On the Erdos-Szekeres n-interior point problem

Subramanya Bharadwaj B V [1] Sathish Govindarajan [2]
Karmveer Sharma [3]

Department of Computer Science and Automation,
Indian Institute of Science, Bangalore, India

Abstract

The n-interior point variant of the Erdos-Szekeres problem is to show the following: For any n, $n \geq 1$, every point set in the plane with sufficient number of interior points contains a convex polygon containing exactly n-interior points. This has been proved only for $n \leq 3$. In this paper, we prove it for pointsets having atmost logarithmic number of convex layers. We also show that any pointset containing atleast n interior points, there exists a 2-convex polygon that contains exactly n-interior points.

Keywords: Convex polygons, interior points, Erdos-Szekeres problem, j-convexity

1 Introduction

Let $P \subset R^2$ be a finite set of points in general position. Let $Conv(P)$ denote the convex hull of P, $C(P)$ the set of points which determine the convex hull of P and $I(P) = |P \setminus C(P)|$ be the number of points lying in the interior of $Conv(P)$. Interesting questions of the form: "*If $|P|$ is large, then certain*

[1] Email: `subramanya@csa.iisc.ernet.in`
[2] Email: `gsat@csa.iisc.ernet.in`
[3] Email: `karm@csa.iisc.ernet.in`

special convex subsets $S \subseteq P$ exist" have been posed and studied in the last few decades.

The earliest classical result in this area is the Erdos-Szekeres Theorem[4] which showed that there exists an integer $f(n)$ such that if $|P| \geq f(n)$ then there exists a convex subset $S \subseteq P$ such that $|C(S)| = n$ i.e. a subset which is a convex polygon of size n. A natural question is to find lower and upper bounds on $f(n)$. This has been well studied and the current best bounds are $2^{n-2} + 1 \leq f(n) \leq \binom{2n-5}{n-2} + 1$ [7]. Many other variants of the Erdos Szekeres problem have been considered. See surveys [6,7] for details.

The following problem pertaining to interior points was posed by Avis et al in 2001 [3]: Is there a smallest integer $g(n)$ such that any point set P with $I(P) \geq g(n)$ has a convex subset $S \subseteq P$ with $I(S) = n$ i.e. a subset which is a convex polygon that contains exactly n interior points. They showed that $g(1) = 1$, $g(2) = 4$. It was recently shown that $g(3) = 9$ by Wei and Ding [9]. It is unknown whether $g(n)$ exists for $n \geq 4$. The best known lower bound is $g(n) \geq 3n$ for $n \geq 3$ [8]. Several variants of this problem have been considered in [2,5].

In this paper, we show the following:

Theorem 1.1 *For any n, $n \geq 4$, every point set P with $I(P)$ internal points and atmost $\frac{1}{6}\log_n I(P)$ convex hull layers has a convex subset $S_i \subseteq P$ with $I(S_i) = i$ for all i, $0 \leq i \leq n$.*

The notion of convexity has been generalized to j-convexity in [1]. A polygon M is said to be j-convex is every line intersects M in atmost j connected components. Note that 1-convex is the standard definition of convex polygon. If the requirement of convex subset $S \subseteq P$ is relaxed to 2-convex subset, the problem becomes easy. More precisely, we show that any pointset containing atleast n interior points, there exists a 2-convex polygon that contains exactly n-interior points for all $n \geq 1$.

2 Point sets with 2 convex layers

Definition 2.1 A convex point set P is called monotonic convex if the points are in monotonic order(increasing or decreasing).

Definition 2.2 Recursively decompose the point set P into disjoint convex hull layers $C(P), C(I(P)) \cdots$. We call this the Convex Hull Decomposition of P or the CHD of P.

Consider a point set P^* whose CHD has two convex layers both of which are monotonically convex(increasing). Let $C_2^* = C(P^*)$ and $C_1^* = I(P^*)$(external

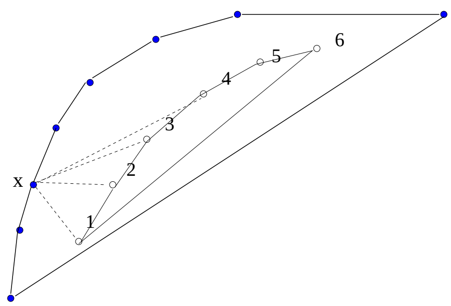

Fig. 1. $arc(x) = \{1, 2, 3, 4\}$

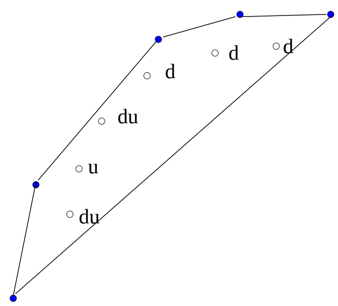

Fig. 2. Labelling points in C_1^*

and internal). See Figure 1.

Definition 2.3 For $x \in C_2^*$, let $arc(x) \subseteq C_1^*$ denote the set of points to which a straight line can be drawn from x without intersecting $Conv(C_1^*)$.

Note that $arc(x)$ is the set of points 'visible' from x. See Figure 1. Let $S_i \subseteq P^*$ denote a set of points in convex position with $I(S_i) = i$. We will show that S_i exists. Let us index all the points in C_1^* as $1, 2...$ and all the points in C_2^* also as $1, 2.....$

Lemma 2.4 *If $|arc(x)| = n$ then for every $0 \leq i \leq n-2$ there exists a $S_i \subseteq P^*$.*

Consider any point $y' \in C_1^*$. Let $P_{y'} \subseteq C_2^*$ be the set of points from which y' is visible. Since y' is visible from at least one point $x \in C_2^*$, $P_{y'} \neq \phi$.

Definition 2.5 We label y' as d if there exists a $x \in P_{y'}$ such that x is above y', as u if there exists a $x \in P_{y'}$ such that x lies below y' and as du if both hold.(See Figure 2).

Lemma 2.6 *If there is a contiguous set of points in C_1^* of size at least $n+2$ all of which are labelled d or all of which are labelled u then there is a $S_i \subseteq P^*$ for $1 \leq i \leq n$.*

Proof. Consider a contiguous set of points $M = \{i', i'+1, ...i'+r'\}$ with $r' \geq n+1$ where all of them have been labelled u. Since $i'+r'$ has been labelled u there exists $x \in P_{i'+r'}$ such that x is below $i'+r'$. x is also below i' as otherwise $i'+s'$ would be labelled du for some $0 \leq s' < r'$. Since i' and $i'+r'$ are both visible from x and x lies below i', all of $i', \cdots, i'+r'$ are visible from x which implies $|arc(x)| \geq n+2$. From Lemma 2.4 there is a $S_i \subseteq P^*$ for $1 \leq i \leq n$. The other case when all the points are labelled d can be argued similarly. □

Lemma 2.7 *If i' is labelled d and $i'+2$ as u or vice versa then $i'+1$ is labelled du.*

Theorem 2.8 *For any n, $n \geq 4$ if $|C_2^*| > 2n^2$, then there exists a $S_i \subseteq P^*$ for all i, $0 \leq i \leq n$.*

Proof. We prove by contradiction. Let, if possible, no subset S_n exists in P^*. We can visualize the labellings of points in C_2^* as a $m * (n+1)$ matrix. The point i' is mapped to the $(i' mod(n+1), i' - i' mod(n+1))$ entry. Since $|C_2^*| > 2n^2$, the number of rows m is greater than $n+2$ for $n \geq 4$. We note the following properties of the matrix from the above lemmas.

- If the (r, c) entry is d or du then $(r+i, c)$ is d for $i \geq 0$. Otherwise let the point i' correspond to the (r, c) label and point j' correspond to $(r+1, c)$ label. There is an $i \in P_{i'}$ such that i is below i' and a $j \in P_{j'}$ such that j is above j'. Now consider the polygon $i \cdots jj'i'$. This is clearly convex and has $j' - i' = n$ points. Hence S_n exists.
- The labellings $(r, c), (r, c+1), \cdots (r+1, 1), \cdots, (r+1, c)$ cannot all be d or u, since otherwise by Lemma 2.6, S_n exists.
- If the (r, c) is d and $(r, c+2)$ entry is u then the $(r, c+1)$ entry is du and vice versa (Lemma 2.7). Note that the column calculations are modulo the appropriate values

We draw the following conclusions from the above properties. Let us assume that we start from row 2 and traverse the matrix row by row.

- There is an entry du in each row.
- If the (r, c) entry is d or du then $(r+i, c)$ is d for $i \geq 0$.
- Two consecutive rows cannot be identical.

Since $m > n+2$, we will traverse at least $n+2$ rows. But we observe that after traversing the first $n+2$ rows all further rows will have entries d in all columns. But then, by Lemma 2.6, S_n exists contradicting our assumption.□

Definition 2.9 Given $a, b \in C_i$ where C_i is the i^{th} convex layer, let $C_{ab} \subseteq C_i$ denote the set of points obtained by traversing $Conv(C_i)$ clockwise and let P_{ab} denote the convex polygon formed by points in C_{ab}.

Theorem 2.10 *For any n, $n \geq 4$, every point set P whose CHD has exactly two layers C_1 and C_2 with $|C_1| \geq 16n^2$ has a subset $S_i \subseteq P$ for $0 \leq i \leq n$.*

Proof. We show that we can obtain either a point set $P^* \subseteq P$ whose CHD has two monotonic layers C_1^* and C_2^* with $|C_1^*| \geq 2n^2$ or a $x \in C_2^*$ with

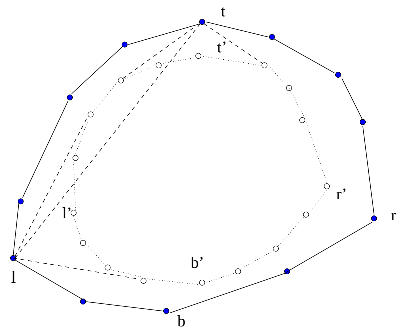

Fig. 3. Decomposing a two layer convex set

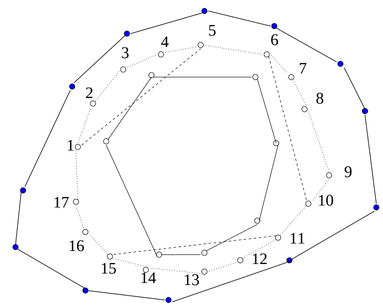

Fig. 4. Obtaining a two layer convex set

$|arc(x)| \geq n^2$. If $|C_1^*| \geq 2n^2$, then S_i exists by Theorem 2.8. Otherwise, S_i exists by Lemma 2.4. The topmost, leftmost, bottommost and rightmost points in C_1, namely l', t', r', b', are visible from the corresponding topmost, leftmost, bottommost and rightmost points in C_2 namely l, t, r, b. See Figure 3. We have $C_{l't'} \cup C_{t'r'} \cup C_{r'b'} \cup C_{b'l'} = C_1$. Therefore without loss of generality $|C_{l't'}| \geq 4n^2$ since $|C_1| \geq 16n^2$. Consider the polygon P_{lt}. We note that $C_{l't'} \subseteq I(P_{lt}) \cup arc(t) \cup arc(l)$ which implies $|C_{l't'}| \leq |I(P_{lt})| + |arc(t)| + |arc(l)|$ and if both $|arc(l')| < n^2$ and $|arc(t')| < n^2$, then $I(P_l) \geq 2n^2$ and P_{lt} is the required polygon P^*. □

3 Point sets with r convex layers

Theorem 3.1 *Let P be a point set whose CHD has r layers $C_1, \cdots, C_r$ (from internal to external) $r \geq 2$. Define $C_0 = 1$ and let $n \geq 4$. If for any i, $1 \leq i \leq r$, $|C_i| \geq 64n^2|C_{i-1}|$, then there exists a subset $S_i \subseteq P$ for $0 \leq i \leq n$.*

Proof. Let $C_i = \{1, \cdots, l\}$ and $|C_{i-1}| = t$. Set $m = 16n^2$. Since $|C_i| \geq 64n^2|C_{i-1}|$, $l > ((t+1)(m+2) + m + 3)$. Consider the set of polygons $\hat{P} = \{P_{(1)(m+2)}, P_{(m+3)(2m+5)}, \cdots, P_{((t+1)(m+2)+1)((t+1)(m+2)+m+3)}\}$. See Figure 4. Note for $P_{ab}, P_{cd} \in \hat{P}$, $I(P_{ab}) \cap I(P_{cd}) = \phi$. This means that there exists P_{ab} such that $I(P_{ab}) = \phi$ as otherwise $\sum_{P_{cd} \in \hat{P}} |I(P_{cd})| \geq t + 1$. Consider such a P_{ab}. The line ab induces a natural partition of $C_i U C_{i+1}$ into two point sets P' and P''. Wlog $P' \cap C_{i-1} = \phi$. Note that P' has two convex layers C_1', C_2' in its CHD and $|C_1'| \geq 16n^2$. By Theorem 2.10 S_i exists. □

Corollary 3.2 *For any n, $n \geq 4$, every point set P that has r layers in its CHD and $I(P)$ interior points, $I(P) \geq \frac{p^r - 1}{p-1} \geq p^r$ where $p = 64n^2$ has a subset $S_i \subseteq P$ for all i, $0 \leq i \leq n$.*

Corollary 3.2 implies Theorem 1.1.

4 2-convex polygon with n interior points

Theorem 4.1 *For any n, $n \geq 1$, let P be a set of points in the plane such that $Conv(P)$ is a triangle and containing atleast n interior points. Then P contains a 2-convex polygon with exactly n-interior points.*

Proof. Wlog, assume that the base AB of the triangle ABC is horizontal and C lies above AB. Let $I(P) = \{p_1, p_2, \ldots, p_m\}, m \geq n$ be the interior points of P that is sorted by decreasing y-coordinate. Let H be the convex hull of $\{B, C, p_{n+1}, p_{n+2}, \ldots, p_m\}$. The polygon $Q = T \setminus H$ is 2-convex and has exactly n interior points, i.e., points $p_1, \ldots, p_n$. □

References

[1] Aichholzer, O., F. Aurenhammer, F. Hurtado, P. A. Ramos and J. Urrutia, *Two-convex polygons*, European Workshop on Computational Geometry (2009), pp. 447 – 451.

[2] Avis, D., K. Hosono and M. Urabe, *On the existence of a point subset with 4 or 5 interior points*, in: J. Akiyama, M. Kano and M. Urabe, editors, *Discrete and Computational Geometry*, Lecture Notes in Computer Science **1763**, 2000 pp. 57–64.

[3] Avis, D., K. Hosono and M. Urabe, *On the existence of a point subset with a specified number of interior points*, Discrete Mathematics **241** (2001), pp. 33–40.

[4] Erdos, P. and G. Szekeres, *A combinatorial problem in geometry*, Compoistio Mathematica **2** (1935), pp. 463–470.

[5] Koshelev, V., *On erdos-szekeres-type problems*, Electronic Notes in Discrete Mathematics **34** (2009), pp. 447 – 451, european Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009).

[6] Morris, W. and V. Soltan, *The erdos-szekeres problem on points in convex positiona survey*, Bull. Amer. Math. Soc. **37** (2000), pp. 437–458.

[7] Toth, G. and P. Valtr, *The erdos-szekeres theorem: upper bounds and related results*, , **52**, 2005.

[8] Wei, X. and R. Ding, *More on planar point subsets with a specified number of interior points*, Mathematical Notes **83** (2008), pp. 684–687.

[9] Wei, X. and R. Ding, *More on an erdos-szekeres-type problem for interior points*, Discrete & Computational Geometry **42** (2009), pp. 640–653.