

On Locally Gabriel Geometric Graphs

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Abstract

Let P be a set of n points in the plane. A geometric graph G on P is said to be *locally Gabriel* if for every edge (u, v) in G , the disk with u and v as diameter does not contain any points of P that are neighbors of u or v in G . A locally Gabriel graph is a generalization of Gabriel graph and is motivated by applications in wireless networks. Unlike a Gabriel graph, there is no unique locally Gabriel graph on a given point set since no edge in a locally Gabriel graph is necessarily included or excluded. Thus the edge set of the graph can be customized to optimize certain network parameters depending on the application. In this paper, we show the following combinatorial bounds on edge complexity and independent sets of locally Gabriel graphs:

- (i) For any n , there exists locally Gabriel graphs with $\Omega(n^{5/4})$ edges. This improves upon the previous best bound of $\Omega(n^{1+\frac{1}{\log \log n}})$.
- (ii) For various subclasses of convex point sets, we show tight linear bounds on the maximum edge complexity of locally Gabriel graphs.
- (iii) For any locally Gabriel graph on any n point set, there exists an independent set of size $\Omega(\sqrt{n} \log n)$.

1 Introduction

A geometric graph $G = (V, E)$ is an embedding of the set V as points in the plane and edges in E as straight-line segments connecting the points in V . Delaunay graphs, Gabriel graphs and Relative Neighborhood graphs (RNG) are fundamental geometric proximity graphs with applications in fields like computer graphics, vision, GIS, wireless networks, etc. For a nice survey on these graphs and their applications, see [11].

The Gabriel graph introduced by Gabriel and Sokal [9] is defined as follows: Given a set of points P in the plane, an edge exists between points u and v iff the Euclidean disk with the segment joining u and v as diameter does not contain any other point of P . Gabriel graphs have been used to model the topology in wireless networks [3, 18]. Motivated by applications in wireless networks, [14, 12] generalized these structures to k -locally delaunay/gabriel graphs. The edge complexity of these structures have been

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studied in [12, 15]. In this paper, we focus on 1-locally Gabriel graphs and call them as *Locally Gabriel Graphs (LGGs)*.

A *locally gabriel graph* is a geometric graph G with the following property: for each edge (u, v) in G , the Euclidean disk with the segment joining u and v as diameter does not contain any point of P that is a neighbor of u or v in G .

Study of these graphs was initially motivated by design of dynamic routing protocols for *ad hoc* wireless networks [13]. An ad-hoc wireless network consists of a collection of wireless transceivers (corresponds to points) and an underlying network topology (corresponds to edges) that is used for communication/routing. Like Gabriel Graphs, *LGGs* can be used to design wireless network topology since they capture the interference patterns well. An interesting point to be noted is that there is no unique *LGG* on a given point set since no edge in *LGG* is necessarily included or excluded. Thus the edge set of the graph (used for wireless communication) can be customized to optimize certain network parameters depending on the application. *LGGs* also provide certain advantages over Gabriel Graphs. While a Gabriel graph has linear number of edges (planar graph), we show in this paper that there exists *LGGs* with $n^{5/4}$ edges. A dense network can be desirable for applications like broadcasting or multicasting where a large number of pairs of nodes need to communicate with each other. Another important parameter in the topology of wireless network is the number of simultaneous transmissions that can be performed. A node in a wireless network cannot transmit and receive in the same time slot. Thus, the set of transmitting nodes at any time slot form an independent set in the underlying graph. We show that there exists an independent set of size $\Omega(\sqrt{n} \log n)$ in any *LGG* of any n pointset.

An interesting combinatorial question, that we address in this paper, is to bound the edge complexity of locally gabriel graphs.

It was observed in [15] that the unit distance graph [7], introduced by Erdos, is also a locally delaunay/gabriel graph. The maximum edge complexity of unit distance graphs has been extensively studied [7, 16, 17]. See [4] for a survey on this problem. There is a significant gap between the lower and upper bounds and improving them is considered a hard open problem in discrete geometry. The edge complexity of unit distance graphs on convex point sets have also been studied. The best lower bound is $2n - 7$ [6] and the best upper bound is $n \log n$ [8, 5]. It has been conjectured in [4] that the edge complexity of unit distance graphs on convex point sets is $2n$ and proving this conjecture is a challenging open problem.

[12] initiated the study of maximum edge complexity of locally delaunay/gabriel graphs by showing non-trivial upper bounds. [15] showed an upper bound of $O(n^{3/2})$ and a lower bound of $\Omega(n^{4/3})$ on the maximum edge complexity of locally delaunay graphs.

For locally gabriel graphs, [12] showed an upper bound of $O(n^{3/2})$ by proving that $K_{2,3}$ is a forbidden subgraph. The best known lower bound is $\Omega(n^{1 + \frac{1}{\log \log n}})$ [7], given by Erdos classic lower bound construction for unit distance graphs. While the gap between the upper and lower bounds for locally delaunay graphs has been narrowed significantly, the gap is quite wide for locally gabriel graphs. In this paper, we improve the lower bound significantly.

We show the following results in this paper:

- (i) For any n , there exists locally gabriel graphs with $\Omega(n^{5/4})$ edges. This improves the previous lower bound of $\Omega(n^{1+\frac{1}{\log \log n}})$ [7].
- (ii) For various subclasses of convex point sets like monotonic convex point set, half convex point set, centrally symmetric convex point set, we prove tight linear bounds on the edge complexity of locally gabriel graphs.
- (iii) For any LGG on any n point set, we show that there exists an independent set of size $\Omega(\sqrt{n} \log n)$.

The paper is organized as follows: Definitions that will be used in the paper is presented in Section 2. We present the lower bound construction and analyze it in Section 3. We prove various upper and lower bounds for convex point sets in Section 4. The independent set construction is presented in Section 5.

2 Preliminaries

Let P be a set of n points in \mathbb{R}^2 . For any $p, q \in P$, we denote by d_{pq} the disk with the segment joining p and q as diameter.

Definition 2.1 (Locally Gabriel condition) *Let G_P be a geometric graph on P . An edge (u, v) of G_P is said to satisfy the locally Gabriel condition if disk d_{uv} does not contain any neighbor of u or v in G_P .*

Definition 2.2 (Locally Gabriel Graph) *A geometric graph G_P on P is said to be Locally Gabriel Graph (LGG) if every edge of G_P satisfies the locally Gabriel condition.*

Let $p = (p^x, p^y)$ be any point in \mathbb{R}^2 .

Definition 2.3 (Upper-right monotonic convex point set) *Let $P = \{p_1, p_2 \dots, p_k\}$ be a set of points in convex position that are ordered in counterclockwise direction. P is called a upper-right monotonic convex point set if $p_i^x \leq p_j^x, p_i^y \geq p_j^y, \forall 1 \leq i < j \leq k$*

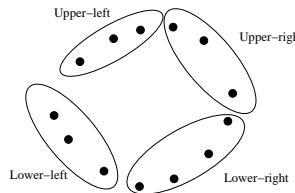


Figure 1: Four types of monotonic convex sets

Similarly, we define the other three types of monotonic convex point sets, i.e., upper-left, lower-right and lower-left. Figure 1 shows the 4 types of monotonic convex point sets. Note that any convex point set can be decomposed into the above 4 types.

Definition 2.4 (Half convex point set) *Let $P = Q \cup R$ be a set of points in convex position that is ordered in counterclockwise direction. P is called a right (resp. left) half convex point set if Q is upper-right monotonic and R is lower-right monotonic (resp. Q is upper-left monotonic and R is lower-left monotonic).*

Definition 2.5 (Centrally symmetric convex point set) *Let P be a set of points in convex position. P is said to be centrally symmetric with respect to the origin, if for every point $p \in P$, point $-p$ also belongs to P .*

Let p, q, r be three points in P .

Lemma 2.1 *If q and r are neighbors of p in an LGG on P , then $\angle pqr, \angle prq < \pi/2$.*

Proof. Since (p, q) is an edge of G_P , r must lie outside the disk d_{pq} . Thus, $\angle prq < \frac{\pi}{2}$. Since (p, r) is also an edge in G_P , q must lie outside the disk d_{pr} . Thus, $\angle pqr < \frac{\pi}{2}$. \square

Conversely, if either $\angle pqr \geq \frac{\pi}{2}$ or $\angle prq \geq \frac{\pi}{2}$, then we call the edges (p, q) and (p, r) as *conflicting*. Two conflicting edges cannot exist simultaneously in an LGG.

3 Lower Bound Construction

In this section, we describe the construction of a LGG with $\Omega(n^{5/4})$ edges. The point set P for this construction is a $\sqrt{n} \times \sqrt{n}$ uniform grid. First, we describe the algorithm that constructs the LGG G_P on the grid point set P . Then, we prove the correctness of our algorithm. Finally, we analyze the edge complexity of G_P .

3.1 Construction

Let us denote the points on the grid as $(x, y), 0 \leq x, y < \sqrt{n}$. The algorithm is an iterative greedy procedure that assigns neighbors to each grid point. First, we describe the procedure that assigns neighbors to an arbitrary point $p = (p^x, p^y)$ on the grid. For technical reasons, we only assign neighbors to p that are in the first and third quadrant w.r.t. p . By applying this procedure to the grid points $(x, y), \sqrt{n}/3 \leq x, y < 2\sqrt{n}/3$ (we choose only these grid points to avoid edge effects), we obtain our LGG G_P .

Now, we describe the iterative procedure that assigns neighbors to p in a counter-clockwise direction. Let q_i be the current neighbor of p that is assigned by the procedure and θ_i be the angle that segment pq_i makes with the positive direction of x-axis. First, we describe how to find the next neighbor q_{i+1} in the counter-clockwise direction. Let us describe the feasibility region for q_{i+1} . Figure 2 shows the points p, q_i , the disk d_{pq_i} and the tangent line l at q_i . Since (p, q_i) is an edge in G_P , q_{i+1} must lie outside d_{pq_i} . Also, since (p, q_{i+1}) will be an edge in G_P , $\angle pq_i q_{i+1} < \frac{\pi}{2}$ (by Lemma 2.1). This implies that q_{i+1} must lie below the tangent line l . Thus the feasible region for q_{i+1} is outside d_{pq_i} and below l (shown as the shaded region in Figure 2). We choose the next neighbor q_{i+1} to be the grid point in the feasible region that is closest (in Euclidean distance) to q_i (See Figure 2). This greedy choice allows us to pack as many neighbors as possible.

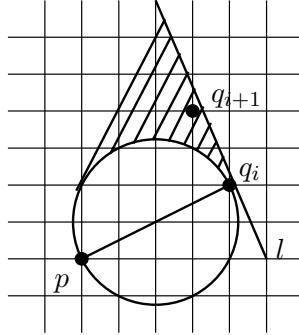


Figure 2: Feasibility region for the next neighbor q_{i+1}

Now, the procedure that assigns neighbors to p is as follows: The first neighbor of p is set as $q_0 = (p^x + s, p^y + s \cdot \tan \theta_0)$, where $s = \sqrt{n}/9$ and $\theta_0, \theta_0 \in (0, \pi/4)$ is a small constant to be fixed later. Starting with this neighbor, we iteratively find the next neighbor using the procedure described above. We continue assigning neighbors as long as the condition $\theta_i \leq \pi/4$ is satisfied. Note that this procedure assigns neighbors only in the first quadrant w.r.t p . Similarly, we find neighbors in the third quadrant w.r.t p by starting with the initial neighbor $(p^x - s, p^y - s \cdot \tan \theta_0)$ and proceeding as long as the condition $\theta_i \leq 5\pi/4$ is satisfied.

3.2 Correctness

In this section, we show that the geometric graph G_P constructed above is a locally gabriel graph.

Remark 1: Since the pointset is an uniform grid, the above deterministic procedure that constructs G_P assigns neighbors in a symmetric consistent manner, i.e., *if the procedure assigns q_i as the i -th neighbor (in 1st quadrant) of p , then it would assign p as the i -th neighbor (in 3rd quadrant) of q_i , if the procedure is applied on q_i .*

By Remark 1, the neighbors of p in G_P are exactly the grid points chosen by the procedure. Let m be the number of neighbors of p in G_P in the first quadrant.

Lemma 3.1 *Let $p \in P$ be any grid point and let $Q = \{q_0, q_1, \dots, q_m\}$ be the neighbors of p in G_P (in counter-clockwise order) in the first quadrant. The disk d_{pq_i} does not contain any neighbor of $p \forall i, 0 \leq i \leq m$.*

Proof. First, we show that d_{pq_i} does not contain any neighbor of p in the first quadrant, i.e., $d_{pq_i} \cap (Q \setminus \{q_i\}) = \emptyset$. Observe that d_{pq_i} does not contain q_{i+1} because the iterative procedure picks q_{i+1} outside the disk d_{pq_i} . Also observe that d_{pq_i} does not contain q_{i-1} because $\angle pq_{i-1}q_i < \frac{\pi}{2}$ (q_i is picked below tangent line of $d_{pq_{i-1}}$). On the contrary, let us assume that d_{pq_i} contains some $q_r, r \neq i-1, i, i+1$. There are 2 cases: (i) $r > i+1$ and (ii) $r < i-1$. We will prove case (i) below. Case (ii) can be proved in a similar manner. Let us assume that k is the smallest index among the neighbors $q_r, r > i+1$ that is contained in d_{pq_i} . Since $q_i, q_{i+1}, \dots, q_{k-1}, q_k$ are in counter-clockwise convex

position, all the disks d_{pq_j} , $i + 1 \leq j \leq k - 1$ also contains q_k (see figure 3). Thus, the disk $d_{pq_{k-1}}$ also contains q_k . This is a contradiction since the iterative procedure picks q_k outside the disk $d_{pq_{k-1}}$.

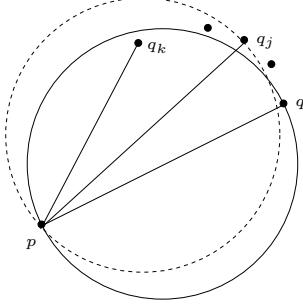


Figure 3: Point p and its neighbors $q_i, \dots, q_j, \dots, q_k$

The disk d_{pq_i} does not contain any neighbor of p in the third quadrant w.r.t p , since d_{pq_i} does not intersect the third quadrant w.r.t p . Thus d_{pq_i} does not contain any neighbor of p . \square

Remark 2: Observe that the grid point set P is a symmetric point set and we use the same deterministic procedure to assign neighbors to all the grid point. Hence Lemma 3.1 is true for all the grid points $p \in P$.

Lemma 3.2 *Edge (p, q_i) of G_P satisfies the locally Gabriel condition $\forall i, 0 \leq i \leq m$.*

Proof. We need to show that the disk d_{pq_i} , $0 \leq i \leq m$, does not contain the neighbors of p or q_i in G_P . By Lemma 3.1, disk d_{pq_i} does not contain any neighbor of p .

By Remark 1, p is the i th neighbor (in 3rd quadrant) of q_i . By Remark 2, we apply Lemma 3.1 for grid point q_i (instead of p) on the neighbors of q_i in the 3rd quadrant (instead of 1st quadrant) to show that disk $d_{q_i p}$ (which is the same as d_{pq_i}) does not contain any neighbors of q_i . \square

Since the procedure assigns neighbors to p in the third quadrant in exactly the same way as the first quadrant, Lemma 3.2 shows that edges from p to its neighbors in the third quadrant also satisfy the locally Gabriel condition. Thus, all the edges from p to neighbors of p satisfies the locally Gabriel condition. Since we use the same deterministic procedure to assign neighbors to all the grid points, the argument for p applies to all grid points $p \in P$. Hence all the edges in G_P satisfy the locally Gabriel condition proving that G_P is locally Gabriel.

3.3 Analysis

In this section, we analyze the lower bound construction described in the previous section. We will show that G_P has $\Omega(n^{5/4})$ edges by proving that the iterative procedure picks $\Omega(n^{1/4})$ neighbors for grid point p . Let q_0, q_1, \dots, q_m be the neighbors

(in counter-clockwise order) of p in the first quadrant. Given the current neighbor q_i , the procedure picks the next neighbor q_{i+1} “close” to q_i . We will prove bounds on the closeness between q_i and q_{i+1} . Using this, we show bounds on m .

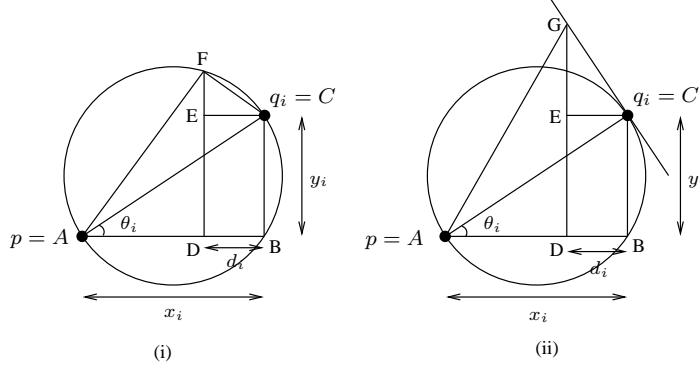


Figure 4: The vertical gridline that contains the next neighbor intersects (i) the diameter disk at F , (ii) the tangent line at G

Figure 4 shows the points p (denoted as A in the figure), the current neighbor q_i (denoted as C), the disk d_{pq_i} and the tangent line l at C . Let the next neighbor q_{i+1} lie at an x-distance d_i from the current neighbor q_i (q_{i+1} lies on the vertical line passing through D and E). Let $|AB| = q_i^x - p^x = x_i$, $|DB| = q_i^x - q_{i+1}^x = d_i$ and $|CB| = q_i^y - p^y = y_i$ (See Figure 4). First, we will prove bounds for d_i in terms of x_i . Let the vertical grid-line passing through q_{i+1} intersect the disk d_{pq_i} at F (See Figure 4(i)) and the tangent line at G (See Figure 4(ii)). Let $|FE| = h_i$ and $|GE| = h'_i$. Since $\triangle AFC, \triangle ADF, \triangle FEC$ are right-angle triangles, (see Figure 4(i)), we have

$$\begin{aligned} |AC|^2 &= |AF|^2 + |FC|^2 \\ &= (|AD|^2 + |DF|^2) + (|FE|^2 + |CE|^2) \\ (x_i \sec \theta_i)^2 &= (x_i - d_i)^2 + (h_i + x_i \tan \theta_i)^2 + h_i^2 + d_i^2 \end{aligned}$$

Simplifying, we get

$$h_i^2 + x_i \tan \theta_i \cdot h_i - d_i(x_i - d_i) = 0 \quad (1)$$

Similarly, since $\triangle ACG$ is right-angled at C (see Figure 4(ii)), we have

$$\begin{aligned} |AG|^2 &= |AC|^2 + |CG|^2 \\ (|AD|^2 + |DG|^2) &= |AC|^2 + (|GE|^2 + |CE|^2) \\ (x_i - d_i)^2 + (h'_i + x_i \tan \theta_i)^2 &= (x_i \sec \theta_i)^2 + h'^2 + d_i^2 \end{aligned}$$

Simplifying, we get $h'_i = d_i \cot \theta_i$.

The next neighbor q_{i+1} lies on the vertical grid line between F and G . To ensure that a grid point exists between F and G , we enforce a condition that the distance between F and G is greater than 1, i.e., $|FG| = h'_i - h_i > 1$. Solving for h_i in Equation 1, substituting for h_i, h'_i , we get

$$d_i \cot \theta_i - \frac{\sqrt{x_i^2 \tan^2 \theta_i + 4d_i(x_i - d_i)} - x_i \tan \theta_i}{2} > 1 \quad (2)$$

Simplifying (2), we get the inequality

$$d_i^2 + \sin^2 \theta_i > x_i \tan \theta_i \sin^2 \theta_i + d_i \sin 2\theta_i$$

Setting $d_i = c_1 \sqrt{x_i}$, $c_1 > 1$ in the above inequality, we get,

$$x_i(c_1^2 - \tan \theta_i \sin^2 \theta_i) - \sqrt{x_i} c_1 \sin 2\theta_i + \sin^2 \theta_i > 0$$

The above inequality is a quadratic in $\sqrt{x_i}$. When $c_1 > 1$, it is satisfied for all $x_i \geq 1, \theta_i \in (0, \pi/4)$ (we assign neighbors to p only till $\theta_i \leq \pi/4$). Therefore, inequality (2) is also satisfied. This gives us a bound on d_i (closeness between q_{i+1} and q_i) in terms of x_i (x-distance of q_i from p).

Now, we will obtain bounds on m , the number of neighbors assigned to p . Note that the procedure assigns neighbors to p as long as $\theta_i \leq \pi/4$, i.e., $y_m \leq x_m$. We will now obtain bounds on x_i and y_i . The x_i 's are related by the following recurrence relation

$$\begin{aligned} x_{i+1} &= x_i - d_i \\ &= x_i - c_1 \sqrt{x_i} \\ &\geq x_i - c_1 \sqrt{\frac{\sqrt{n}}{9}} \quad (x_i \leq \frac{\sqrt{n}}{9}) \end{aligned}$$

Expanding this recurrence with $x_0 = \sqrt{n}/9$, we get

$$x_k \geq \frac{\sqrt{n}}{9} - \frac{k \cdot c_1 n^{1/4}}{3}, 0 < k \leq m \quad (3)$$

Next, we obtain bounds on y_i . The y_i 's are related by the recurrence relation $y_{i+1} = y_i + \lfloor h_i + 1 \rfloor$ (since we pick q_{i+1} as the closest grid point to F). Expanding this recurrence, we get

$$y_k = y_0 + \sum_0^{k-1} \lfloor h_i + 1 \rfloor \quad (4)$$

$$\leq y_0 + k + \sum_0^{k-1} h_i \quad (5)$$

where h_i is given by the solution to Equation 1

$$\begin{aligned}
\sum_0^{k-1} h_i &= \frac{1}{2} \sum_0^{k-1} \sqrt{x_i^2 \tan^2 \theta_i + 4d_i(x_i - d_i)} - x_i \tan \theta_i \\
&= \frac{1}{2} \sum_0^{k-1} \sqrt{x_i^2 \tan^2 \theta_i + 4c_1 \sqrt{x_i}(x_i - c_1 \sqrt{x_i})} - x_i \tan \theta_i \\
&= \frac{1}{2} \sum_0^{k-1} x_i \tan \theta_i \left(\sqrt{1 + \frac{4c_1 \sqrt{x_i}(x_i - c_1 \sqrt{x_i})}{x_i^2 \tan^2 \theta_i}} - 1 \right) \\
&\leq \frac{1}{2} \sum_0^{k-1} x_i \tan \theta_i \left(\sqrt{1 + \frac{4c_1}{\sqrt{x_i} \tan^2 \theta_i}} - 1 \right) \\
&\leq \frac{1}{2} \sum_0^{k-1} x_i \tan \theta_i \left(\left(1 + \frac{2c_1}{\sqrt{x_i} \tan^2 \theta_i} \right) - 1 \right) \\
&\leq \sum_0^{k-1} \frac{c_1 \sqrt{x_i}}{\tan \theta_i}
\end{aligned}$$

Since $\theta_i > \theta_0$ and $x_i \leq \sqrt{n}/9$, we have

$$\sum_0^{k-1} h_i \leq \frac{c_1 \cdot k \cdot n^{1/4}}{3 \tan \theta_0}$$

Hence, from Equation (5), y_k is upper bounded as follows:

$$y_k \leq \frac{\tan \theta_0 \cdot \sqrt{n}}{9} + \frac{c_1 \cdot k \cdot n^{1/4}}{3 \tan \theta_0} + k \quad (6)$$

Setting $c_1 = 1.01$, $\theta_0 = 1.74 \times 10^{-3}$, it can be verified analytically in Equation (3) and Equation (6) that $y_k \leq x_k$ for all $0 \leq k \leq 10^{-4}n^{1/4}$. Thus, $y_m \leq x_m$ for $m = 10^{-4}n^{1/4}$. The number of neighbors of p is at least $10^{-4}n^{1/4}$. The edge complexity of G_P is therefore $\Omega(n \cdot n^{1/4}) = \Omega(n^{5/4})$.

4 Convex Point Sets

In this section, we show edge complexity for LGG on various classes of convex point sets. First, we show exact bounds for half convex point sets. Then, we show asymptotic tight linear bounds for special subclasses of convex point sets. Finally, we show $O(n \log n)$ bounds for arbitrary convex point sets.

4.1 Exact Bounds for Half Convex Point Sets

First, let us consider the special case when P is a monotonic convex point set. W.l.o.g., let us assume that P is of the upper-right type.

Lemma 4.1 Let $P = \{p_1, p_2, \dots, p_n\}$ be a upper-right monotonic convex point set and let G_P be any locally gabriel graph on P . p_1, p_n has atmost 1 neighbor in G_P and hence G_P has atmost $n - 1$ edges.

Proof. We show that the first point p_1 has atmost one neighbor. Let if possible, p_i and p_j be neighbors of p_1 , $j > i$. p_1, p_i, p_j are in monotonic convex position. Thus $\angle p_1 p_i p_j \geq 90^\circ$. Since p_i and p_j are neighbors of p_1 , $\angle p_1 p_i p_j < 90^\circ$ (by Lemma 2.1). Hence a contradiction. By a similar argument, we can also show that p_n has atmost 1 neighbor in G_P .

Removing p_1 from P and applying induction on the remaining points, we see that G_P has atmost $n - 1$ edges. \square

Next we consider the special case when P is a half convex point set. W.l.o.g., let us assume that P is a right half convex point set.

Lemma 4.2 Let $P = Q \cup R$ be a right half convex point set with n points, where Q is upper-right monotonic and R is lower-right monotonic. Let G_P be any locally gabriel graph on P . G_P has atmost $2n - 3$ edges.

Proof. Let p be the point with maximum x-coordinate (rightmost point) in P . $Q \cup \{p\}$ is upper-right monotonic and $R \cup \{p\}$ is lower-right monotonic. By Lemma 4.1, p has degree atmost two(atmost one neighbor in Q and one in R). Removing p from P and applying induction on the remaining points, we get $P(n) \leq P(n - 1) + 2$; $P(2) = 1$. This gives $P(n) \leq 2n - 3$. \square

The above bounds are tight, i.e., it is easy to construct locally gabriel graphs for monotonic and half convex sets that match the above bounds. For any monotonic convex sets, construct a path (of length $n - 1$) connecting all the vertices.

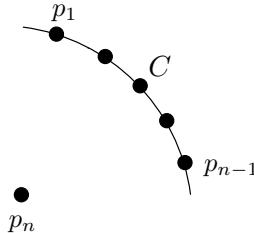


Figure 5: Portion of circle C centered at p_n and points p_1, \dots, p_{n-1} placed equidistant on C

For right half convex sets, we can achieve the exact bound using the following construction:

Let C be a circle with center at p_n . We place points p_1, p_2, \dots, p_{n-1} equidistant along the first quadrant of C (See Figure 5). The point set constructed is right half convex. The edges of G_P are defined as follows:

(i) Add edges (p_i, p_{i+1}) , $1 \leq i \leq n - 2$. This forms a path of length $n - 2$.

(ii) Add edges (p_n, p_i) , $1 \leq i \leq n - 1$. This forms a star of size $n - 1$.

It can be verified that these edges satisfy the locally gabriel condition. Thus, the edge complexity of G_P is $2n - 3$.

4.2 Tight Linear Bounds for Various Subclasses

In this section, we prove asymptotic tight linear bounds for some special subclasses of convex point sets.

4.2.1 Points on a Circle

First, we consider the special case of n points lying on a circle.

Lemma 4.3 *Let C be any circle and $P = \{p_1, p_2, \dots, p_n\}$ be n points that lie on C . Let G_P be any locally gabriel graph on P . G_P has atmost n edges*

Proof. Let p_i be any point in P and p'_i be the point on C that is diametrically opposite to p_i . The diameter $p_ip'_i$ divides the circle C into two halves. We claim that p_i has atmost 1 neighbor in each half. Let, if possible, p_i have two neighbors p_j and p_k in the same half (see Figure 6(i)). We can see that $\angle p_ip_jp'_i = 90^\circ$. Since p_i, p_j, p_k, p'_i are in convex position, we have $\angle p_ip_jp_k > \angle p_ip_jp'_i$. Thus, $\angle p_ip_jp_k > 90^\circ$. But, since (p_i, p_k) is an edge, $\angle p_ip_jp_k < 90^\circ$ (by Lemma 2.1). Hence a contradiction.

Since, each point $p_i \in P$ has atmost 2 neighbors (atmost one in each half), the edge complexity of G_P is atmost n . \square

This bound is exact, since we can always construct a G_P with n edges.

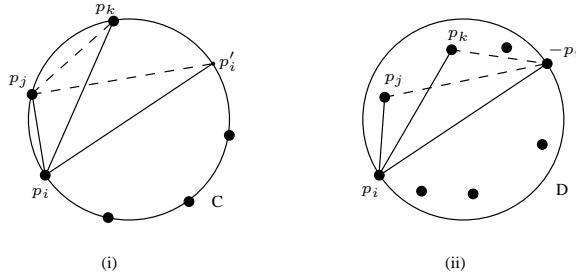


Figure 6: (i) Points on a circle C (ii) Centrally symmetric point set with diameter pair $p_i, -p_i$

4.2.2 Centrally symmetric convex point set

Next, we consider the case of P being in centrally symmetric convex position. We prove that any locally gabriel graph on P has atmost $2n - 3$ edges. Our proof is an adaptation of [1], where it was proved that the unit distance graph on centrally symmetric convex point sets has atmost $2n - 3$ edges.

Lemma 4.4 *Let $P = \{p_1, -p_1, p_2, -p_2 \dots, p_{n/2}, -p_{n/2}\}$ be n points in centrally symmetric convex position. Let G_P be any locally gabriel graph on P . G_P has atmost $2n - 3$ edges*

Proof. In [1], it is shown that the diameter pair (pair that is furthest apart) in any centrally symmetric convex point set must be of the form $(p_r, -p_r)$, for some r . Let $(p_i, -p_i)$ be the diameter pair in P .

If $(p_i, -p_i)$ is an edge in G_P , we can show (by a similar argument as below) that $p_i, -p_i$ has atmost 1 neighbors in P . Thus, G_P would have atmost $n - 1$ edges. Therefore, let us assume that $(p_i, -p_i)$ is not an edge in G_P .

Consider the closed disk D with p_i and $-p_i$ as diameter. Since P is centrally symmetric, all the points in P must lie in D . The diameter $p_i, -p_i$ divides the disk D into two halves. We claim that p_i has atmost 1 neighbor in each half. Let, if possible, p_i have two neighbors p_j and p_k in the same half (see Figure 6(ii)). Since, p_j lies in D , $\angle p_i p_j - p_i \geq 90^\circ$. Also, since $p_i, p_j, p_k, -p_i$ are in convex position, $\angle p_i p_j p_k > \angle p_i p_j - p_i$. Thus, $\angle p_i p_j p_k > 90^\circ$. Since (p_i, p_k) is an edge, $\angle p_i p_j p_k < 90^\circ$ (by Lemma 2.1). Hence a contradiction.

p_i has atmost 2 neighbors in G_P . By the same argument, $-p_i$ also has atmost 2 neighbors. Removing p_i and $-p_i$ from P and recursing on the remaining point set (which is also centrally symmetric), we have $P(n) \leq P(n-2) + 4$; $P(2) = 1$. This gives $P(n) \leq 2n - 3$. \square

We can achieve an almost tight lower bound using the following construction: Let P be a set of n points defined by $P = \{(-1, i) \cup (1, i), -n/4 \leq i < n/4\}$. P consists of equally spaced integer gridpoints on the vertical lines $x = -1$ and $x = 1$ ($n/2$ points in each line). It is easy to see that P is centrally symmetric about the origin. The edges of G_P are defined as follows:

- (i) Add $n - 4$ edges of the form $((-1, i), (-1, i + 2))$ and $((1, i), (1, i + 2))$ for all $-\frac{n}{4} \leq i < \frac{n}{4} - 2$.
- (ii) Add $n - 4$ edges of the form $((-1, i), (1, i + 1))$ and $((-1, i), (1, i - 1))$ for all $-\frac{n}{4} - 1 \leq i < \frac{n}{4} - 1$.

It can be easily verified that these edges satisfy the locally gabriel condition. Thus, the edge complexity of G_P is $2n - 8$.

4.3 Bounds for Convex Point Sets

In this subsection, we consider an arbitrary convex point set P . We prove that the edge complexity of any LGG on P is $O(n \log n)$. The proof is a straightforward extension

of the argument given in [5], which proved that the unit distance graph on convex point sets has $O(n \log n)$ edges.

Lemma 4.5 *Let P be a set of n points in convex point set and let G_P be any locally gabriel graph on P . G_P has $O(n \log n)$ edges.*

Proof. We use the clever recursive method given in [5]. We will describe the method briefly, for sake of completeness. Refer to [5] for details. We can partition P into Q and R using the topmost and bottommost point of P (antipodal pair). Note that Q is left half convex and R is right half convex. In fact, we can perform a partition using any of the antipodal pairs, such that the two parts are half convex sets (for an appropriate reference axis). The basic idea behind the recursive method in [5] is to use the above fact to divide P using two such partitions such that we have two subproblems of size atmost $3n/4$ and the edges at this level of recursion are edges within the four half convex sets. The number of such edges is $O(n)$ using Lemma 4.2. The edge complexity of G_P is thus $O(n \log n)$ \square

For convex point sets, the best known lower bound is $2n - 3$.

5 Independent Sets

In this section, we show that any *LGG* on any n point set contains an independent set of size at least $\Omega(\sqrt{n} \log n)$.

We first show an elementary argument that constructs an independent set of size at least $\frac{\sqrt{n}}{2}$ in a n point set. A set of points ordered by their abscissa is called a monotonic sequence if the ordinates of the points are either monotonically non-increasing or monotonically non-decreasing.

Lemma 5.1 *Let G_P be any *LGG* on a monotonic sequence P with n points. G_P has an independent set of size at least $\frac{n}{2}$.*

Proof. Let us denote the first and the last vertices of the monotonic sequence P as terminal vertices. We show that in any *LGG* on P , a terminal vertex has degree at most one. On the contrary let us assume that a terminal vertex v is incident to vertices v_1 and v_2 and the vertices appear in the sequence as v, v_1 and v_2 . An axis parallel rectangle with vv_2 as diagonal will contain v_1 inside or on the boundary of it. It implies that edges (v, v_2) and (v, v_1) conflict with each other. Thus, v has at most one edge incident to it. Now, add the terminal vertex to the independent set and remove it along with its neighbor (if it exists) from the sequence. In each iteration at most two vertices are removed and one vertex is added to the independent set. Thus, the independent set has size at least $\frac{n}{2}$. \square

Erdos and Szekeres [7] showed that a set of n points will have a monotonic sequence of size at least \sqrt{n} . One such sequence can be computed in $O(n \log n)$ time by an algorithm proposed by Hunt and Szymanski [10]. By Lemma 5.1, any induced *LGG* on this monotonic sequence has an independent set of size at least $\frac{\sqrt{n}}{2}$.

Now, we show that any *LGG* on any point set with n points contains an independent set of size at least $\Omega(\sqrt{n} \log n)$. In a graph $G = (V, E)$ for any $u \in V$, let us define $N(u) = \{v \mid (u, v) \in E\}$. A graph is said to have sparse neighborhood if for any $u \in V$, the chromatic number of the subgraph induced over vertices $\{u\} \cup N(u)$ is a constant. We show that any *LGG* with n vertices will have an independent set of size $\Omega(\sqrt{n} \log n)$ by using Theorem 5.1, where the sparse neighborhood property of *LGGs* (shown in Lemma 5.2) is applied.

Theorem 5.1 (Alon [2]) *Let $G = (V, E)$ be a graph on n vertices with average degree $t \geq 1$ in which for every vertex $v \in V$ the induced subgraph on the set of all neighbors of v is r -colorable. Then, the independence number of G is at least $\frac{c}{\log(r+1)} \frac{n}{t} \log t$, for some absolute positive constant c .*

Lemma 5.2 *Let G_P be any *LGG* on any point set P and u be an arbitrary vertex in G . The induced subgraph over the vertices $\{u\} \cup \{N(u)\}$ is 4-colorable.*

Proof. Let vertex u be adjacent to v_1, v_2, \dots, v_k . Let us consider the induced subgraph over these vertices. We show that any vertex say v_1 has at most one incident edge on either side of the line passing through u and v_1 . On the contrary let us assume that there are two vertices v_2 and v_3 adjacent to v_1 on the same side of line $\overline{uv_1}$. Let us analyze all the possible cases.

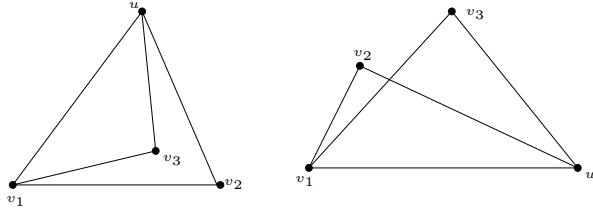


Figure 7: Possible placement of neighborhood in *LGG*

- All the four vertices (u, v_1, v_2 and v_3) cannot be collinear otherwise at least two vertices (say v_1 and v_2 w.l.o.g.) lie on the same side of u and the edges (u, v_1) and (u, v_2) would conflict with each other.
- Let us consider the case when three vertices are collinear. It can be trivially verified that v_1, v_2 and v_3 cannot be collinear due to *LGG* constraints. Similarly u, v_1 and v_2 (or v_3) also cannot be collinear due to *LGG* constraints. If v_2, v_3 and u are collinear then u must lie in between v_2 and v_3 . It contradicts with the assumption that v_2 and v_3 lie on the same side of $\overline{uv_1}$.
- Let us consider the case when convex hull of these four vertices is a triangle and another vertex lies inside this triangle as shown in Figure 7(a). Since it is assumed that v_2 and v_3 lie on the same side of $\overline{uv_1}$, u and v_1 must be the vertices of this triangle. Let us assume that vertex v_3 lies inside $\triangle uv_1v_2$. Since $(u, v_1), (u, v_2)$ and (u, v_3) do not conflict with each other, both $\angle uv_3v_1$ and $\angle uv_3v_2$ should be less than $\frac{\pi}{2}$, which is not possible in this configuration.

- The last case is when all the vertices are in convex position and form a quadrilateral. Lets assume w.l.o.g. that $uv_1v_2v_3$ is a convex quadrilateral as shown in Figure 7(b). By Lemma 2.1, $\angle uv_1v_2 < \frac{\pi}{2}$ (due to edges uv_1 and uv_2), $\angle v_1v_2v_3 < \frac{\pi}{2}$ (due to edges v_1v_3 and v_1v_2), $\angle v_2v_3u < \frac{\pi}{2}$ (due to edges uv_2 and uv_3), $\angle v_3uv_1 < \frac{\pi}{2}$ (due to edges v_1u and v_1v_3) and . But in a quadrilateral at least one of the internal angle should be greater than or equal to $\frac{\pi}{2}$. Hence, it leads to a contradiction.

Hence any vertex $v_i \in N(u)$ has at most two neighbors apart from u in the induced subgraph on neighborhood of u . Thus, the degree of any vertex v_i for $1 \leq i \leq k$ is at most 3. Therefore, this induced subgraph is 4-colorable. \square

Theorem 5.2 *Let G_P be any LGG on a n point set. G_P has an independent set of size $\Omega(\sqrt{n} \log n)$.*

Proof. Since an LGG has a maximum of $O(n^{\frac{3}{2}})$ edges [12], the average degree of a vertex is $O(\sqrt{n})$. Substituting $t = O(\sqrt{n})$ and $r = 4$ in Theorem 5.1, the desired bound is obtained. \square

Conclusion

In this paper, we have shown improved bounds on the maximum edge complexity of locally gabriel graphs. There is still a gap between our lower bound of $\Omega(n^{5/4})$ and the best known upper bound of $O(n^{3/2})$. It is an interesting problem to narrow this gap. We have shown tight linear bounds for various subclasses of convex pointsets. But, for a general convex point sets, the best lower bound on edge complexity of locally gabriel graphs is $2n - 3$, while the upper bound is $O(n \log n)$. Can one obtain tight bounds? Finally, we have shown that any LGG on any n pointset has an independent set of size $\Omega(\sqrt{n} \log n)$. There is no known non-trivial upper bound.

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