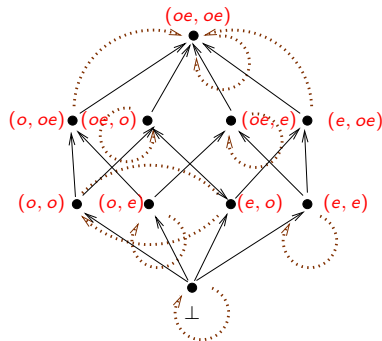
Check that claims of K-T theorem hold here.



Exercise II

Consider the complete lattice and function f depicted below.

- 1 Is the lattice **complete**? Is the function **monotone**?
- 2 Mark the pre-fixpoints with up-triangles ($\triangle$).
- 3 What is the lub of the pre-fixpoints?
- 4 Mark post-fixpoints with down-triangles (∇).
- 5 Fixpoints are the stars ($\star$).



Check that claims of K-T theorem hold here.

Exercise

If you drop one of the conditions of the K-T theorem

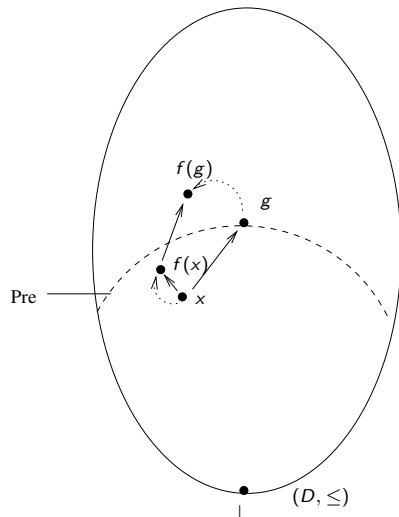
- Monotonicity of the function f
- Completeness of the lattice

does the conclusion of the theorem still hold?

Proof of Knaster-Tarski theorem

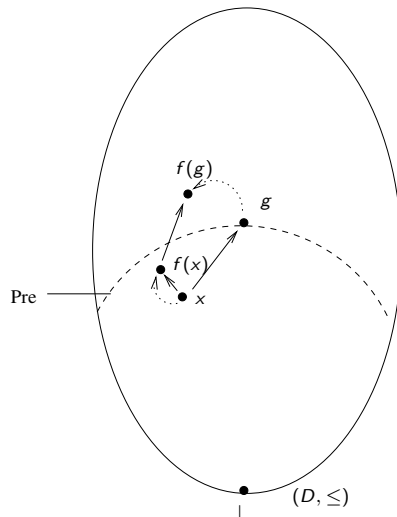
- (a) $g = \bigsqcup Pre$ is a fixpoint of f .
- (b) g is the greatest fixpoint of f .
- (c) Similarly $l = \bigsqcap Post$ is the least fixpoint of f .
- (d) Let P be the set of fixpoints of f . Then $(P, \leq)$ is a *complete* lattice.

Proof of K-T theorem: (a) $g = \sqcup Pre$ is a fixpoint of f .



Proof of K-T theorem: (a) $g = \sqcup Pre$ is a fixpoint of f .

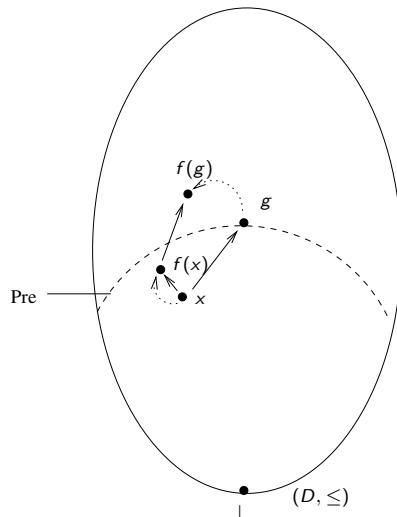
To show $g = f(g)$:



Proof of K-T theorem: (a) $g = \sqcup Pre$ is a fixpoint of f .

To show $g = f(g)$:

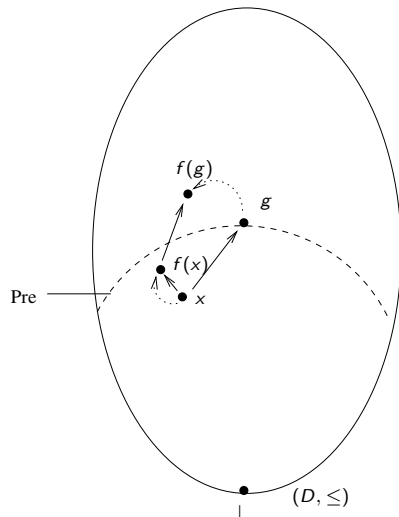
- $g \leq f(g)$



Proof of K-T theorem: (a) $g = \sqcup Pre$ is a fixpoint of f .

To show $g = f(g)$:

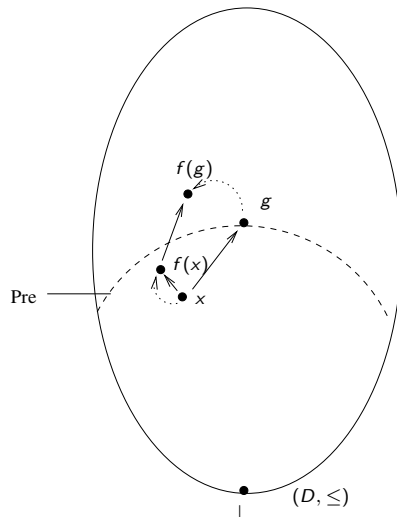
- $g \leq f(g)$
 - Since $f(g)$ can be seen to be u.b. of Pre .



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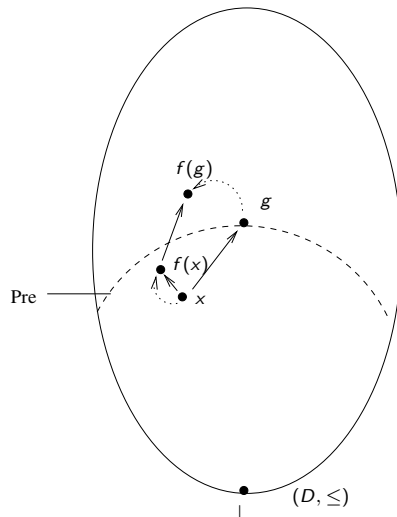
- $g \leq f(g)$
 - Since $f(g)$ can be seen to be u.b. of Pre .
- $f(g) \leq g$



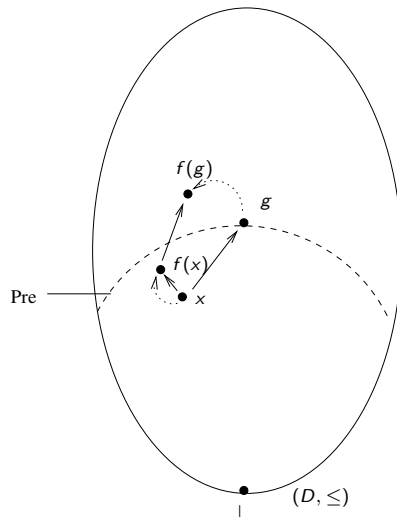
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To show $g = f(g)$:

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- $f(g) \leq g$
 - Since $f(g)$ can be seen to be prefixpoint of f .

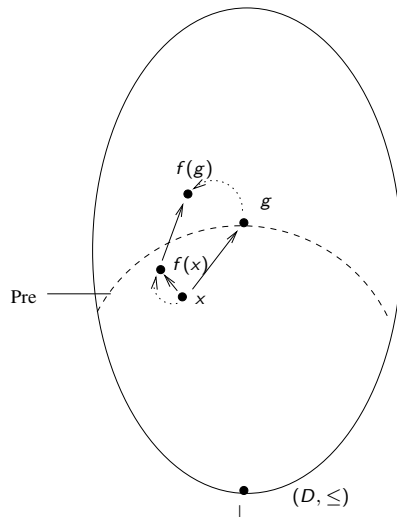


Proof of K-T theorem: (b) g is the greatest fixpoint of f



Proof of K-T theorem: (b) g is the greatest fixpoint of f

Any other fixpoint is also a pre-fixpoint of f , and hence g must dominate it.



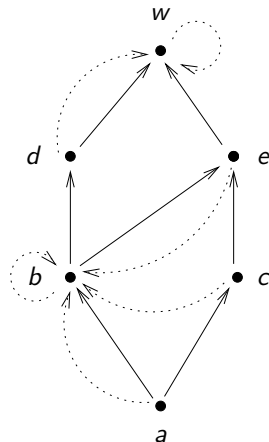
Exercise: intervals and closure

Let $(D, \leq)$ be a partial order, and let $f : D \rightarrow D$.

- Let $a, b \in D$. The **interval** from a to b , written $[a, b]$, is the set $\{d \mid a \leq d \leq b\}$.
- A subset $X \subseteq D$ is said to be **closed** wrt to f , if $f(x) \in X$ for each $x \in X$.

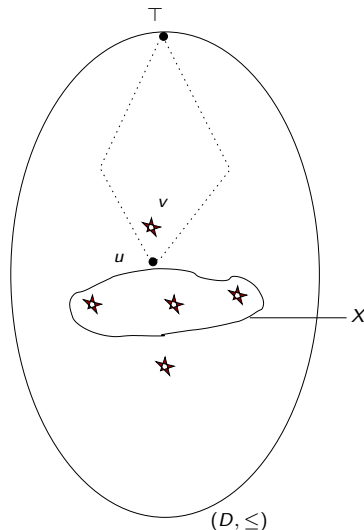
Exercise: Let $(D, \leq)$ be a partial order with a $\top$ element, and let $f : D \rightarrow D$ be a monotone function on D .

- 1 Show that an interval in D need *not* be closed wrt f .
- 2 Let $u \in D$ be the lub of a set X of fixpoints of f . Prove that the interval $[u, \top]$ is closed wrt f .



Proof of K-T theorem: (d) Fixpoints form a complete lattice

- Let P be the set of fixpoints of f .
- $(P, \leq)$ is also a partial order.
- $(P, \leq)$ is a complete lattice
 - Let $X \subseteq P$. We show there is an lub of X in $(P, \leq)$.
 - Let u be lub of X in $(D, \leq)$.
 - Consider “interval” $I = [u, \top] = \{x \in D \mid u \leq x\}$. $(I, \leq)$ is also a complete lattice.
 - $f : I \rightarrow I$ as well, and monotonic on $(I, \leq)$.
 - Hence by part (a) f has a least fixpoint in I , say v .
 - Argue that v is the lub of X in $(P, \leq)$.



Chains in partial orders

Let $(D, \leq)$ be a partial order.

- A **chain** in $(D, \leq)$ is a totally ordered subset of D .
- An **ascending chain** in $(D, \leq)$ is an infinite sequence of elements of D of the form:

$$d_0 \leq d_1 \leq d_2 \leq \cdots .$$

- An ascending chain $\langle d_i \rangle$ is **eventually stable** if there exists n_0 such that $d_i = d_{n_0}$ for each $i \geq n_0$.
- $(D, \leq)$ has **finite** height if each chain in it is finite.
- $(D, \leq)$ has **bounded** height if there exists k such that each chain in D has height at most k (i.e. number of elements in each chain is at most $k + 1$.)

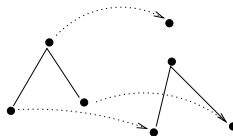
Monotonicity, distributivity, and continuity of a function f on a complete lattice $(D, \leq)$

- f is monotone:

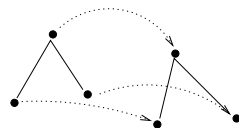
$$x \leq y \implies f(x) \leq f(y).$$

- f is distributive:

$$f(x \sqcup y) = f(x) \sqcup f(y).$$



Monotonic



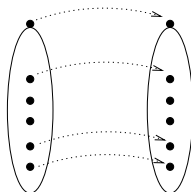
Distributive

- f is continuous: For any asc chain X :

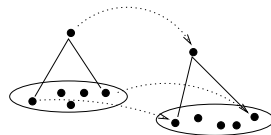
$$f(\bigsqcup X) = \bigsqcup (f(X)).$$

- f is inf distributive: For any $X \subseteq D$:

$$f(\bigsqcup X) = \bigsqcup (f(X)).$$



Continuous



Inf-Distributive

Characterising LFP of a function f in a complete lattice $(D, \leq)$

- If f is **monotonic**

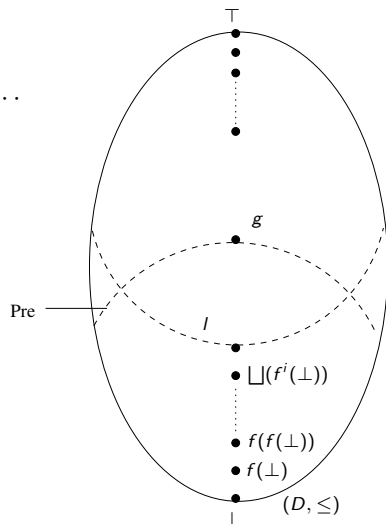
- Then

$$\perp \leq f(\perp) \leq f^2(\perp) \leq f^3(\perp) \leq \dots$$

is an ascending chain.

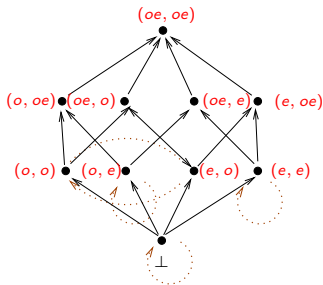
- If this chain stabilizes the stable value will be $\text{lfp}(f)$.
- If $(D, \leq)$ has **finite height** then we can compute $\text{lfp}(f)$ by finding the stable value of this chain.
- If f is **continuous** then

$$\text{lfp}(f) = \bigsqcup_{i \geq 0} (f^i(\perp)).$$



Exercise

Consider the statement " $p := p + q$ ". Show the transfer function of this statement in the parity lattice below.



Is it monotonic/distributive/continuous?