On piercing (pseudo)lines and boxes

Subramanya Bharadwaj B. V[*], Chintan H. Rao[†], Pradeesha Ashok[‡], Sathish Govindarajan [§]

Abstract

We say a family of geometric objects C has (l, k)-property if every subfamily $C' \subseteq C$ of cardinality at most l is k-piercable. In this paper we investigate the existence of $g(k, d)$ such that if any family of objects C in $\mathbb{R}^d$ has the $(g(k, d), k)$-property, then C is k-piercable. Danzer and Grünbaum showed that $g(k, d)$ is infinite for families of boxes and translates of centrally symmetric convex hexagons. In this paper we show that any family of pseudolines(lines) with $(k^2 + k + 1, k)$-property is k-piercable and extend this result to certain families of objects with discrete intersections. This is the first positive result for arbitrary k for a general family of objects. We also pose a relaxed version of the above question and show that any family of boxes in $\mathbb{R}^d$ with (k^{2d}, k)-property is $2^d k$-piercable.

1 Introduction

A family of geometric objects C in $\mathbb{R}^d$ is said to be k-piercable if there exists a set of points $P \subset \mathbb{R}^d$ of cardinality k such that every object in C contains (is pierced by) at least one of the points of P.

Definition 1 *We say a family of geometric objects C has (l, k)-property if every subfamily $C' \subseteq C$ of cardinality at most l is k-piercable.*

The classical Helly's theorem [8] stated in this notation is as follows: *Any family of convex objects C in $\mathbb{R}^d$ having $(d + 1, 1)$-property is 1-piercable.*

Helly-type theorems have been widely studied for different settings (see surveys [5, 6]). Danzer and Grünbaum [4] considered the following generalised version of Helly's theorem:

For every positive integer k, does there exist a finite $g(k, d)$ such that if any family of convex objects C in $\mathbb{R}^d$ has $(g(k, d), k)$-property, then C is k-piercable?

They showed that $g(k, d)$ is infinite even for families of boxes in $\mathbb{R}^d$. Specifically, they gave a generic construction and showed that $g(k, d)$ is infinite for all $k \geq 3$, $d \geq 2$ and $(k, d) \neq (3, 2)$. The same construction also works as a counterexample for hypercubes in $\mathbb{R}^d$. Katchalski et al [9] showed that $g(k, d)$ is infinite for translates of a symmetric convex hexagon.

Positive results are known for small values of k (i.e. $k = 2$). Danzer and Grünbaum [4] showed that for a family of boxes in $\mathbb{R}^d$, $g(2, d) = 3d - 1$ if d is even and $g(2, d) = 3d$ if d is odd. They also proved that $g(3, 2) = 16$ for a family of rectangles in $\mathbb{R}^2$. Katchalski et al [9] showed that for a family of homothetic triangles in $\mathbb{R}^2$, $g(2, 2) = 9$.

In this paper, we obtain the first positive results for general k. We show that for a family of pseudolines in $\mathbb{R}^2$, $g(k, 2)$ is finite for all $k \geq 2$. Specifically, we prove the following:

Theorem 1 *Let C be a family of pseudolines in $\mathbb{R}^2$ with $|C| \geq k^2 + k + 1$. For any integer $k \geq 2$, if C has $(k^2 + k + 1, k)$-property then C is k-piercable.*

We extend the above theorem for families of objects C with the following property: any subfamily of $p + 1$ distinct objects in C intersect in at most one point. Note that $p = 1$ for a family of pseudolines.

Theorem 2 *Let C be a family of objects with the property that any subfamily of $p + 1$ distinct objects in C intersect in at most one point. Let $|C| \geq k(kp + 1) + 1$. For any integer $k \geq 2$, if C has $(k(kp + 1), k)$-property then C is k-piercable.*

The proof of Theorem 1 and 2 are combinatorial and exploit only the intersection property. In fact, Theorem 2 is true for set systems with the property that any subfamily of $p + 1$ distinct sets intersect in at most one element. Also the proofs lead naturally to a FPT algorithm for the minimum piercing problem on these objects. Note that the minimum piercing problem is NP-hard and APX-hard even for lines in $\mathbb{R}^2$ [12, 3].

Since $g(k, d)$ is infinite for most families of geometric objects in the above problem, we define the following relaxed variant, which we refer to as the k-Helly problem:

k-Helly problem: *For every positive integer k, determine the smallest $f(k, d)$ such that if any family of convex objects C in $\mathbb{R}^d$ has $(g(k, d), k)$-property for some $g(k, d)$, then C is $f(k, d)$-piercable.*

The k-Helly problem is related to the weak ϵ-net [1] and Hadwiger-Debrunner (p, q)-problem [7] as follows:

[*]Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, subramanya@csa.iisc.ernet.in

[†]Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, chintanraoh@csa.iisc.ernet.in

[‡]Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, pradeesha@csa.iisc.ernet.in

[§]Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India gsat@csa.iisc.ernet.in

Weak ϵ-nets is a special case of the k-Helly problem : In the weak ϵ-net problem, we ask for a piercing set for objects containing $> \epsilon n$ points. By pigeon hole principle, in any subcollection of $\frac{1}{\epsilon}+1$ objects, two will intersect. Therefore the objects satisfy $(\frac{1}{\epsilon}+1, \frac{1}{\epsilon})$-property.

The k-Helly problem is a special case of the Hadwiger-Debrunner (p,q)-problem since $(g(k,d),k)$-property implies Hadwiger-Debrunner (p,q)-property for $p = g(k,d), q = g(k,d)/k$. Also, the finiteness of $g(k,d), f(k,d)$ is implied by the Hadwiger Debrunner (p,q) theorem [2], which shows a finite piercing set. However, the bounds given by the Hadwiger Debrunner (p,q) theorem are large (roughly $O(p^6)$ for convex objects in $\mathbb{R}^2$).

We show the following result for boxes in $\mathbb{R}^d$:

Theorem 3 *Let C be a family of boxes in $\mathbb{R}^d$. For any $k \geq 2$, if C has (k^{2d},k)-property, then C is $2^d k$-piercable.*

Note that for boxes in $\mathbb{R}^d$, $f(k,d) > k$ since otherwise $g(k,d)$ is infinite. The proof of Theorem 3 directly leads to a 2^d-approximate FPT algorithm for the minimum piercing problem on boxes. We note that the minimum piercing problem for boxes is NP-hard as well as W[1]-hard [11].

2 Lines and Pseudolines

Any two lines in $\mathbb{R}^2$ intersect in at most one point. This can be generalized in the following way.

Definition 2 *A family of geometric objects C in $\mathbb{R}^2$ is called a family of pseudolines if for every $l_i, l_j \in C$, l_i and l_j intersect in at most one point.*

Let C be a finite family of pseudolines in $\mathbb{R}^2$.

Definition 3 *Let a point x lie in the intersection of a set of pseudolines $l_1, l_2, \cdots, l_s \in C$. We call x k-degenerate in C if $s > k$.*

Lemma 4 *Let H be a set of points that pierces C. If x is k-degenerate in C and $x \notin H$, then $|H| \geq k+1$.*

Proof. If $x \notin H$ then we need at least s points to hit the s pseudolines passing through x. Since $s > k$ the lemma follows. □

Lemma 5 *Let $|C| \geq (k^2+k+1)$ and G be the set of all k-degenerate points in C. If C has (k^2+k+1,k)-property, then $1 \leq |G| \leq k$*

Proof. Let S be a subset of C such that $|S| = k^2+k+1$. S is k-piercable. By pigeon hole principle, there exists a point x that pierces at least $k+1$ pseudolines in S. Hence $|G| \geq 1$. Also if $G \geq k+1$, there exists $S' \subset C$ which contains $k+1$ pseudolines passing through each of the first k points in G and one pseudoline passing through the $(k+1)$th point which does not pass through the first k points. Clearly $|S'| \leq k^2+k+1$ and S' is not k-piercable, a contradiction. Hence $1 \leq |G| \leq k$. □

Proof of Theorem 1. Let G be the set of all k-degenerate points in C. From the Lemma 5 $1 \leq |G| \leq k$. Let C' be the set of pseudolines not pierced by any of the points in G. We claim that if $|G| = k$ then $C' = \emptyset$. For if $C' \neq \emptyset$ then there is a $l \in C'$ such that it is not pierced by any point in G. For each point in G, pick $k+1$ pseudolines passing through it. This together with l gives a set C'' of at most $k(k+1)+1 = k^2+k+1$ pseudolines which is not k-piercable, a contradiction.

Hence let $|G| = k-r$ where $k > r \geq 1$ and $C' \neq \emptyset$. We claim $|C'| \leq rk$. Assume, for contradiction, that $|C'| > rk$. Then, for each point in G, pick $k+1$ pseudolines passing through it. This together with $rk+1$ lines from C' to give a set C'' of (at most) $(k-r)(k+1)+rk+1 = k^2+(k-r)+1 < k^2+k+1$ pseudolines. C'', being a subset of C, has (k^2+k,k)-property and hence can be pierced by k points. Any point in G can pierce only $k+1$ pseudolines in C'' and no r points outside G can pierce the remaining $rk+1$ pseudolines in C'', a contradiction.

Now as before we pick $k+1$ pseudolines from each of the $k-r$ k-degenerate points together with at most rk pseudolines from C' to get a system of (at most) $(k-r)(k+1)+rk = k^2+k-r < k^2+k+1$ pseudolines. This can be pierced by k points. We have to choose each of the $k-r$ k-degenerate points in a piercing set for this system. This means that the rk pseudolines from C'(none of them are pierced by the degenerate points) have to be pierced by r points. This implies that C is k-piercable.

Lemma 6 *Let C be a family of pseudolines with $|C| \geq 6$. If C has $(6,2)$-property then C is 2-piercable.*

Proof. As C has $(6,2)$-property, there exist two cases. There exist some 6 pseudolines out of which 5 do not intersect or out of every 6 pseudolines 5 intersect.

In the first case there are two sub cases. There exist $l_1, \ldots, l_6 \in C$ such that l_1, l_2, l_3, l_4 intersect or in the second sub case l_1, l_2, l_3 and l_4, l_5, l_6 intersect respectively. Let $l \in C$. If l_1, l_2, l_3, l_4 intersect, then l is incident on the intersection of l_1, l_2, l_3 or on the intersection of l_5, l_6. Otherwise $l_1, l_2, l_3, l_5, l_6, l$ is a set of 6 pseudolines which are not 2-piercable. If l_1, l_2, l_3 and l_4, l_5, l_6 intersect, then l is incident on the intersection of l_1, l_2, l_3 or l_5, l_6. Otherwise $l_1, l_2, l_3, l_5, l_6, l$ is a set of 6 lines which are not 2-piercable. Hence in both sub cases C is 2-piercable.

In the second case when out of every 6 pseudolines 5 intersect, all the lines except one have a common intersection and hence C is 2-piercable.

Hence in either case C is 2-piercable. □

The above result is tight since there is a family of 6 lines with $(5,2)$-property which is not 2-piercable (shown in Figure 1).

Consider a collection of pseudolines C. We wish to determine if C is k-piercable or not. There is a naive FPT algorithm which is implied by the above combinatorial result

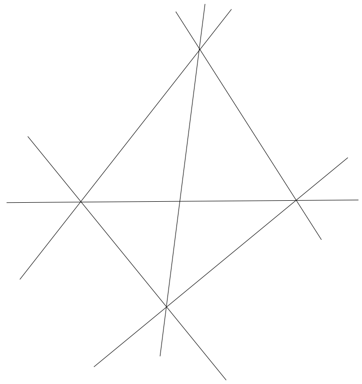

Figure 1: A family of 6 lines with $(5, 2)$-property which is not 2-piercable.

which takes $O(n^2 + k^3k^{4k})$ time. However one can use the techniques given in [10] to get a faster FPT algorithm which takes $O(n^2 + k^{2k+2})$ time.

3 Objects with discrete intersection

We extend the results of the previous section to a more general family of objects. We consider families of objects C with the following property : any subfamily of $p+1$ distinct objects intersect in at most one point. This is a notion similar to the one in [10]. Unit circles, curves in the plane obtained by polynomial equations of bounded degree are some examples of objects with the above property.

Definition 4 *Let a point x lie in the intersection of a set of objects $c_1, c_2, \cdots, c_s \in C$. We call x k-degenerate in C if $s > kp$.*

Lemma 7 *Let H be a set of points that pierces C. If x is k-degenerate in C and $x \notin H$, then $|H| \geq k+1$.*

Proof. Any point $y \neq x$ can pierce at most $p-1$ of the objects passing through x. Hence we need at least $k+1$ points to pierce $kp+1$ objects passing thorugh x. □

Consider a set of objects C with $|C| \geq k(kp+1)+1$ which has $(k(kp+1)+1, k)$-property. Consider a subset $S \subseteq C$ with $|S| = k(kp+1)$. Note that it can be pierced by a set H of size k. Then there is some point $x_1 \in H$ which pierces at least $kp+1$ objects in C. We construct a set of degenerate points as follows. Let $C_1 = C$, $G_1 = \{x_1\}$ where x_1 is obtained as before. Construct G_{i+1}, $i \geq 1$, as long as possible, in the following way: $C_{i+1} = C_i \setminus \{c \in C_i : c \text{ pierced by } x_i\}$. Let x_{i+1} be any k-degenerate point in C_{i+1}. Now $G_{i+1} = G_i \cup x_{i+1}$.

Lemma 8 *Consider a family of objects C with $|C| \geq k(kp+1)+1$ which has the $(k(kp+1), k)$ property. Let G be a set of k-degenerate points in C with maximum cardinality. Then $1 \leq |G| \leq k$.*

Proof. Clearly $|G| \geq 1$. Suppose $|G| \geq k+1$. Let C^i be a subset of objects pierced by $x_i \in G$ with $|C^i| = kp+1$. Consider $X = \cup_{1 \leq i \leq k} C^i \cup \{c\}$ where $c \in C^{k+1}$. Clearly $|X| \leq k(kp+1)+1$. Hence X is k-piercable. Any k-piercing set for C' must contain $\{x_1, \cdots x_k\}$ (by Lemma 7). This is a contradiction as $\forall x_i \in G, 1 \leq i \leq k$, x_i cannot pierce c. □

Proof of Theorem 2. Let G be a set of k-degenerate points in C with maximum cardinality and let $|G| = k - r$ where $k > r \geq 0$ (by Lemma 8). Let C^i be a subset of objects pierced by $x_i \in G$ with $|C^i| = kp+1$.

Let C' be the set of remaining objects not pierced by any of these points. If $C' = \emptyset$ then C is k-piercable. Hence let us assume $C' \neq \emptyset$. We claim that $|C'| \leq r(kp+1)$.

Assume, for contradiction, that $|C'| = r(kp+1)+1$. Then a subset of objects $X = C^1 \cup \cdots C^{k-r} \cup C'$ is k-piercable since $|X| \leq k(kp+1)+1$. Any k-piercing set for X must contain all $k-r$ points in G. If $r = 0$ this means that a object in C' is not pierced, a contradiction. Else if $r > 0$ this implies $r(kp+1)$ objects in C' is pierced by r points, all of which are not k-degenerate, a contradiction.

Hence $|C'| \leq r(kp+1)$. Again as before a subset of objects $X = C^1 \cup \cdots C^{k-r} \cup C'$ is k-piercable since $|X| \leq k(kp+1)+1$. Any k-piercing set for X must contain all $k-r$ points in G. This implies C' is r-piercable(if $r = 0$ this means $C' = \emptyset$). Hence C is k-piercable.

We extend the result on lines in the previous section to hyperplanes in 3 dimensions. The idea of replacing degenerate hyperplanes by a line is from [10].

Lemma 9 *Let C be a family of hyperplanes in R^3 with $(k(k+1)^3, k)$ property. Then C is k-piercable.*

Proof. We obtain a family of objects C' from C as follows. If at least $k+1$ hyperplanes intersect in a line then we replace them with the line.

It is obvious that if C' is k-piercable then C is k-piercable. We note that if any $k+1$ hyperplanes in C intersect in a line l then any k-piercing set must contain a point from l. Hence C' is k-piercable if and only if C is k-piercable.

We claim that any set of $k+2$ objects in C' intersect in at most 1 point. There are two cases - the set contains at least two lines or the set contains at most one line. The claim is true if there are at least two lines in this set of $k+2$ objects. In the other case, the set contains at least $k+1$ hyperplanes and these cannot intersect in a line. Hence the $k+2$ objects intersect at most 1 point.

From Theorem 2 if C' has $(k(k(k+2)+1), k)$ property then C' is k-piercable. Any line in C' can be realized as the intersection of at most $k+1$ hyperplanes in C. Hence if C has $((k+1)k(k(k+2)+1), k)$ property then C is k-piercable which proves the claim. □

This result can be extended to higher dimensions.

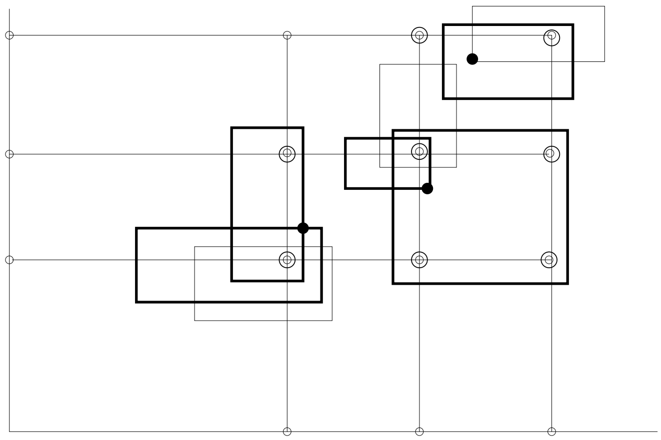

Figure 2: Grid points, representative rectangles and piercing set for a collection of rectangles.

4 Boxes in $\mathbb{R}^d$

In this section we consider the k-Helly problem for families of boxes in $\mathbb{R}^d$.

Lemma 10 *Let I be a family of intervals in $\mathbb{R}$ with $(k+1, k)$ property. Then I is k-piercable.*

Proof. We note that I satisfies the Hadwiger Debrunner $HD(k+1, 2)$ property. Hence I has a piercing set of size k [7]. □

Lemma 11 *Let S be a family of vertical and horizontal slabs with $(k+1, k)$-property. S is k-piercable.*

Proof. Let S_1 be the set of vertical slabs and S_2 the set of horizontal slabs. Clearly from Lemma 10, S_1 and S_2 are k-piercable. Without loss of generality let S_1 and S_2 be 'pierced' by k points on the x axis $v_1, \dots, v_k$ and k points on the y axis $h_1, \dots, h_k$ respectively. $(v_1, h_1), \dots, (v_k, h_k)$ is a k-piercing set for S. □

Proof of Theorem 3. We prove the claim for $d = 2$. The proof for $d > 2$ is a straightforward generalization. Let C be a family of rectangles in $\mathbb{R}^2$ with (k^4, k)-property. We orthogonally project each rectangle $r \in C$ to the two coordinate axes. For each axis i, $1 \le i \le 2$, we get a set of intervals C_i with (k^4, k)-property. Hence C_i has $(k+1, k)$-property and is k-piercable (Lemma 10). Let H_i, $|H_i| \le k$, be such a piercing set (the small hollow points on the x and y axes in Figure 2). Consider the grid points $H = \{(x, y) : x \in H_1, y \in H_2\}$ (the small hollow circles in Figure 2). For $r \in C$, let r_1 be the projection of r on the x axis and r_2 be the projection of r on the y axis. There exist $x \in H_1, y \in H_2$ such that x pierces r_1 and y pierces r_2. Thus $(x, y) \in H$ pierces r. Hence every $r \in C$ can be pierced by one of the (at most k^2) grid points in H.

For $X \subseteq H$ we define $S_X \subseteq C$ as follows:

$$S_X = \{r \in C : r \cap H = X\}$$

We note that C is partitioned into the sets S_X, i.e. $C = \dot{\bigcup}_{X \subseteq H} S_X$.

The subset of H 'induced' by a rectangle $r \in C$ will be of the form of a rectangular 'sub block' of H. Any rectangular sub block of H is uniquely determined by its diagonal endpoints. Hence there are at most $\binom{k^2}{2}$ distinct subsets of H induced by rectangles. Therefore there are at most $\binom{k^2}{2} \le k^4$ distinct nonempty S_X.

Let $S' \subseteq C$ be a set of representative rectangles obtained by picking exactly one rectangle from each of the nonempty sets S_X, $X \subseteq H$ (the bold rectangles in Figure 2). Note that $|S'| \le k^4$. Since C has (k^4, k) property, S' can be pierced by a set of points $W \subset \mathbb{R}^2$, $|W| \le k$ (the filled points in Figure 2). For $p \in W$, let $N(p)$ denote the set of (at most 4) grid points of H which form the gridcell containing p. Let $P = \cup_{p \in W} N(p)$, $|P| \le 4k$ (the big hollow points in Figure 2). If p pierces some rectangle $r \in S_X$, then the points in $N(p)$ pierce all rectangles in S_X. Since points in W pierce all the rectangles in S', points in P pierce all the rectangles in $C = \dot{\bigcup}_{X \subseteq H} S_X$. Thus C is $4k$-piercable.

Algorithm 1 FPT algorithm to give a 2^d approximation for piercing boxes in R^d

```
Orthogonally project each box r ∈ C to the d axes to get
a set of intervals C_i for each axis i
if All the C_i are k-piercable then
   Obtain S'
   Bruteforce check if S' is k-piercable

   if S' is k-piercable then
      return Grid neighbours of piercing set
   else
      return false
   end if
else
   return false
end if
```

The proof of Theorem 3 directly leads to a 2^d-approximate FPT algorithm for the minimum piercing problem on boxes. Given a collection of boxes C Algorithm 1 returns no if C is not k-piercable and returns a piercing set of size atmost $2^d k$ otherwise.

Obtaining C_i takes $O(dn)$ time. Checking if each C_i is k-piercable takes $O(dn \log n)$ time. Obtaining S' takes $O(dn \log k)$ time. The bruteforce check takes $O(k^{4k})$ time. Hence the whole algorithm takes $O(dn \log n + k^{4k})$ time.

5 Conclusion

In this paper we prove that any family of pseudolines with (k^2+k+1,k)-property is k-piercable. We extend this result for other families of geometric objects with discrete intersection, i.e., polynomial curves and hyperplanes. It is an interesting question to fully charaterise such families of objects for which $g(k,d)$ is finite. We also pose a relaxed variant of this problem as the k-Helly problem and show non-trivial bounds for a family of boxes in $\mathbb{R}^d$. An interesting open problem is to obtain tight bounds on the k-Helly problem for other families of geometric objects in $\mathbb{R}^d$.

References

[1] N. ALON, I. BARANY, Z. FUREDI, AND D. J. KLEITMAN, *Point selections and weak ϵ-nets for convex hulls*, Combinatorics, Probability and Computing, 1 (1992), pp. 189–200.

[2] N. ALON AND D. KLEITMAN, *Piercing convex sets and the Hadwiger Debrunner (p,q)-problem*, Advances in Mathematics, 92 (1992), pp. 103–112.

[3] B. BRODEN, M. HAMMAR, AND B. J. NILSSON, *Guarding lines and 2-link polygons is APX-hard*, in Proceedings of the Canadian Conference on Computational Geometry, 2001, pp. 45–48.

[4] L. DANZER AND B. GRUNBAUM, *Intersection properties of boxes in $\mathbb{R}^d$*, Combinatorica, 2 (1982), pp. 237–246.

[5] L. DANZER, B. GRUNBAUM, AND V. KLEE, *Helly's theorem and its relatives*, in Convexity, vol. 7, American Mathematical Society, 1963, pp. 101–179.

[6] J. ECKHOFF, *Helly, Radon, and Caratheodory type theorems*, in Handbook of Convex Geometry, 1993, pp. 389–448.

[7] H. HADWIGER AND H. DEBRUNNER, *Uber eine variante zum hellyschen satz*, Archiv der Mathematik, 8 (1957), pp. 309–313.

[8] E. HELLY, *Uber mengen konvexer kŏrper mit gemeinschaftlichen punkten*, Jber. Deutsch. Math. Verein., 32 (1923), pp. 175–176.

[9] M. KATCHALSKI AND D. NASHTIR, *On a conjecture of Danzer and Grunbaum*, AMS, 124 (1996).

[10] S. LANGERMAN AND P. MORIN, *Covering things with things*, Discrete & Computational Geometry, 33 (2005), pp. 717–729.

[11] D. MARX, *Efficient approximation schemes for geometric problems?*, in Proceedings of the 13th annual European Symposium on Algorithms, 2005, pp. 448–459.

[12] N. MEGIDDO AND A. TAMIR, *On the complexity of locating linear facilities in the plane*, Operations Rsearch Letters, 1 (1982), pp. 194–197.