Small Strong Epsilon Nets[☆]

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Abstract

Let P be a set of n points in $\mathbb{R}^d$. A point x is said to be a *centerpoint* of P if x is contained in every convex object that contains more than $\frac{dn}{d+1}$ points of P. We call a point x a *strong centerpoint* for a family of objects $\mathcal{C}$ if $x \in P$ is contained in every object $C \in \mathcal{C}$ that contains more than a constant fraction of points of P. A strong centerpoint does not exist even for halfspaces in $\mathbb{R}^2$. We prove that a strong centerpoint exists for axis-parallel boxes in $\mathbb{R}^d$ and give exact bounds. We then extend this to small strong ϵ-nets in the plane and prove upper and lower bounds for $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is the family of axis-parallel rectangles, halfspaces and disks. Here $\epsilon_i^{\mathcal{S}}$ represents the smallest real number in $[0, 1]$ such that there exists an $\epsilon_i^{\mathcal{S}}$-net of size i with respect to $\mathcal{S}$.

Keywords: centerpoint, ϵ-nets, axis-parallel rectangles

1. Introduction

Let P be a set of n points in $\mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is said to be a *centerpoint* of P if any halfspace that contains x contains at least $\frac{n}{d+1}$ points of P. Equivalently, x is a centerpoint if and only if x is contained in every convex object that contains more than $\frac{d}{d+1}n$ points of P. It has been proved that a centerpoint exists for any point set P and the constant $\frac{d}{d+1}$ is tight [24]. The computational aspects related to centerpoint have been studied in [8, 12, 16]. The notion of centerpoint has found many applications in statistics, combinatorial geometry, geometric algorithms etc [17, 18, 26].

The centerpoint question has also been studied for certain special classes of convex objects. [2] shows exact constants on centerpoints for halfspaces, axis-parallel rectangles and disks in $\mathbb{R}^2$.

By the definition of centerpoint, x need not be a point in P. A natural question to ask is the following: Does there exist a *strong centerpoint* i.e., the

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centerpoint must belong to P. In other words, for all point sets P, does there exist a point $p \in P$ such that p is contained in every convex object that contains more than a constant fraction of points of P? It can be clearly seen that a strong centerpoint does not exist for convex objects by considering a point set with points in convex position. By the same example, a strong centerpoint does not exist even for objects like disks and halfspaces. The notion of strong centerpoints has been studied with respect to wedges in [19] where it was shown that there always exists a point $p \in P$ such that any $\frac{3\pi}{2}$-wedge anchored at p contains at least $\frac{n}{d+1}$ points. We show that a strong centerpoint exists for axis-parallel boxes in $\mathbb{R}^d$ i.e. for any point set P with n points, there exists a point $x \in P$ such that x is contained in every axis-parallel box that contains more than $\frac{2d-1}{2d}n$ points of P. A natural extension of strong centerpoint will be to allow more number of points and see what the bounds are. This question is related to a well studied area called ϵ-nets. First we shall define ϵ-nets.

Definition 1. Let P be a set of n points in $\mathbb{R}^d$ and $\mathcal{R}$ be a family of geometric objects. $N \subset P$ is called a (strong) ϵ-net of P with respect to $\mathcal{R}$ if $N \cap R \neq \emptyset$ for all subsets $R \in \mathcal{R}$ that has more than ϵn points of P, that is, $|R \cap P| > \epsilon n$. Moreover, N is called a weak ϵ-net if $N \subset \mathbb{R}^d$, i.e. N need not be a subset of P.

The concept of ϵ-nets was introduced by Haussler and Welzl [11] and has found many applications in computational geometry, approximation algorithms, learning theory etc. It has been proved that for geometric objects with finite VC dimension d, there exist ϵ-nets of size $O(\frac{d}{\epsilon}\log\frac{1}{\epsilon})$ [11]. It has been proved that ϵ-nets of size $O(\frac{1}{\epsilon})$ exist for halfspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ and pseudo-disks in $\mathbb{R}^2$ [13, 14, 23]. Alon [1] has shown a slightly super-linear lowerbound for ϵ-nets with respect to lines in $\mathbb{R}^2$ thereby disproving the conjecture that there exists linear-sized ϵ-nets for families of geometric objects. Recently, it has been shown that ϵ-nets of size $\Omega(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ is needed for the family of axis-parallel rectangles in $\mathbb{R}^2$ [22]. This bound is also tight by a previous result by Aronov *et al.* [3].

The dual version of ϵ-net is also well studied. Here, the input is a collection of n geometric objects and we have to find a sub-collection of objects that cover all points that are contained in more than ϵn objects. This question is extensively studied in [3, 9, 25]. Recently some tight lower bounds for dual ϵ-nets are proved in [22].

Strong ϵ-nets of size independent of n do not exist for convex objects since they have infinite VC dimension. However, it is shown that weak ϵ-nets of size polynomial in $\frac{1}{\epsilon}$ exist for convex objects in $\mathbb{R}^d$ [7, 15, 20]. Also, a lower bound of $\Omega(\frac{1}{\epsilon}\log^{d-1}\frac{1}{\epsilon})$ is proved in [6].

Weak ϵ-nets are an extension of centerpoint. A centerpoint is precisely a weak $\frac{d}{d+1}$-net of size one. Small weak ϵ-nets have been studied for convex objects, disks, axis-parallel rectangles and halfspaces in [2, 5, 10, 21]. Here the size of the weak ϵ-net is fixed as a small integer i and the value of ϵ_i is bounded.

In this paper, we initiate the study of small strong ϵ-nets. Let $\mathcal{S}$ be a family of geometric objects. Let $\epsilon_i^{\mathcal{S}} \in [0,1]$ represent the smallest real number such that, for any set of points P, there exists a set $Q \subset P$ of size i which is an $\epsilon_i^{\mathcal{S}}$-net with respect to $\mathcal{S}$. Thus a strong centerpoint will be an $\epsilon_1^{\mathcal{S}}$-net. We investigate bounds on $\epsilon_i^{\mathcal{S}}$ for small values of i where $\mathcal{S}$ is the family of axis-parallel rectangles, halfspaces and disks.

1.1. Our Results

Let $\mathcal{R}, \mathcal{H}, \mathcal{D}$ be the family of axis-parallel rectangles, halfspaces and disks respectively.

1. For axis-parallel boxes in $\mathbb{R}^d$, we show that strong centerpoint exists and obtain tight bound of $\epsilon_1^{\mathcal{R}} = \frac{2d-1}{2d}$. Note that strong centerpoint does not exist for halfspaces, disks or convex objects.
2. We give upper and lower bounds for $\epsilon_i^{\mathcal{R}}$ in the plane for small values of i. We also prove some general upper bounds for $\epsilon_i^{\mathcal{R}}$ which works for all values of i.
3. We give bounds for $\epsilon_i^{\mathcal{H}}$ which are tight for even values of i and almost tight for odd values of i. These bounds are based on a result from [13].
4. For the family of disks, we give a non-trivial upper bound for $\epsilon_2^{\mathcal{D}}$.

In section 2, we give a general lower bound construction which will be used in subsequent sections. Section 3 discusses small strong ϵ-nets for axis-parallel rectangles. Sections 4 and 5 give bounds for halfspaces and disks respectively.

2. General lower bound construction

In this section, we give a recursive construction to obtain lower bounds for $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is a family of compact convex objects in $\mathbb{R}^d$. More precisely, we give a lower bound construction for $\epsilon_{j+k}^{\mathcal{S}}$ based on the lower bound constructions of $\epsilon_j^{\mathcal{S}}$ and $\epsilon_k^{\mathcal{S}}$. These lower bounds work for weak ϵ-nets as well.

Theorem 1. $\epsilon_{j+k}^{\mathcal{S}} \geq \frac{\epsilon_j^{\mathcal{S}} \epsilon_k^{\mathcal{S}}}{\epsilon_j^{\mathcal{S}} + \epsilon_k^{\mathcal{S}}}$ *for all* $j, k \geq 1$

Proof. Let P be a set of n points in $\mathbb{R}^d$, arranged as two subsets, P_1 and P_2, containing $\frac{\epsilon_k^{\mathcal{S}}}{\epsilon_j^{\mathcal{S}} + \epsilon_k^{\mathcal{S}}} n$ and $\frac{\epsilon_j^{\mathcal{S}}}{\epsilon_j^{\mathcal{S}} + \epsilon_k^{\mathcal{S}}} n$ points respectively. Let P_1 be arranged corresponding to the lower bound construction for $\epsilon_j^{\mathcal{S}}$ and P_2 be arranged corresponding to the lower bound construction for $\epsilon_k^{\mathcal{S}}$. These two subsets are placed sufficiently far from each other. Therefore, if N_1 is an $\epsilon_j^{\mathcal{S}}$-net for P_1, there exists some $S \in \mathcal{S}$ avoiding N_1 such that $|S \cap P_1| = \epsilon_j^{\mathcal{S}} |P_1|$. Since S is compact and P_2 is placed sufficiently far from P_1, S does not contain any point of P_2. Similarly, if N_2 is an $\epsilon_k^{\mathcal{S}}$-net for P_2, there exists some $S \in \mathcal{S}$ avoiding N_2 such that $|S \cap P_2| = \epsilon_k^{\mathcal{S}} |P_2|$ and $S \cap P_1 = \emptyset$. Let N be any $\epsilon_{j+k}^{\mathcal{S}}$-net. Then N contains either $\leq j$ points of P_1 or $\leq k$ points of P_2. Therefore there always exists some $S \in \mathcal{S}$ that avoids N and contains $\frac{\epsilon_j^{\mathcal{S}} \epsilon_k^{\mathcal{S}}}{\epsilon_j^{\mathcal{S}} + \epsilon_k^{\mathcal{S}}} n$ points.

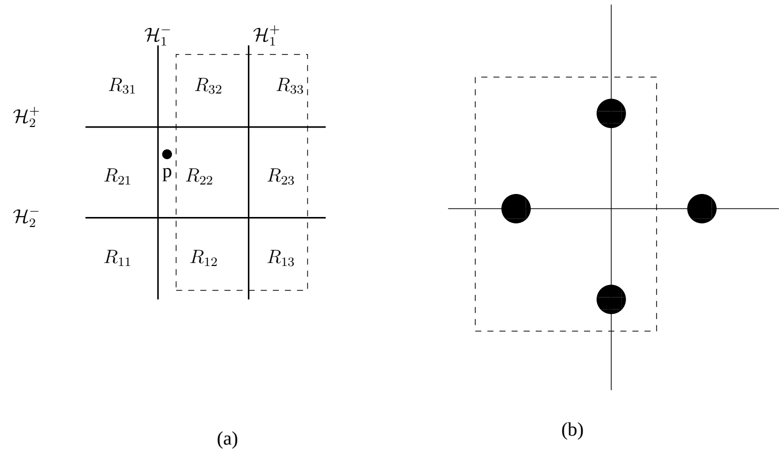


Figure 1: Bounds for $\epsilon_1^{\mathcal{R}}$ in $\mathbb{R}^2$

Based on this theorem, we give improved lower bounds on ϵ_i for axis-parallel rectangles in section 3.

3. Axis-parallel Rectangles

In this section, we show bounds on $\epsilon_i^{\mathcal{R}}$.

Let P be a set of n points in $\mathbb{R}^d$. Assume that all points in P have distinct co-ordinates, i.e. if $p = (p_1, p_2, ..., p_d)$ and $q = (q_1, q_2, ..., q_d)$ are two points in P, then $p_i \neq q_i$ for all i, $1 \leq i \leq d$. This assumption can be easily removed by slightly perturbing the input point set such that the co-ordinates are distinct. It can be seen that an ϵ-net for the perturbed set acts as an ϵ-net for the original set also (see section 4 in [2]).

3.1. Strong centerpoints in $\mathbb{R}^d$

Let $\mathcal{R}$ be the family of axis-parallel boxes in $\mathbb{R}^d$.

Theorem 2. $\epsilon_1^{\mathcal{R}} = \frac{2d-1}{2d}$

Proof. Let $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$, $1 \leq i \leq d$, be two axis-parallel hyperplanes orthogonal to the i^{th} dimension that divide P into three slabs. Let $\mathcal{P}_i^+$ be the subset of P contained in the positive hyperspace defined by $\mathcal{H}_i^+$ and $\mathcal{P}_i^-$ be the subset of P contained in the negative hyperspace defined by $\mathcal{H}_i^-$. $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ are placed such that $|\mathcal{P}_i^+| = |\mathcal{P}_i^-| = \frac{n}{2d} - 1$. The hyperplanes $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$, $1 \leq i \leq d$, partition $\mathbb{R}^d$ into 3^d axis-parallel d-dimensional boxes. Indexing the partition along each dimension, these boxes are denoted as $R_{x_1x_2...x_d}$, where $x_i \in \{1, 2, 3\}$ (see figure 1(a) for the upper bound construction in $\mathbb{R}^2$). Let $P_{x_1x_2...x_d} = R_{x_1x_2...x_d} \cap P$. We claim that $P_{22...2} \neq \emptyset$.

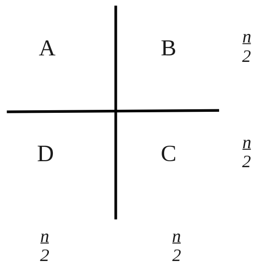


Figure 2: Bisection of P by vertical and horizontal lines

Let $\mathcal{K} = \sum_{i=1}^{d}(|\mathcal{P}_i^-| + |\mathcal{P}_i^+|) = n - 2d$. Since none of the points in $P_{22...2}$ is counted in any $\mathcal{P}_i^+$ or $\mathcal{P}_i^-$, $\mathcal{K} \geq n - |P_{22...2}|$. This implies that $|P_{22...2}| \geq 2d$.

Let p be any point in $P_{22...2}$. We claim that $\{p\}$ is a $\frac{2d-1}{2d}$-net. Any d-dimensional box that does not contain p has to avoid some $\mathcal{P}_i(\mathcal{P}_i^+$ or $\mathcal{P}_i^-)$ containing $\frac{n}{2d} - 1$ points. Hence it contains at most $\frac{2d-1}{2d}n$ points.

For the lower bound, place $2d$ subsets of $\frac{n}{2d}$ points such that each axis has two subsets at unit distance on either side of the origin. The lower bound construction for $\epsilon_1^{\mathcal{R}}$ in $\mathbb{R}^2$ is shown in figure 1(b). Let $\{q\}$ be an ϵ-net. Without loss of generality, assume that q is chosen from the subset placed at coordinates $(1, 0, 0, ...0)$. Now the d-dimensional axis-parallel box $x \leq 0.5$ avoids q but contains all the remaining $2d - 1$ subsets thereby containing $\frac{2d-1}{2d}n$ points.

$\square$

3.2. Upper Bounds on $\epsilon_i^{\mathcal{R}}$

Let P be a set of n points and $\mathcal{R}$ be the family of axis-parallel rectangles in $\mathbb{R}^2$. We prove upper bounds for $\epsilon_i^{\mathcal{R}}$.

Lemma 3. *There exists a point $p \in P$ with coordinates (x', y') such that the halfspaces $x \geq x'$ and $y \geq y'$ contain at least $\frac{n}{2}$ points of P.*

PROOF. Divide P into two horizontal and two vertical slabs such that each slab contains $\frac{n}{2}$ points (see figure 2).

- Case 1: $D \cap P \neq \emptyset$. Any point $p \in D \cap P$ has the desired property.
- Case 2: $D \cap P = \emptyset$. In this case, $|A \cap P| = |C \cap P| = \frac{n}{2}$ and $B \cap P = \emptyset$. Then the point $p \in A \cap P$ with the smallest perpendicular distance to the horizontal line has the desired property.

$\square$

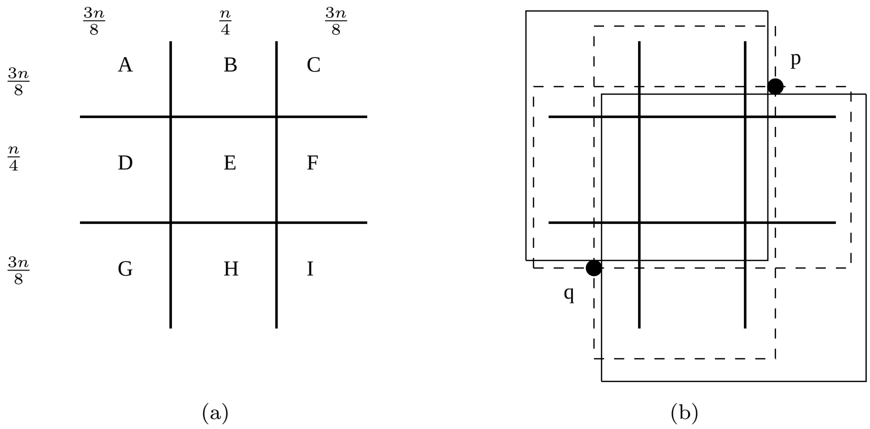


Figure 3: Upper bound for $\epsilon_2^{\mathcal{R}}$

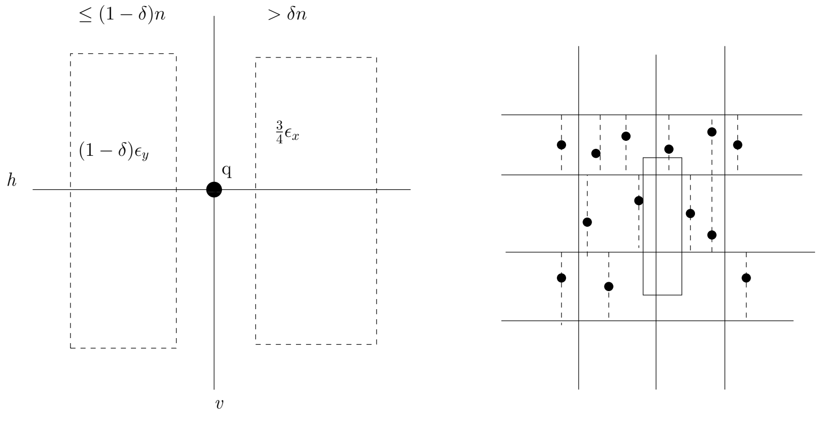
Lemma 4. $\epsilon_2^{\mathcal{R}} \leq \frac{5}{8}$

PROOF. Divide P into three horizontal slabs containing $\frac{3n}{8}$, $\frac{n}{4}$ and $\frac{3n}{8}$ points respectively. Similarly, divide P into three vertical slabs in the same proportion to get a grid with nine rectangular regions (figure 3(a)).

- Case 1: $E \cap P \neq \emptyset$: Let x be any point in E. Now $\{x\}$ is a $\frac{5}{8}$-net since any axis-parallel rectangle that avoids x will avoid an extreme slab which contains $\frac{3}{8}n$ points.

- Case 2: $E \cap P = \emptyset$: Let R_1, R_2, R_3, R_4 be the regions $A \cup B \cup D \cup E, B \cup C \cup E \cup F, D \cup E \cup G \cup H, E \cup F \cup H \cup I$ respectively and let P_i denote $R_i \cap P$ for all i, $1 \leq i \leq 4$. Since $E \cap P = \emptyset$, we have $|P_1| + |P_2| + |P_3| + |P_4| = n + |(B \cup D \cup F \cup H) \cap P| = \frac{3n}{2}$. Therefore either $(|P_1| + |P_4|) \geq \frac{3n}{4}$ or $(|P_2| + |P_3|) \geq \frac{3n}{4}$. Without loss of generality assume that $(|P_2| + |P_3|) \geq \frac{3n}{4}$. Using lemma 3, choose a point $p = (p_x, p_y) \in P_2$ such that the halfspaces $x \geq p_x$ and $y \geq p_y$ contain at least half of the points in P_2. Similarly, choose a point $q = (q_x, q_y) \in P_3$ such that the halfspaces $x \leq q_x$ and $y \leq q_y$ contain at least half of the points in P_3. We claim that $\{p, q\}$ is a $\frac{5}{8}$-net. Any axis-parallel rectangle that does not take points from all the three rows and columns contains at most $\frac{5n}{8}$ points of P. So assume that R is an axis-parallel rectangle that takes points from all the rows and columns. To avoid $\{p, q\}$, R must avoid at least half of the points from P_2 as well as P_3 (figure 3(b)). So it must avoid at least $\frac{|P_2|+|P_3|}{2} = \frac{3n}{8}$ points. Therefore any axis-parallel rectangle that avoids $\{p, q\}$ contains at most $\frac{5n}{8}$ points.

$\square$



(a) (b)

Figure 4: General constructions for axis-parallel rectangles

Now we discuss two general recursive constructions for $\epsilon_i^{\mathcal{R}}$.

Theorem 5. $\epsilon_{2(x+y)+1}^{\mathcal{R}} \leq \max(\frac{3}{4}\epsilon_x^{\mathcal{R}}, \frac{\epsilon_y^{\mathcal{R}}\epsilon_z^{\mathcal{R}}}{\epsilon_y^{\mathcal{R}}+\epsilon_z^{\mathcal{R}}})$ *for* $x, y \geq 0, x \geq y$ *and* $z = \lfloor \frac{x+y}{2} \rfloor$

Proof. Construct an $\epsilon_1^{\mathcal{R}}$-net $\{q\}$ for P as described in Theorem 2. Let v and h be vertical and horizontal lines through q that divide P into two vertical and two horizontal slabs respectively. Let $\delta \in [0, 1]$ be a parameter that will be fixed later. If either of the two vertical slabs contains at least δn points of P then construct an $\epsilon_x^{\mathcal{R}}$-net for the points in the slab containing at least δn points and $\epsilon_y^{\mathcal{R}}$-net for the points in the other slab. If both the vertical slabs contain less than δn points, then construct an $\epsilon_z^{\mathcal{R}}$-net for the points in each of the two vertical slabs. Repeat the same construction for the horizontal slabs also. We have thus added at most $2(x + y) + 1$ points to our ϵ-net Q.

Since $q \in Q$, any axis-parallel rectangle that avoids Q is contained in one of the (vertical or horizontal) slabs and this slab has at most $\frac{3n}{4}$ points.

First consider the case where there is a vertical or horizontal slab with at least δn points. After adding an $\epsilon_x^{\mathcal{R}}$-net to Q, any axis-parallel rectangle that avoids Q contains at most $\frac{3}{4}\epsilon_x^{\mathcal{R}} n$ points of P from this slab. Similarly, the other slab has at most $(1 - \delta)n$ points of P and any axis-parallel rectangle that avoids Q contains at most $(1 - \delta)\epsilon_y^{\mathcal{R}} n$ points of P from this slab. Thus an axis-parallel rectangle that avoids Q and is contained in one of these slabs has at most $\max(\frac{3}{4}\epsilon_x^{\mathcal{R}} n, (1 - \delta)\epsilon_y^{\mathcal{R}} n)$ points (see figure 4 (a)). In the case where both the slabs have less than δn points, any axis-parallel rectangle that avoids Q has at most $\delta\epsilon_z^{\mathcal{R}} n$ points. Thus any axis-parallel rectangle that avoids Q has at most $max(\frac{3}{4}\epsilon_x^{\mathcal{R}} n, (1 - \delta)\epsilon_y^{\mathcal{R}} n, \delta\epsilon_z^{\mathcal{R}} n)$ points. Setting $\delta = \frac{\epsilon_y^{\mathcal{R}}}{\epsilon_y^{\mathcal{R}}+\epsilon_z^{\mathcal{R}}}$ so that $(1 - \delta)\epsilon_y^{\mathcal{R}} = \delta\epsilon_z^{\mathcal{R}}$, we get $\epsilon_{2(x+y)+1}^{\mathcal{R}} \leq \max(\frac{3}{4}\epsilon_x^{\mathcal{R}}, \frac{\epsilon_y^{\mathcal{R}}\epsilon_z^{\mathcal{R}}}{\epsilon_y^{\mathcal{R}}+\epsilon_z^{\mathcal{R}}})$.

Theorem 6. $\epsilon^{\mathcal{R}}_{2(x-2)(y-1)+jx+ky} \leq \max(\frac{2\epsilon^{\mathcal{R}}_j}{x}, \frac{\epsilon^{\mathcal{R}}_k}{y})$ *for all* $x, y \geq 2$ *and* $j, k \geq 0$.

Proof. Divide P into x horizontal slabs and y vertical slabs to get a grid with each horizontal slab containing $\frac{n}{x}$ points and each vertical slab containing $\frac{n}{y}$ points. Let the horizontal slabs be denoted as $H_1, H_2, \ldots, H_x$. For a horizontal slab H_i, there are $y-1$ vertical lines of the grid intersecting it. For each of these lines, find two points(if present) of P in H_i that has the least perpendicular distance from that line on either side. Repeat this for all horizontal slabs $H_i, 2 \leq i \leq x-1$ (see figure 4 (b)). Add these (at most) $2(x-2)(y-1)$ points to the $\epsilon^{\mathcal{R}}$-net Q.

We claim that any axis-parallel rectangle that avoids these points has at most $max(\frac{2n}{x}, \frac{n}{y})$ points. If an axis-parallel rectangle intersects at most one vertical slab or at most two horizontal slabs, then it contains at most $max(\frac{2n}{x}, \frac{n}{y})$ points. Let R be an axis-parallel rectangle that intersects at least two vertical slabs and at least three horizontal slabs. Let $H_i, H_{i+1}, \ldots, H_m$ be the horizontal slabs intersected by R. R also intersects at least one vertical line and avoids the nearest points to these vertical lines in all H_l, $i+1 \leq l \leq m-1$. Therefore R cannot take points from any such H_l. Thus R can only take points from H_i and H_m and hence contains at most $\frac{2n}{x}$ points of P.

Now add an $\epsilon^{\mathcal{R}}_j$-net for points in each horizontal slab and an $\epsilon^{\mathcal{R}}_k$-net for points in each vertical slab and add these ϵ-net points to Q. Now $|Q| \leq 2(x-2)(y-1) + jx + ky$ and the result follows. $\square$

Lemma 7. $\epsilon^{\mathcal{R}}_3 \leq \frac{9}{16}$; $\epsilon^{\mathcal{R}}_5 \leq \frac{15}{32}$; $\epsilon^{\mathcal{R}}_7 \leq \frac{3}{7}$; $\epsilon^{\mathcal{R}}_9 \leq \frac{5}{13}$

Proof. The results follow from the fact that $\epsilon^{\mathcal{R}}_0 = 1$ and Theorem 5 with $x = 1, y = 0$ for $\epsilon^{\mathcal{R}}_3$; $x = 2, y = 0$ for $\epsilon^{\mathcal{R}}_5$; $x = 3, y = 0$ for $\epsilon^{\mathcal{R}}_7$; $x = 4, y = 0$ for $\epsilon^{\mathcal{R}}_9$. $\square$

Lemma 8. $\epsilon^{\mathcal{R}}_4 \leq \frac{1}{2}$; $\epsilon^{\mathcal{R}}_8 \leq \frac{2}{5}$; $\epsilon^{\mathcal{R}}_{10} \leq \frac{3}{8}$

Proof. The results follow from the fact that $\epsilon^{\mathcal{R}}_0 = 1$ and Theorem 6 with $x = 4, y = 2, j = k = 0$ for $\epsilon^{\mathcal{R}}_4$; with $x = 5, y = 2, j = 0, k = 1$ for $\epsilon^{\mathcal{R}}_8$; $x = 4, y = 2, j = 1, k = 1$ for $\epsilon^{\mathcal{R}}_{10}$. $\square$

3.3. Lower Bounds on $\epsilon^{\mathcal{R}}_i$

In this subsection, we call an axis-parallel rectangle R an α-big rectangle if $|P \cap R| \geq \alpha|P|$. Let Q be an ϵ-net and $P_1 \subset P$. We call P_1 as free if $P_1 \cap Q = \emptyset$.

Lemma 9. $\epsilon^{\mathcal{R}}_2 \geq \frac{5}{9}$

Proof. Divide the n points into three equal subsets of $\frac{n}{3}$ points each and place each subset uniformly on the bold segments as shown in figure 5. Let Q be an ϵ-net of size two. The different cases of choosing Q from this set are shown in figure 5. Let x, y denote the fraction of points from a subset which lie on one side of the point in Q as shown in the corresponding figure for each case. Let

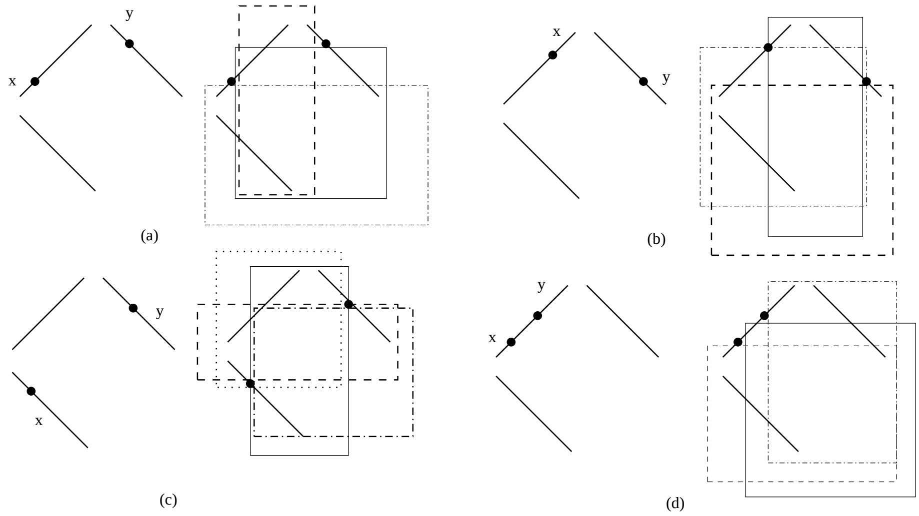


Figure 5: Lower bound for $\epsilon_2^{\mathcal{R}}$

f be a function that represents the maximum fraction of points of P that an axis-parallel rectangle contains without containing any of the points in Q. In each case, we compute the values of x and y that minimize f by considering all the rectangles that avoids the ϵ-net points.

(a) $f = max(\frac{2x}{3} + \frac{1}{3}, 1 - \frac{2(x+y)}{3}, \frac{2(1-x)}{3} + \frac{y}{3})$. f is minimized when $x = y = \frac{1}{3}$, which results in $f = \frac{5}{9}$.

(b) $f = max(\frac{2y}{3} + \frac{1}{3}, 1 - \frac{2x+y}{3}, \frac{2x}{3} + \frac{1-y}{3})$. f is minimized when $x = \frac{1}{2}, y = \frac{1}{3}$, which results in $f = \frac{5}{9}$.

(c) $f = max(1 - \frac{x+y}{3}, \frac{2x}{3} + \frac{1-y}{3}, \frac{2y}{3} + \frac{1-x}{3}, \frac{2(x+y)-1}{3})$. f is minimized when $x = y = \frac{2}{3}$, which results in $f = \frac{5}{9}$.

(d) $f = max(\frac{1}{3} + \frac{2x}{3}, \frac{1}{3} + \frac{2y}{3}, 1 - \frac{2(x+y)}{3})$. f is minimized when $x = y = \frac{1}{3}$, which results in $f = \frac{5}{9}$.

So there always exists an axis-parallel rectangle that avoids all the points in the ϵ-net and contains at least $\frac{5n}{9}$ points. $\square$

Lemma 10. $\epsilon_3^{\mathcal{R}} \geq \frac{2}{5}$

PROOF. Let P be arranged into 20 equal subsets. Each quadrant has a group of five subsets, we will denote these groups as A, B, C, D as shown in figure 6.

Clearly, all the subsets of one of the groups will be free. Assume that all the subsets in A are free. The three points in Q can be chosen in two ways.

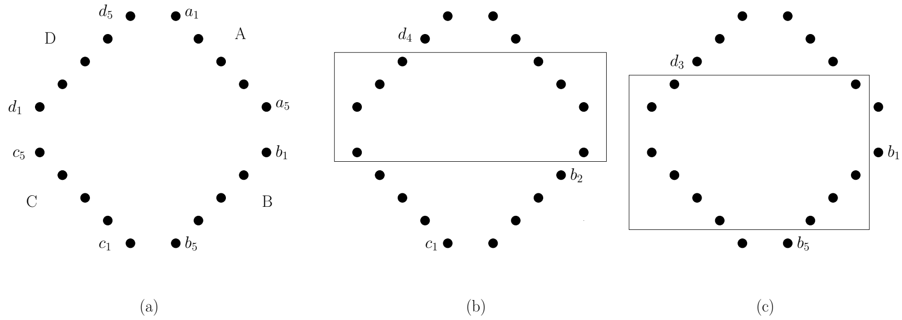


Figure 6: Lower bound for $\epsilon_3^{\mathcal{R}}$

- *Case 1: One point is chosen from each of the other three groups:* Any set of eight consecutive subsets is contained in a $\frac{2}{5}$-big axis-parallel rectangle. Since subsets in A are free, all the three subsets near A in B and D cannot be free. This means a point has to be chosen from one of the subsets b_1, b_2 or b_3. Now if a point is chosen from subset b_1 then the remaining subsets in B are free and there exists a $\frac{2}{5}$-big axis-parallel rectangle containing all those subsets and a_1, a_2, a_3, a_4 from A. Therefore b_1 has to be free. Symmetrically, d_5 also has to be free. Clearly two points have to be selected from the subsets b_2 and d_4 to avoid the $\frac{2}{5}$-big axis-parallel rectangle with consecutive subsets from D, B and A ($d_5, a_1, \ldots, a_5, b_1, b_2$ and $d_4, d_5, a_1, \ldots, a_5, b_1$). If the subset c_1 is free then there is a $\frac{2}{5}$-big axis-parallel rectangle containing the free subsets $d_5, a_1, a_2, a_3, b_3, b_4, b_5, c_1$. Therefore the three points have to be chosen from the subsets b_2, c_1, d_4. Now there exists a $\frac{2}{5}$-big axis-parallel rectangle as shown in figure 6(b) that avoids these subsets.

- *Case 2: Two points are chosen from a group and the third point is chosen from the diagonally opposite group:* Assume that two points are chosen from B and one point from D. Clearly, a point has to be chosen from the subset d_3 to avoid the $\frac{2}{5}$-big axis-parallel rectangles with consecutive subsets from D, A and D, C ($d_3, d_4, d_5, a_1, \ldots, a_5$ and $c_1, \ldots, c_5, d_1, d_2, d_3$). The other subsets in D are free. The next two points chosen have to be from the subsets b_1 and b_5 to avoid the $\frac{2}{5}$-big axis-parallel rectangles containing consecutive subsets from B, C, D ($b_5, c_1, \ldots, c_5, d_1, d_2$) and D, A, B ($d_4, d_5, a_1, \ldots, a_5, b_1$). Now there exists a $\frac{2}{5}$-big axis-parallel rectangle as shown in figure 6(c) which avoids the subsets b_1, b_5, d_3.

□

Lemma 11. $\epsilon_4^{\mathcal{R}} \geq \frac{3}{10}$

PROOF. Let P be arranged as ten equal subsets as shown in figure 7. The eight subsets in between the topmost and bottommost subsets are arranged as two

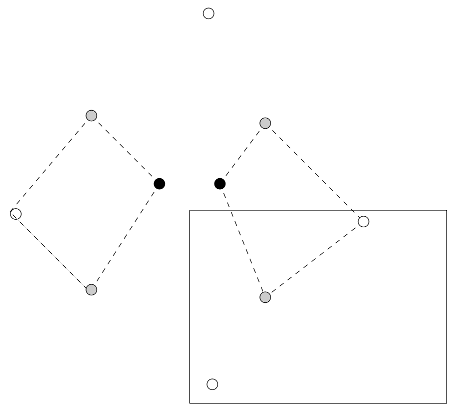


Figure 7: Lower bound for $\epsilon_4^{\mathcal{R}}$

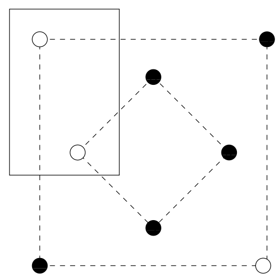


Figure 8: Lower bound for $\epsilon_5^{\mathcal{R}}$

quadrilaterals. Let Q be an ϵ-net of size four. We claim that there always exists a $\frac{3}{10}$-big axis-parallel rectangle that avoids Q.

It is easy to see that there can be at most two free subsets in each of the quadrilaterals since there is a $\frac{3}{10}$-big axis-parallel rectangle that contains any three free subsets from a quadrilateral. Hence all the four points of Q have to be chosen from the subsets forming the two quadrilaterals(two points from each quadrilateral). Therefore the top and bottom subsets are free. Both these subsets form a $\frac{3}{10}$-big axis-parallel rectangle with the black subsets in the quadrilateral (see figure 7). Therefore two of the points in Q have to be chosen from the black subsets. The two gray subsets in the lower(resp. upper) half form a $\frac{3}{10}$-big axis-parallel rectangle with the bottommost(resp. topmost) subset. Hence the remaining two points in Q have to be chosen from the gray subsets(one point from the top gray subsets and one from the bottom gray subsets). Thus one gray subset in the lower half is free which forms a $\frac{3}{10}$-big axis-parallel rectangle that avoids Q as shown in figure 7. $\square$

Lemma 12. $\epsilon_5^{\mathcal{R}} \geq \frac{1}{4}$

Proof. Let P be arranged as eight equal subsets, four in the outer layer and four in the inner layer (see figure 8). Let Q be an ϵ-net of size five. We claim that there always exists a $\frac{1}{4}$-big axis-parallel rectangle that avoids Q.

It is clear that we have to choose a point each from at least three of the inner layer subsets, otherwise there can be an axis-parallel rectangle that contains all the points of two free subsets. Also, a point each has to be chosen from at least two of the outer subsets to avoid the axis-parallel rectangles that contains all the points of two consecutive subsets in the outer layer. Moreover, these two points are to be chosen from diagonally opposite subsets. Let the ϵ-net points be chosen from the five black subsets as shown in figure 8. Now there exists a $\frac{1}{4}$-big axis-parallel rectangle containing the free subset in the inner layer and the nearest free subset in the outer layer.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8	ϵ_9	ϵ_{10}
LB	3/4	5/9	2/5	3/10	1/4	1/5	5/29	2/13	3/22	1/8
UB	3/4	5/8	9/16	1/2	15/32	15/32	3/7	2/5	5/13	3/8

Table 1: Summary of lower and upper bounds for $\epsilon_i^{\mathcal{R}}$.

Lemma 13. $\epsilon_6^{\mathcal{R}} \geq \frac{1}{5}$; $\epsilon_7^{\mathcal{R}} \geq \frac{5}{29}$; $\epsilon_8^{\mathcal{R}} \geq \frac{2}{13}$; $\epsilon_9^{\mathcal{R}} \geq \frac{3}{22}$; $\epsilon_{10}^{\mathcal{R}} \geq \frac{1}{8}$

PROOF. The results follow from theorem 1 with $j = k = 3$ for $\epsilon_6^{\mathcal{R}}$; $j = 2, k = 5$ for $\epsilon_7^{\mathcal{R}}$; $j = 3, k = 5$ for $\epsilon_8^{\mathcal{R}}$; $j = 4, k = 5$ for $\epsilon_9^{\mathcal{R}}$; $j = 5, k = 5$ for $\epsilon_{10}^{\mathcal{R}}$.

Lemma 14. $\epsilon_i^{\mathcal{R}} \geq \frac{10}{9i}$, *for* $i \geq 2$

Proof. We use mathematical induction to prove this. The lemma holds for $i = 2$ and $i = 3$ since $\epsilon_2^{\mathcal{R}} \geq \frac{5}{9} \geq \frac{10}{9\times 2}$ (by lemma 9) and $\epsilon_3^{\mathcal{R}} \geq \frac{2}{5} \geq \frac{10}{9\times 3}$ (by lemma 10). Now assume the lemma holds for $i = k$, i.e., $\epsilon_k^{\mathcal{R}} \geq \frac{10}{9k}$ or $\frac{9}{10}k \geq \frac{1}{\epsilon_k^{\mathcal{R}}}$. Thus, $\frac{9}{10}k + \frac{9}{5} \geq \frac{1}{\epsilon_k^{\mathcal{R}}} + \frac{1}{\epsilon_2^{\mathcal{R}}}$, i.e., $\frac{9}{10}(k+2) \geq \frac{\epsilon_k^{\mathcal{R}} + \epsilon_2^{\mathcal{R}}}{\epsilon_k^{\mathcal{R}} \epsilon_2^{\mathcal{R}}}$. Therefore, $\epsilon_{k+2}^{\mathcal{R}} \geq \frac{\epsilon_k^{\mathcal{R}} \epsilon_2^{\mathcal{R}}}{\epsilon_k^{\mathcal{R}} + \epsilon_2^{\mathcal{R}}} \geq \frac{10}{9(k+2)}$ (from theorem 1). Hence the result follows for all values of i.

□

The summary of lower and upper bounds for $\epsilon_i^{\mathcal{R}}$ is given in table 1.

4. Half spaces

Let P be a set of n points in $\mathbb{R}^2$ and $\mathcal{H}$ be the family of halfspaces. In this section, we prove bounds on $\epsilon_i^{\mathcal{H}}$.

Lemma 15. $\epsilon_i^{\mathcal{H}} \leq \frac{2}{i+1}$

Proof. The proof of this lemma is similar to that of Theorem 4.1 in [13]. For $i = 2$, [13] gives a constructive proof to show that $\epsilon_2^{\mathcal{H}} \leq \frac{2}{3}$. For $i \geq 3$, [13] gives an existential proof showing that a strong ϵ-net of size $\lceil \frac{2}{\epsilon} \rceil - 1$ exists. Based on this proof, we give an explicit construction for $\frac{2}{i+1}$-net of size i.

Let $c_1, c_2, \ldots, c_m$ be the vertices of the convex hull of P, in clockwise order. Let N represent the ϵ-net. We will denote the chosen ϵ-net points as $n_1, n_2, \ldots$. Initially $N = \{c_1\}$ and $x = c_1$.

Let c_z be the first vertex encountered in the clockwise direction such that the halfspace formed by $\overline{xc_{z+1}}$ and containing c_z has $> \frac{2}{i+1}n$ points. Add c_z to N and set $x = c_z$. Repeat this procedure till we have considered all vertices of the convex hull i.e, we have added the point n_{k+1} to N such that n_{k+1} is c_1 or the halfspace $\overline{n_k n_{k+1}}$ contains c_1. In the latter case, there are two possibilities:

- $n_{k+1} = n_2$: Now n_1 is redundant and is removed from N. Rename n_k as n_1.
- n_{k+1} lies between n_1 and n_2 : n_{k+1} is a redundant point and is removed from N.

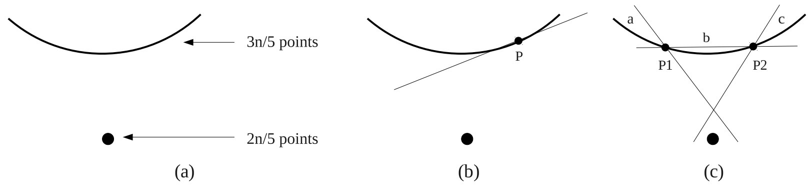


Figure 9: Lower bound for $\epsilon_2^{\mathcal{H}}$

Clearly any halfspace that avoids N has at most $\frac{2}{i+1}n$ points. Hence N is a $\frac{2}{i+1}$-net. We claim that N has at most i points.

Let N have l points and let H_j be the halfspace formed by the line $\overline{n_{j-1}n_{j+1}}$ and containing n_j. Let $h_j = |H_j \cap P|$. From the above construction, we know that $h_j > \frac{2n}{i+1}$ for all j, $1 \leq j \leq l$. Therefore $\sum_{j=1}^{l} h_j > \frac{2l}{i+1}n$. Since any point in P is present in at most two H_j's, $\sum_{j=1}^{l} h_j \leq 2n$. Combining the above two inequalities, we have $l < i + 1$ and the result follows.

□

Lemma 16. *[13]*

$$\epsilon_i^{\mathcal{H}} \geq \begin{cases} \frac{2}{i+1} & \textit{for } i, \textit{odd} \\ \frac{2}{i+2} & \textit{for } i, \textit{even} \end{cases}$$

Proof. The result follows from the lower bounds given in [13]. For odd values of i, consider a point set divided into $\frac{i+1}{2}$ equal subsets, as described in [13]. For any ϵ-net N of size i, there exists a halfspace that avoids N and contains all the $\frac{2}{i+1}n$ points from one subset. For even values, the lower bound for $\epsilon_i^{\mathcal{H}}$ follows from that for $\epsilon_{i+1}^{\mathcal{H}}$.

□

Lemma 17. $\epsilon_2^{\mathcal{H}} \geq \frac{3}{5}$

Proof. Let P be a set of n points arranged as a subset S of $\frac{3n}{5}$ points uniformly placed on the arc of a circle with angle less than π and a subset T of $\frac{2n}{5}$ points placed close together, at a sufficiently large distance from S so that the tangent to the arc at any point $p \in S$ has $S \backslash \{p\}$ and T on different sides of it (see figure 9(a)).

If only one point in the ϵ-net is chosen from S, the tangent to the arc passing through it results in a halfspace containing $\frac{3n}{5} - 1$ points (figure 9(b)). So assume that both the ϵ-net points are chosen from S. Let p_1 and p_2 be the ϵ_2-net points chosen from S. The points in S are divided into three arcs by p_1 and p_2. One of the arcs contain at least $\frac{n}{5}$ points. There exists a halfspace avoiding p_1 and p_2 that contains all the points of this arc and all the points of T (see figure 9(c)).

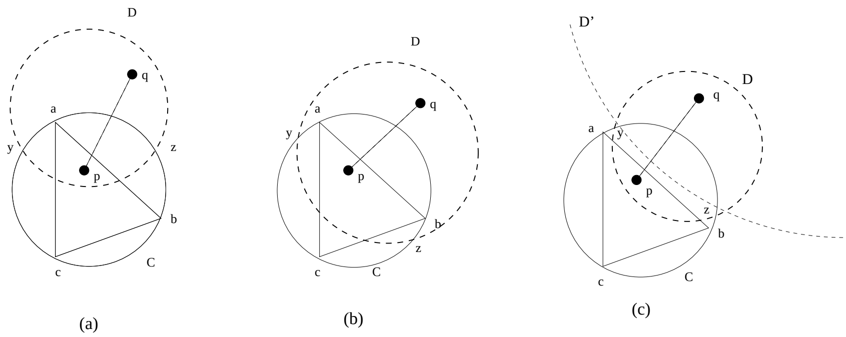


Figure 10: Upper bound for $\epsilon_2^{\mathcal{D}}$

5. Disks

In this section, we consider the family of disks. We show an upper bound for $\epsilon_2^{\mathcal{D}}$. The proof is motivated by the construction for $\epsilon_2^{\mathcal{H}}$ given in [13].

Theorem 18. $\epsilon_2^{\mathcal{D}} \leq \frac{2}{3}$

Proof. Let P be a set of n points and p be the centerpoint of P. Let T be the Delaunay triangulation of P. Let a, b, c be points in P such that they are the vertices of the delaunay triangle in T that contains p. For all $x \in P \setminus \{a, b, c\}$, consider the line segment connecting x and p. Each of these $n-3$ line segments intersect one of the edges ab, bc or ac and at least one of them, say ab, has at least one-third of these line segments intersecting it. We claim that $\{a, b\}$ is a $\frac{2}{3}$-net.

Let $P_1 \subset P$ be the set of points q such that the line segment pq intersects the edge ab. Consider the circumcircle C of triangle abc and let C_{ab} be the arc between a and b that is intersected by the line segment pq, for any $q \in P_1$. By the definition of delaunay triangulation, C does not contain any point of P. Any disk D that avoids $\{a, b\}$ and contains more than $\frac{2n}{3}$ points of P contains the centerpoint p and at least one point $q \in P_1$, since $|P_1| \geq \frac{n-3}{3}$. Therefore D intersects C at two points y, z. If exactly one of these points lie on the arc C_{ab} then D contains either a or b (see figure 10(a)). Similarly if both y and z are outside the arc C_{ab}, then D contains both the points a and b (see figure 10(b)). Therefore assume that both y and z lie on the arc C_{ab}. In this case, D contains only points from P_1. Then we can have another disk D' such that $(D \cap P) \subset (D' \cap P)$ and $p \notin D'$ (see figure 10(c)). This implies that D' contains more than $\frac{2n}{3}$ points and does not contain the centerpoint, a contradiction.

$\square$

The lower bounds $\epsilon_i^{\mathcal{D}} \geq \frac{2}{i+1}$ for odd values of i and $\epsilon_i^{\mathcal{D}} \geq \frac{2}{i+2}$ for even values of i follow from the lower bound results for halfspaces.

	Rectangles		Half Spaces		Disks	
	LB	UB	LB	UB	LB	UB
ϵ_1	3/4		1		1	
ϵ_2	5/9	5/8	3/5	2/3	3/5	2/3
ϵ_3	2/5	9/16	1/2		1/2	2/3

Table 2: Summary of lower and upper bounds for ϵ_i.

Conclusions and Open Questions

In this paper, we have shown lower and upper bounds for $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is the family of axis-parallel rectangles, halfspaces and disks. A summary of the bounds for $i \leq 3$ are given in Table 2. An interesting open question is to find the exact value of $\epsilon_i^{\mathcal{S}}$ for small values of i.

Acknowledgements

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References

[1] N. Alon. A non-linear lower bound for planar epsilon-nets. *Discrete & Computational Geometry*, 47(2):235–244, 2012.

[2] B. Aronov, F. Aurenhammer, F. Hurtado, S. Langerman, D. Rappaport, C. Seara, and S. Smorodinsky. Small weak epsilon-nets. *Computational Geometry*, 42(5):455–462, 2009.

[3] B. Aronov, E. Ezra, and M. Sharir. Small-size epsilon-nets for axis-parallel rectangles and boxes. *SIAM Journal on Computing*, 39(7):3248–3282, 2010.

[4] P. Ashok, S. Govindarajan, and J. Kulkarni. Small strong epsilon nets. In *Proceedings of the 22nd Canadian Conference on Computational Geometry (CCCG2010)*, pages 155–158, 2010.

[5] M. Babazadeh and H. Zarrabi-Zadeh. Small weak epsilon-nets in three dimensions. In *Proceedings of the 18th Canadian Conference on Computational Geometry*, pages 47–50, 2006.

[6] B. Bukh, J. Matoušek, and G. Nivasch. Lower bounds for weak epsilon-nets and stair-convexity. *Israel Journal of Mathematics*, 182(1):199–228, 2011.

[7] B. Chazelle, H. Edelsbrunner, M. Grigni, L. Guibas, M. Sharir, and E. Welzl. Improved bounds on weak ε-nets for convex sets. *Discrete & Computational Geometry*, 13(1):1–15, 1995.

[8] K. Clarkson, D. Eppstein, G. Miller, C. Sturtivant, and S. Teng. Approximating center points with iterated radon points. In *Proceedings of the ninth annual symposium on Computational geometry*, pages 91–98, 1993.

[9] K. Clarkson and K. Varadarajan. Improved approximation algorithms for geometric set cover. *Discrete & Computational Geometry*, 37(1):43–58, 2007.

[10] M. Dulieu. ϵ-nets faibles. Master's thesis, Mèmoire de Licence, Universitè Libre de Bruxelles,, 2006.

[11] D. Haussler and E. Welzl. ϵ-nets and simplex range queries. *Discrete & Computational Geometry*, 2(1):127–151, 1987.

[12] S. Jadhav and A. Mukhopadhyay. Computing a centerpoint of a finite planar set of points in linear time. *Discrete & Computational Geometry*, 12(1):291–312, 1994.

[13] J. Komlós, J. Pach, and G. Woeginger. Almost tight bounds for ϵ-nets. *Discrete & Computational Geometry*, 7(1):163–173, 1992.

[14] J. Matoušek, R. Seidel, and E. Welzl. How to net a lot with little: small *epsilon*-nets for disks and halfspaces. In *Proceedings of the sixth annual symposium on Computational geometry*, pages 16–22, 1990.

[15] J. Matousek and U. Wagner. New constructions of weak ϵ-nets. *Discrete & Computational Geometry*, 32(2):195–206, 2004.

[16] G. Miller and D. Sheehy. Approximate centerpoints with proofs. *Computational Geometry*, 43(8):647–654, 2010.

[17] G. Miller, S. Teng, W. Thurston, and S. Vavasis. Automatic mesh partitioning. *Institute for Mathematics and Its Applications*, 56:57, 1993.

[18] G. Miller, S. Teng, W. Thurston, and S. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *Journal of the ACM (JACM)*, 44(1):1–29, 1997.

[19] G. L. Miller, T. Phillips, and D. R. Sheehy. The centervertex theorem for wedge depth. In *Proceedings of the 21st Canadian Conference on Computational Geometry (CCCG2009)*, pages 79–82, 2009.

[20] N. Mustafa and S. Ray. Weak ϵ-nets have basis of size $o(\frac{1}{\epsilon} log \frac{1}{\epsilon})$ in any dimension. *Computational Geometry*, 40(1):84–91, 2008.

[21] N. Mustafa and S. Ray. An optimal extension of the centerpoint theorem. *Computational Geometry*, 42(6-7):505–510, 2009.

[22] J. Pach and G. Tardos. Tight lower bounds for the size of epsilon-nets. In *Proceedings of the 27th annual ACM symposium on Computational geometry*, pages 458–463, 2011.

[23] E. Pyrga and S. Ray. New existence proofs for ε-nets. In *Proceedings of the twenty-fourth annual symposium on Computational geometry*, pages 199–207, 2008.

[24] R. Rado. A theorem on general measure. *Journal of the London Mathematical Society*, 1(4):291–300, 1946.

[25] K. Varadarajan. Epsilon nets and union complexity. In *Proceedings of the 25th annual symposium on Computational geometry*, pages 11–16, 2009.

[26] F. Yao. A 3-space partition and its applications. In *Proceedings of the fifteenth annual ACM symposium on Theory of computing*, pages 258–263, 1983.