On the Erdős-Szekeres n-interior point problem

Subramanya Bharadwaj B V [1] Sathish Govindarajan [2]
Karmveer Sharma [3]

Department of Computer Science and Automation,
Indian Institute of Science, Bangalore, India

Abstract

The n-interior point variant of the Erdős-Szekeres problem is the following: for any n, $n \geq 1$, does there exist a $g(n)$ such that every point set in the plane with at least $g(n)$ interior points has a convex polygon containing exactly n-interior points. The existence of $g(n)$ has been proved only for $n \leq 3$. In this paper, we show that, for point sets having at most logarithmic number of convex layers, $g(n)$ exists for all $n \geq 5$. We also consider a relaxation of the notion of convex polygons and show that for all $n, n \geq 1$, any point set with at least n interior points has an almost convex polygon (simple polygon with at most one concave vertex) that contains exactly n-interior points.

Keywords: Convex polygons, interior points, Erdős-Szekeres problem, j-convexity

1 Introduction

Let $P \subset R^2$ be a finite set of points in general position. Let $Conv(P)$ denote the convex hull of P, $C(P)$ the set of points which determine the convex hull

[1] Email: `subramanya@csa.iisc.ernet.in`
[2] Email: `gsat@csa.iisc.ernet.in`
[3] Email: `karm@csa.iisc.ernet.in`

of P and $I(P) = P \setminus C(P)$ be the set of points lying in the interior of $Conv(P)$. Interesting questions of the form: "*If $|P|$ is large, then certain special convex subsets $S \subseteq P$ exist*" have been posed and studied in the last few decades.

The earliest classical result in this area is the Erdős-Szekeres Theorem [6] which showed that there exists an integer $f(n)$ such that if $|P| \geq f(n)$ then there exists a convex subset $S \subseteq P, |S| = n$ i.e., a subset which is a convex polygon of size n. In fact they showed that $2^{n-2}+1 \leq f(n) \leq \binom{2n-4}{n-2}+1$ [6,7]. They conjectured that the lower bound is tight and this has been proved for $n \leq 6$ [12,16]. There have been improvements on the upper bound over the years [5,13,17]. The current best bounds are $2^{n-2}+1 \leq f(n) \leq \binom{2n-5}{n-2}+1$ [18].

A convex subset $S \subseteq P$ is called a n-hole if $|S| = n$ and $|I(S)| = 0$. Let $g(n)$ be the smallest integer such that if $|P| \geq g(n)$, then there is a $S \subseteq P$ such that S is a n-hole. The exact value of $g(n)$ has been found for $n \leq 5$ [10]. Horton showed that for $n \geq 7$, $g(n)$ does not exist [11]. It was recently shown that $g(6)$ is finite by Gerken and Nicholas independently [9,15].

Given n, q the problem of finding $f(n,q)$ such that any point set P with $|P| \geq f(n,q)$ has a convex subset $S \subseteq P, |S| = n$ such that $|I(S)| = 0 (\text{mod } q)$ was considered by Bialostocki et al [4]. They showed that such a $f(n,q)$ exists for $n = 2 (\text{mod } q)$ or $n \geq q+3$. Many other variants of the Erdős-Szekeres problem have been considered. See surveys [14,18] for details.

The following problem pertaining to interior points was posed by Avis et al in 2001 [3]: Is there a smallest integer $g(n)$ such that any point set P with $|I(P)| \geq g(n)$ has a convex subset $S \subseteq P$ with $|I(S)| = n$ i.e. a subset which is a convex polygon that contains exactly n interior points. They showed that $g(1) = 1$, $g(2) = 4$. It was recently shown that $g(3) = 9$ by Wei and Ding [20]. It is not known whether $g(n)$ exists for $n \geq 4$. Lower bounds on $g(n)$ has been considered in [3,8]. It was shown in [3] that $g(n) \geq n+2$ for $n \geq 4$. This was improved to $g(n) \geq 3n-1$ for $n \geq 3$ [8]. The best known lower bound is $g(n) \geq 3n$ for $n \geq 3$ [19]. A related problem is to find a $h(n)$ which guarantees the existence of a convex polygon with n or $n+1$ interior points. This was addressed by Avis et al in [2] where they showed $h(4) = 7$. Another variant is find a $h(n)$ which guarantees the existence of a convex polygon with n or $n+2$ interior points. This was considered in [21] where it was shown that $h(3) = 8$.

For the n-interior point problem posed by Avis et al.[3], existence of $g(n)$ is known only for $n \leq 3$. In this paper, we consider several special cases of the n-interior point problem and show that $g(n)$ exists for all n. First, we shall show that $g(n)$ exists for point sets with a small number of convex layers. More specifically, we prove the following:

Theorem 1.1 *For any n, $n \geq 5$, every point set P with $r \geq 2$ convex layers and $|I(P)| \geq (64n^2)^r$ interior points, has a convex subset $S_n \subseteq P$ with $|I(S_n)| = n$.*

In other words Theorem 1.1 states that any point set where the number of convex layers is at most logarithmic in the number of interior points has a convex subset with n interior points.

We also consider the n-interior point problem where the notion of convex polygon is relaxed. Specifically we consider the following relaxations:

Definition 1.2 A simple polygon S is said to be *almost convex* if S contains at most one concave vertex.

Note that any convex polygon is also almost convex.

Theorem 1.3 *For any $n, n \geq 1$, every point set P with $|I(P)| \geq n$ interior points has an almost convex polygon that contains exactly n interior points.*

The notion of convexity has been generalized to j-convexity in [1].

Definition 1.4 A simple polygon S is said to be j-convex is every line intersects S in at most j connected components.

Note that 1-convex is the standard definition of convex polygon. It can be seen that an almost convex polygon is 2-convex.

Finally, we consider another relaxation where we ask for the existence of two convex polygons such that the sum of their interior points is exactly n. We show that the almost convex polygon constructed by the proof of Theorem 1.3 can be partitioned into two convex polygons whose interior points sum up to n.

Any point set considered in the rest of the paper is assumed to be in general position i.e. no three points are collinear.

2 Point sets with small number of convex layers

In this section we show that that there exists $g(n)$, such that, any point set P with a small number of convex layers and at least $g(n)$ interior points has a convex subset S_n that contains exactly n interior points.

In Section 2.1 we show that S_n exists in points sets with two monotonic convex layers. In Section 2.2 we extend this argument to show that S_n exists in point sets with two convex layers. Finally, in Section 2.3, we show that any point set with r convex layers contains S_n.

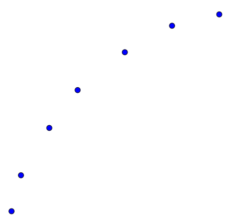

Fig. 1. An increasing monotonic convex set

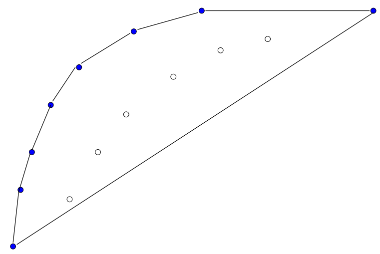

Fig. 2. Point set P^*

2.1 Point sets with 2 monotonic convex layers

Definition 2.1 Let $p = (p_x, p_y)$ be a point. A convex point set $P = \{p^1, \ldots, p^k\}$ is called increasing monotonic convex if $p_x^i \leq p_x^j$ and $p_y^i \leq p_y^j$ for all i, j, $1 \leq i \leq j \leq k$ (See Figure 1).

Definition 2.2 Recursively decompose a given point set P into disjoint convex hull layers $C(P), C(I(P)), \ldots$. We call this the Convex Hull Decomposition of P or the CHD of P.

Let P^* denote a point set whose CHD has two convex layers both of which are increasing monotonic convex. Let $C_2^* = C(P^*)$ and $C_1^* = I(P^*)$ (See Figure 2). Let us index all the points in C_1^* as $1, 2, \ldots$ and all the points in C_2^* also as $1, 2, \ldots$ assuming a natural order of points. We will also denote a point in C_1^* as $x', y', \ldots$ and a point in C_2^* as $x, y, \ldots$.

Definition 2.3 For any $x \in C_2^*$, let $arc(x) \subseteq C_1^*$ denote the set of points to which a straight line can be drawn from x without intersecting $Conv(C_1^*)$.

Definition 2.4 We say a point y' is visible from x if $y' \in arc(x)$.

Note that $arc(x)$ is the set of points visible from x (See Figure 3).

Definition 2.5 For any point $y' \in C_1^*$, let $arci(y') \subseteq C_2^*$ be the set of points from which y' is visible (See Figure 4).

Since y' is visible from at least one point $x \in C_2^*$, $arci(y') \neq \phi$.

Definition 2.6 We label $y' \in C_1^*$ as d (down) if there exists a $x \in arci(y')$ such that x is above y', as u (up) if there exists a $x \in arci(y')$ such that x lies below y' and as du if both hold (See Figure 4).

Let $S_n \subseteq P^*$ denote a set of points in convex position with $|I(S_n)| = n$.

Lemma 2.7 *For any $x \in C_2^*$, if $|arc(x)| = n + 2$, then S_n exists.*

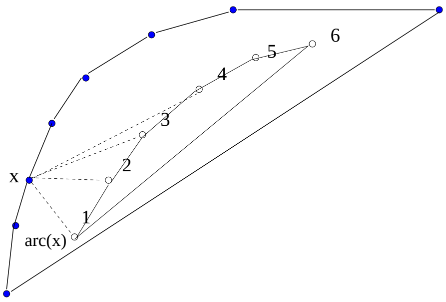

Fig. 3. $arc(x) = \{1, 2, 3, 4\}$

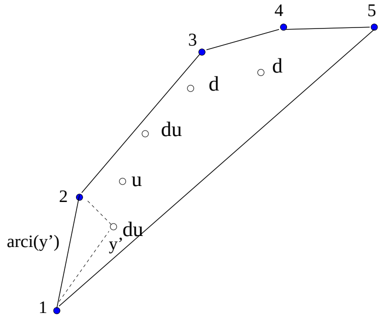

Fig. 4. Labelling points in C_1^* and $arci(y') = \{1, 2\}$

Proof. Let $y', z' \in arc(x)$. Consider triangle $xy'z'$. It contains $|y' - z'| - 1$ vertices. Since this is true for any $y', z' \in arc(x)$, the claim follows. □

Lemma 2.8 *If i' is labelled u, then $i' + 1$ cannot be labelled d.*

Proof. Let $x = max(arci(i'))$ and $y = min(arci(i' + 1))$. Clearly either i' is visible from y or $i' + 1$ is visible from x, which is a contradiction. □

Lemma 2.9 *If i' is labelled d (or du) and $i' + n + 1$ labelled u (or du) then S_n exists.*

Proof. Without loss of generality let i' be labelled d and $j' = i' + n + 1$ labelled u. There is an $i \in arci(i')$ such that i is above i' and a $j \in arci(j')$ such that j is below j'. Now consider the polygon $i \ldots jj'i'$. This is clearly convex and has $j' - i' - 1 = n$ points. Hence S_n exists. □

Lemma 2.10 *If there is a contiguous set of points in C_1^* of size at least $n+2$, all of which are labelled d, or all of which are labelled u, then S_n exists.*

Proof. Consider a contiguous set of points $M = \{i', i' + 1, ...i' + r'\} \subseteq C_1^*$ with $r' \geq n + 1$, where each point in M is labelled u. Since $i' + r'$ has been labelled u there exists $x \in arci(i' + r')$ such that x is below $i' + r'$. x is also below i' as otherwise $i' + s'$ would be labelled du for some $0 \leq s' < r'$. Since i' and $i' + r'$ are both visible from x and x lies below i', all of $i', \ldots, i' + r'$ are visible from x which implies $|arc(x)| \geq n + 2$. From Lemma 2.7 S_n exists. The argument for the case of all points in M labelled d is similar. □

Lemma 2.11 *If there exists a contiguous set of points in C_1^* of size at least $n + 2$ with no point in it labelled du, then S_n exists.*

Proof. Consider a contiguous set of points $M = \{i', i'+1, ...i'+r'\} \subseteq C_1^*$ with $r' = n + 1$, where each point in M is labelled u or d (but not du). Suppose

$$\begin{bmatrix} u & \cdots & u & u & u & du \\ du & \cdots & du & u & du & d \\ d & \cdots & d & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ d & \cdots u \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Fig. 5. Matrix of labels

$i', \ldots, i' + r'_1$ is a maximal contiguous subset of points in which each point has been labelled u, with $r'_1 < r'$. From Lemma 2.8, $i' + r'_1 + 1$ cannot be labelled d, which is a contradiction. Hence if i' is labelled u, every point in M is labelled u and from Lemma 2.10 S_n exists. Suppose $i', \ldots, i' + r'_1$ is a maximal contiguous subset of points in which each point has been labelled d, with $r'_1 < r'$ and let points $i' + r'_1 + 1, \ldots, i' + r'_2$ be labelled u, with $r'_2 \leq r'$. If $r'_2 = r'$, S_n exists (by Lemma 2.9). Else if $r'_2 < r'$, then $i' + r'_2 + 1$ is labelled d which is a contradiction because of Lemma 2.8.

□

We show that S_n always exists if the number of points in C_1^* is large enough. Consider the list of labels of points in C_1^*. The ith entry in the list corresponds to the label of point i'. We create a matrix M of labels corresponding to this list in a natural way. The first $n+1$ entries form the first row, the next $n+1$ entries the next row and so on. This is a $m * (n+1)$ matrix. Formally point i' is mapped to $(i(mod\ (n+1)), i - i(mod\ (n+1)))$ entry in the matrix (See Figure 5). Note that the last row might not be complete and we leave the rest of the entries in the matrix undefined..

Let P^* be a point set not containing S_n and M be the corresponding matrix of labels formed as above. We observe the following:

Remark 1: If the (r, c) entry is d or du then $(r + i, c)$ is d for all $i \geq 1$ (Lemma 2.9).

Remark 2: The labels $(r, c), (r, c+1), \ldots, (r+1, 1), \ldots, (r+1, c)$ cannot all be d or cannot be all u (Lemma 2.10).

Theorem 2.12 *For any n, $n \geq 5$, if $|C_1^*| \geq 2n^2$, then S_n exists.*

Proof. We prove by contradiction. Let, if possible, no such subset $S_n \subseteq P^*$ exist. We construct the corresponding matrix M of labels as given above. Let r_1 be the smallest row for which (r_1, c_1) is labelled du for some c_1. Without loss of generality let $r_1 = 2$ and let $c_1 = 1$ (See Figure 5). Consider the

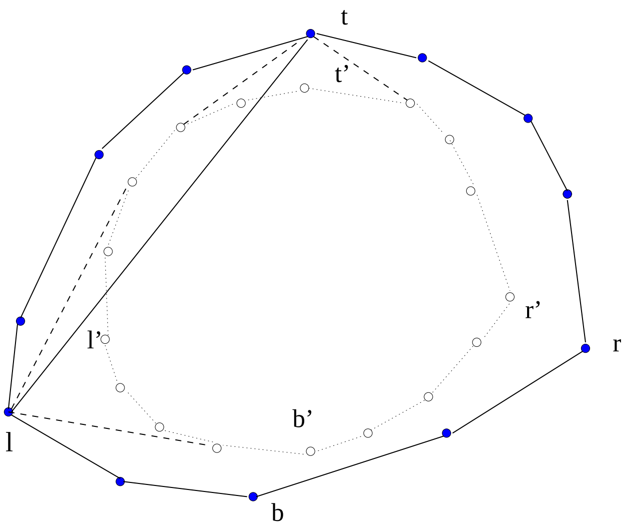

Fig. 6. Decomposing a two layer convex set

sequence of rows $r_1, \ldots, r_{n+3}$, where $r_{i+1} = r_i + 1$, $1 \leq i < n+3$. For $i < n+2$ if $(r_i, c) = du$ (or d) then $(r_j, c) = d$ for all j, $i < j \leq n+2$ (Remark 1). Hence $(r_i, 1) = d$ for all i, $2 \leq i \leq n+3$. Also if for any i, $2 \leq i \leq n+1$, $(r_i, c) = d$ for all c, $1 \leq c \leq n+1$, then S_n exists (Remark 2). Otherwise we note that $(r_i, 1) = d$ and $(r_{i+1}, 1) = d$, and $(r_i, 1), \ldots, (r_{i+1}, 1)$ is a contiguous sequence of $n+2$ labels. Therefore by Lemma 2.11 there exists a c such that $(r_i, c) = du$. Let $d(r)$ represent the number of entries labelled d in row r. If for some i, $1 \leq i \leq n+1$, $(r_i, c) = d$ for all c, $1 \leq c \leq n+1$ then S_n exists (Remark 2). Otherwise $d(r_i) < d(r_{i+1})$ for all i, $1 \leq i \leq n+1$ since a du or d entry in (r_i, c) implies a d entry in (r_{i+1}, c) (Remark 1). Since $d(r_2) \geq 1$, $d(r_{n+2}) \geq n+1$ i.e. $(r_{n+2}, c) = d$ for all c, $1 \leq c \leq n+1$. Also $(r_{n+3}, 1) = d$. By Remark 2 S_n exists which is a contradiction. We note that the entries in the matrix are well defined for the above ranges if $n \geq 5$

□

2.2 Point sets with two convex layers

Definition 2.13 Given $a, b \in C_i$ where C_i is the ith convex layer, let $C_{ab} \subseteq C_i$ denote the set of points obtained by traversing $Conv(C_i)$ clockwise from a to b and let P_{ab} denote the convex polygon formed by points in C_{ab}.

Theorem 2.14 *For any n, $n \geq 5$, every point set P whose CHD has exactly two layers C_1 and C_2 (internal and external) with $|C_1| \geq 8n^2 + 8n + 8$ has a*

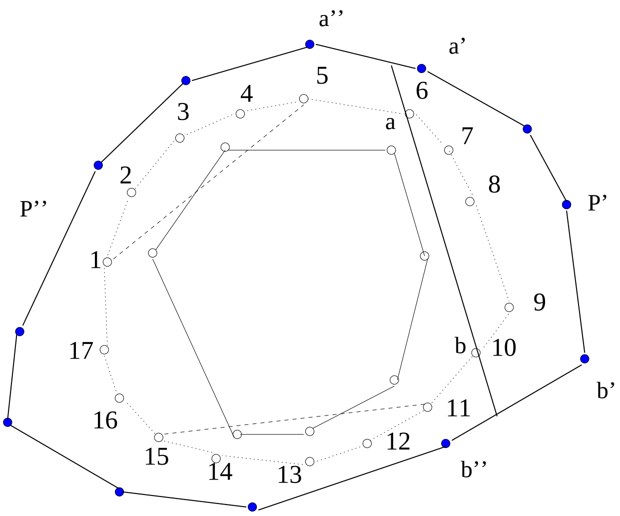

Fig. 7. Obtaining a two layer convex set

subset S_n.

Proof. The topmost, leftmost, bottommost and rightmost points in C_1, namely l', t', r', b', are visible from the corresponding topmost, leftmost, bottommost and rightmost points in C_2 namely l, t, r, b(See Figure 6). We have $C_{l't'} \cup C_{t'r'} \cup C_{r'b'} \cup C_{b'l'} = C_1$. Therefore without loss of generality $|C_{l't'}| \geq 2n^2 + 2n + 2$ since $|C_1| \geq 8n^2 + 8n + 8$. Consider the polygon P_{lt} (the corresponding set is C_{lt}). We note that $C_{l't'} \subseteq I(C_{lt}) \cup arc(t) \cup arc(l)$ which implies $|C_{l't'}| \leq |I(C_{lt})| + |arc(t)| + |arc(l)|$. If $|arc(l)| \geq n + 2$ or $|arc(t)| \geq n + 2$ then S_n exists.(Lemma 2.7). Else if $|arc(l)| \leq n + 1$ and $|arc(t)| \leq n + 1$, then $|I(C_{lt})| \geq 2n^2$. The point set $P^* = C_{lt} \cup I(C_{lt})$ has two increasing monotonic convex layers C_1^* and C_2^* with $|C_1^*| \geq 2n^2$. Hence by Theorem 2.12 S_n exists.□

2.3 Point sets with r convex layers

Theorem 2.15 *Let P be a point set whose CHD has r layers $C_1, \cdots, C_r$ (from internal to external) $r \geq 2$. Define $|C_0| = 1$ and let $n \geq 5$. If for any i, $1 \leq i \leq r$, $|C_i| \geq 64n^2|C_{i-1}|$, then S_n exists.*

Proof. Let $C_i = \{1, \cdots, l\}$ and $|C_{i-1}| = t$. Set $m = 8n^2 + 12n + 16$. Since $|C_i| \geq 64n^2|C_{i-1}|$, $l > ((t+1)(m+2) + m + 3)$. Consider the set of polygons $\hat{P} = \{P_{(1)(m+3)}, P_{(m+3)(2m+5)}, \cdots, P_{((t+1)(m+2)+1)((t+1)(m+2)+m+3)}\}$ (See Figure 7). Let C_{ab} be the set of points coresponding to P_{ab} for any a, b. Let $\hat{C} = \{C_{(1)(m+3)}, C_{(m+3)(2m+5)}, \cdots, C_{((t+1)(m+2)+1)((t+1)(m+2)+m+3)}\}$. Note

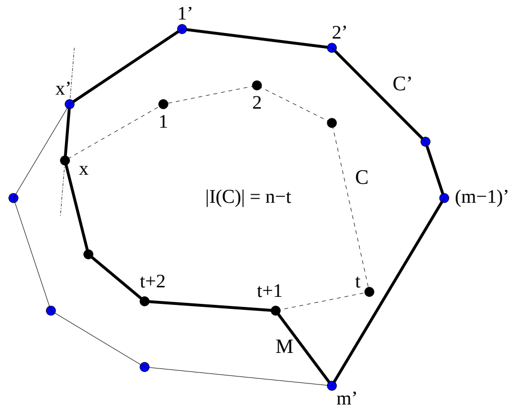

Fig. 8. Convex layers C and C' and the almost convex polygon M (shown in dark boundary)

that for $C_{ab}, C_{cd} \in \hat{C}$, $I(C_{ab}) \cap I(C_{cd}) = \phi$. This means that there exists C_{ab} such that $I(C_{ab}) = \phi$ as otherwise $\sum_{C_{cd} \in \hat{C}} |I(C_{cd})| \geq t + 1$. Consider such a C_{ab}. The line ab induces a natural partition of $C_i \cup C_{i+1}$ into two point sets P' and P''. Wlog $P' \cap C_{i-1} = \phi$. The line ab intersects $Conv(C_{i+1})$ in two line segments. Let the endpoints of the line segments be $a' \in P', a'' \in P''$ and $b' \in P', b'' \in P''$ respectively. Let $arc_S(x)$ denote $arc(x) \cap S$ for a set S of points. $C_{ab} \subseteq I(C_{a'b'}) \cup arc_{P'}(a'') \cup arc_{P'}(b'') \cup arc_{P'}(a') \cup arc_{P'}(b')$. If $arc_{P'}(a'') \geq n+2$ or $arc_{P'}(b'') \geq n+2$ or $arc_{P'}(a') \geq n+2$ or $arc_{P'}(b') \geq n+2$ then S_n exists (Lemma 2.7). Else $P^* = C_{a'b'} \cup I(C_{a'b'})$ is a pointset with two convex layers with $|I(P^*)| = |I(C_{a'b'})| \geq 8n^2 + 8n + 8$. Hence by Theorem 2.14 S_n exists. □

Theorem 2.15 implies Theorem 1.1.

3 Almost convex polygon with n interior points

In this section we show that any point set P with at least n interior points contains an almost convex polygon with n points in its interior.

Proof of Theorem 1.3:

Proof. Consider the convex hull layers in the convex hull decomposition $CHD(P)$. Let C and C' be the convex hull layers such that they are consecutive in $CHD(P)$(inside to outside) and $|I(C)| \leq n$ and $|I(C')| > n$.(This will always happen as $|I(P)| \geq n$) If $|I(C)| = n$, then C is the required convex polygon. Let us assume that $|I(C)| = n - t$, $t > 0$.

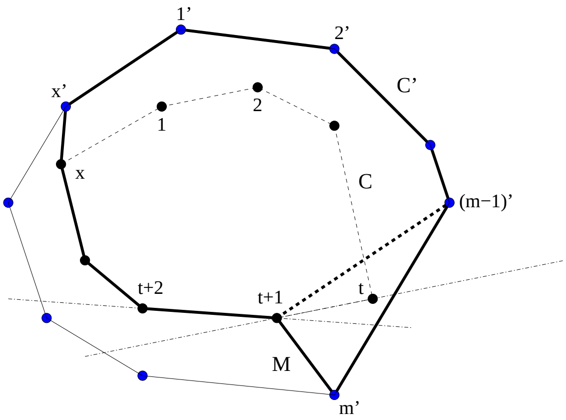

Fig. 9. The almost convex polygon M is split into a convex polygon and a triangle using the chord connecting vertex $(t+1)$ and $(m-1)'$.

We construct an almost convex polygon M using the vertices of C and C' as follows: Let x' be any vertex of C' and $x'x$ be the tangent from x' to C (as shown in Figure 8). We number the vertices of C' (resp. C) as $1', 2', \ldots x'$ (resp. $1, 2, \ldots, x$) in clockwise order starting with the clockwise adjacent vertex of x' (resp. x). Let $m' \in C'$ be the vertex with smallest index such that vertex $(t+1) \in arc(m')$. Since every vertex of C is present in the arc of some vertex of C', vertex m' exists. M is given by the simple polygon $x', 1', 2', \ldots, m', (t+1), (t+2) \ldots, x$ (boundary of M is shown as solid line in figure 8). M contains exactly n interior points (t points of C and $n-t$ points of $I(C)$). It can seen that M is almost convex since all vertices of M, except vertex $(t+1)$, are convex. □

Since any almost convex polygon is 2-convex, we have the following corollary:

Corollary 3.1 *For any $n, n \geq 1$, every point set P with $I(P) \geq n$ contains a 2-convex polygon with exactly n interior points.*

Finally, we prove the following corollary that shows the existence of two convex polygons that contain n-interior points in total.

Corollary 3.2 *For any $n, n \geq 1$ and every point set P with $I(P) \geq n$, there exists two convex polygons M_1 and M_2 such that $|I(M_1)| + |I(M_2)| = n$.*

Proof. We show that the almost convex polygon M constructed by the proof of Theorem 1.3 can be partition into two convex polygons M_1 and M_2. If vertex $t+1 \in C$ is convex, then M is convex and we are done. Let us assume that vertex $t+1 \in C$ is concave. Since $m' \in C'$ is picked as the smallest index vertex such that vertex $(t+1) \in arc(m')$, m' lies to the right of line connecting $t+1$ and t (See figure 9). Also, m' lies below the line connecting $t+2$ and $t+1$ since $t+1 \in C$ is assumed to be concave. Now, the chord connecting $t+1 \in C$ and $(m-1)' \in C'$ partitions M into $M_1 = x', 1', 2', \ldots, (m-1)', (t+1), (t+2), \ldots, x$ and triangle $M_2 = (m-1)', m', (t+1)$. M_1 is convex since vertex $t+1$ is convex. Also, $|I(M_1)| + |I(M_2)| = n$ since every interior point of M is an interior point of either M_1 or M_2. □

4 Conclusion

In this paper, we have considered two special cases of the n-interior point variant of the Erdős-Szekeres problem. We have shown that $g(n) \leq (64n^2)^r$ for all $n, n \geq 5$, for point sets with $r \geq 2$ convex layers. We have also shown that $g(n) = n$ for all $n, n \geq 1$, when the notion of convex polygon is relaxed to almost convex polygon.

For arbitrary point sets, existence of $g(n)$ is known only for $n \leq 3$. The main open problem is to show that $g(n)$ exists for all n.

References

[1] Aichholzer, O., F. Aurenhammer, F. Hurtado, P. A. Ramos and J. Urrutia, *Two-convex polygons*, European Workshop on Computational Geometry (2009), pp. 447 – 451.

[2] Avis, D., K. Hosono and M. Urabe, *On the existence of a point subset with 4 or 5 interior points*, in: *Japanese Conference on Discrete and Computational Geometry*, Lecture Notes in Computer Science **1763**, 1998 pp. 57–64.

[3] Avis, D., K. Hosono and M. Urabe, *On the existence of a point subset with a specified number of interior points*, Discrete Mathematics **241** (2001), pp. 33–40.

[4] Bialostocki, A., P. Dierker and B. Voxman, *Some notes on the erdös-szekeres theorem*, Discrete Mathematics **91** (1990), pp. 231–238.

[5] Chung, F. R. K. and R. L. Graham, *Forced convex n-gons in the plane*, Discrete and Computational Geometry **19** (1998), pp. 367–371.

[6] Erdos, P. and G. Szekeres, *A combinatorial problem in geometry*, Compoistio Mathematica **2** (1935), pp. 463–470.

[7] Erdos, P. and G. Szekeres, *On some extremum problems in elementary geometry*, Ann. Universitatis Scientiarum Budapestinensis **3/4** (1960), pp. 463–470.

[8] Fevens, T., *A note on point subsets with a specified number of interior points*, in: *Japanese Conference on Discrete and Computational Geometry*, Lecture Notes in Computer Science **2866**, 2002 pp. 152–158.

[9] Gerken, T., *Empty convex hexagons in planar point sets*, Discrete and Computational Geometry **39** (2008), pp. 239–272.

[10] Haborth, H., *Konvexe f nfecke in ebenen punktmengen*, Elementary Mathematics **33** (1978), pp. 116–118.

[11] Horton, J. D., *Sets with no empty convex 7-gons*, Canadian Mathematical Bulletin **26** (1983), pp. 482–484.

[12] Kalbfeisch, J. D., J. G. Kalbfleisch and R. G. Stanton, *A combinatorial problem on convex regions.*, in: *Proc. Louisiana Conf. Combinatorics, Graph Theory and Computing,* (1970), pp. 180–188.

[13] Kleitman, D. J. and L. Pachter, *Finding convex sets among points in the plane*, Discrete and Computational Geometry **19** (1998), pp. 405–410.

[14] Morris, W. and V. Soltan, *The erdos-szekeres problem on points in convex position - a survey*, Bulletin of the American Mathematical Society **37** (2000), pp. 437–458.

[15] Nicolas, C. M., *The empty hexagon theorem*, Discrete and Computational Geometry **38** (2007), pp. 389–397.

[16] Szekeres, G. and L.Peters, *Computer solutions to be the 17-point erös- szekeres problem*, The Anazian journal **48** (2006), pp. 151–164.

[17] Toth, G. and P. Valtr, *Note on the erdos-szekeres theorem*, Discrete and Computational Geometry **19** (1998), pp. 457–459.

[18] Toth, G. and P. Valtr, *The erdos-szekeres theorem: upper bounds and related results*, Combinatorial and Computational Geometry **52** (2005), pp. 557–568.

[19] Wei, X. and R. Ding, *More on planar point subsets with a specified number of interior points*, Mathematical Notes **83** (2008), pp. 684–687.

[20] Wei, X. and R. Ding, *More on an erdos-szekeres-type problem for interior points*, Discrete and Computational Geometry **42** (2009), pp. 640–653.

[21] Wei, X., W. Lan and R. Ding, *On the existence of a point subset with three or five interior points*, Mathematical Notes **88** (2010), pp. 103–111.