Small Strong Epsilon Nets

Pradeesha Ashok* Sathish Govindarajan† Janardhan Kulkarni‡

Abstract

In this paper, we initiate the study of small strong ϵ-nets for axis-parallel rectangles, half spaces, strips and wedges. We also give some improved bounds for small weak ϵ-nets.

1 Introduction

Let P be a set of n points in the plane. $N \subset \mathbb{R}^2$ is a *weak ϵ-net* for a family of sets $\mathcal{S}$ if $S \cap N$ is not empty for any $S \in \mathcal{S}$ such that $S \cap P > \epsilon n$. Moreover, N is a *strong ϵ-net* if $N \subset P$. The concept of ϵ-nets was introduced by Haussler and Welzl [4] and has found many applications in computational geometry, approximation algorithms, learning theory etc.

It has been proved that for a range space $(P, \mathcal{S})$ with finite VC Dimension d, there exist ϵ-nets of size $O(\frac{d}{\epsilon}\log\frac{1}{\epsilon})$ [4]. Also, ϵ-nets of size $O(\frac{1}{\epsilon})$ exist for half spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ [5] and pseudodisks in the plane [6, 7]. Recently, it is shown that ϵ-nets of size $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ exist for axis-parallel rectangles in the plane [2].

Small weak ϵ-nets have been studied for convex objects, axis-parallel rectangles, disks and half spaces in [2, 8, 3]. In this problem, the size of weak ϵ-net is fixed as i and the value of ϵ_i is bounded. Small nets are especially interesting for the range spaces where tight bounds for epsilon nets are not known. Also, small epsilon nets can be seen as a generalization of center point theorem for different range spaces.

In this paper, we initiate the study of small strong ϵ-nets. Let $\epsilon_i^{\mathcal{S}} \in [0, 1]$ represent the smallest real number such that, for any set of points P in the plane, there exists a set $Q \subset P$ of size i which is an $\epsilon_i^{\mathcal{S}}$-net for P with respect to $\mathcal{S}$. We obtain bounds on $\epsilon_i^{\mathcal{S}}$ where $\mathcal{S}$ is the family of axis-parallel rectangles, half spaces, strips or wedges. We also improve some of the small weak ϵ-net bounds given in [1].

*Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, pradeesha@csa.iisc.ernet.in

†Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, gsat@csa.iisc.ernet.in

‡Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India, janardhan@csa.iisc.ernet.in

	Rectangles		Halfspaces		Strips	Wedges
	LB	UB	LB	UB	LB	LB
ϵ_1	3/4		1		1	1
ϵ_2	1/2	2/3	5/9	2/3	2/3	1
ϵ_3	3/8	9/16	1/2		1/2	2/3
ϵ_4	2/7	1/2	1/3	1/2	2/5	1/2

Table 1: Summary of bounds for small strong epsilon nets.

2 General lower bounds for Axis-parallel rectangles and Half spaces

Let $\mathcal{S}$ represents the family of axis-parallel rectangles/ half spaces.

Theorem 1 *$\epsilon_i^{\mathcal{S}} \geq \frac{1}{i}$, for $i \geq 2$*

Proof. Let P be a set of n points arranged along the boundary of a circle with center c. Let $N = \{p_1, ..., p_i\} \subset P$ be an $\epsilon_i^{\mathcal{S}}$-net. Connect c with p_j for $1 \leq j \leq i$. These lines divide the circle into i sectors and at least one of these sectors contains $\frac{n-i}{i}$ points from P. We can have a range $S \in \mathcal{S}$ such that S contains all points from this sector and does not include any point from N. Therefore, for large values of n, $\epsilon_i^{\mathcal{S}} \geq \frac{1}{i}$. □

In Section 3, we give improved (better than $\frac{1}{i}$) lower bounds for axis-parallel rectangles for $i \leq 4$. For $i \geq 5$, the above bounds improve upon the previously known bounds of $\frac{1}{i+1}$. Section 4 gives better bounds for half spaces.

The above general bound also applies to weak ϵ-nets for the family of axis-parallel rectangles, half spaces and convex sets. The proof for this is given in Appendix A.

3 Axis-parallel Rectangles

In this section, we show bounds on $\epsilon_i^{\mathcal{R}}$ for axis-parallel rectangles. The summary of the bounds are given in Table 1.

Let P be a set of n points in $\mathbb{R}^2$.

Lemma 2 $\epsilon_1^{\mathcal{R}} = \frac{3}{4}$

Proof : Let V_1 and V_2 be vertical lines that divide a point set P such that V_1 has $\frac{|P|}{4} - 1$ points of P to the left of it and V_2 has $\frac{|P|}{4} - 1$ points of P to the right of it. Similarly, let H_1 and H_2 be horizontal lines that divide

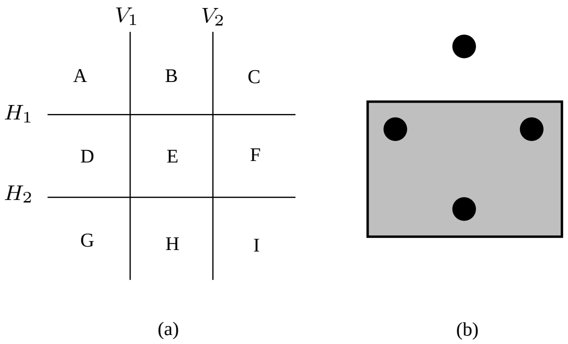


Figure 1: Upper and lower bound for $\epsilon_1^{\mathcal{R}}$

P such that H_1 has $\frac{|P|}{4} - 1$ points of P above it and H_2 has $\frac{|P|}{4} - 1$ points of P below it(See figure 1(a)). Now region E is not empty. Otherwise, since each of the first and third columns contains $\frac{|P|}{4} - 1$ points, regions B and H contain $\frac{|P|}{2} + 2$ points between them. But this is not possible since each of them can contain at most $\frac{|P|}{4} - 1$ points.

Let x be any point in region E. Any axis-parallel rectangle that does not contain x avoids at least one row or column thereby missing out at least $\frac{|P|}{4}$ points. Thus $\{x\}$ is a $\frac{3}{4}$-net.

To show the lower bound, consider the point set shown in figure 1(b). The n points are arranged as four subsets of equal size. For any point x, there exists an axis-parallel rectangle that does not contain x but includes $\frac{3n}{4}$ points from the other three subsets.

Lemma 3 $\frac{1}{2} \leq \epsilon_2^{\mathcal{R}} \leq \frac{2}{3}$

Proof : Partition the input points by two horizontal lines H_1 and H_2 such that each of the three regions contain $\frac{n}{3}$ points. Similarly, partition the given points by two vertical lines V_1 and V_2. This results in nine axis-parallel rectangles $A_{ij}, 1 \leq i, j \leq 3$ which contain all the input points as shown in figure 2 (a). If A_{22} is not empty, choose an arbitrary point p belonging to A_{22} as $\frac{2}{3}$-net. If A_{22} is empty, let p_1 denote the point which is at the least perpendicular distance from V_1 in A_{21}. Similarly, let p_2 denote the point which is at the least perpendicular distance from V_2 in A_{23}. We claim that $\{p_1, p_2\}$ forms a $\frac{2}{3}$-net.

Consider an axis-parallel rectangle R containing more than $\frac{2n}{3}$ points. R has to include points from three horizontal partitions and three vertical partitions. Hence, R contains A_{22}. If A_{22} is not empty, R contains p.

Let VP_1, VP_2 be vertical lines passing through p_1, p_2 respectively. If A_{22} is empty and R does not contain p_1 or p_2, then R is restricted to the region between lines VP_1 and VP_2, Therefore R can have at most $\frac{2n}{3}$ points since it cannot include any points from A_{21} or A_{23}.

To prove the lower bound, consider the point set shown in figure 2(b). The point set contains four subsets of equal size. For any two points selected, there exists an axis-parallel rectangle that contains $\frac{n}{2}$ points and neither of the selected points.

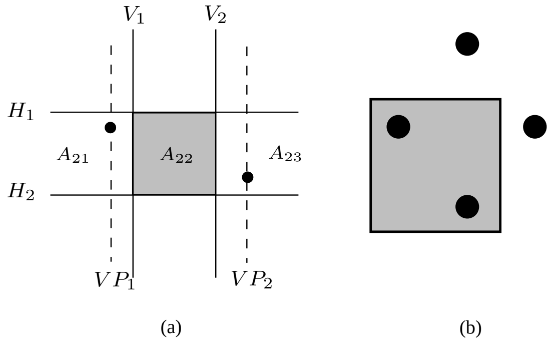


Figure 2: Upper and lower bounds for $\epsilon_2^{\mathcal{R}}$

Lemma 4 $\frac{3}{8} \leq \epsilon_3^{\mathcal{R}} \leq \frac{9}{16}$

Proof : Construct an $\epsilon_1^{\mathcal{R}}$-net for P as described in Lemma 2. The vertical line passing through p divides P into two regions of which, at most one, say Q, can contain more than $\frac{n}{2}$ points. Similarly, the horizontal line through p divides P into two regions of which, at most one, say R, can have more than $\frac{n}{2}$ points. Also, Q and R can have at most $\frac{3n}{4}$ points. Let $\{q\}$ and $\{r\}$ be $\epsilon_1^{\mathcal{R}}$-net for Q and R respectively. Now any axis-parallel rectangle that excludes $\{p, q, r\}$ can contain at most $\frac{3}{4}.\frac{3n}{4} = \frac{9n}{16}$ points. Therefore $\{p, q, r\}$ is a $\frac{9}{16}$-net.

To prove the lower bound, consider the point set as shown in figure 3. The point set consists of n points arranged into eight subsets of equal size. We claim that wherever we choose three points there exists an axis-parallel rectangle that avoids these three points and contains three out of the eight subsets. Index these subsets as $1, 2, 3...8$ starting from any subset and moving in the clockwise direction. Consider the axis-parallel rectangles that contain three consecutive subsets. There are eight such rectangles and any point can cover only three of them. Hence, three points are needed to cover all the eight rectangles. Figure 3 shows the two ways of picking these three points. In both the cases, there exists an axis-parallel rectangle with at least $\frac{3n}{8}$ points and not containing any of them.

Lemma 5 $\frac{2}{7} \leq \epsilon_4^{\mathcal{R}} \leq \frac{1}{2}$

Proof : Let V_1 and H_1 be lines that bisect P vertically and horizontally respectively and p be the point of intersection. Now bisect the two vertical and the two horizontal slabs so that we get a grid with all the rows and columns containing $\frac{n}{4}$ points each. Let a and b be

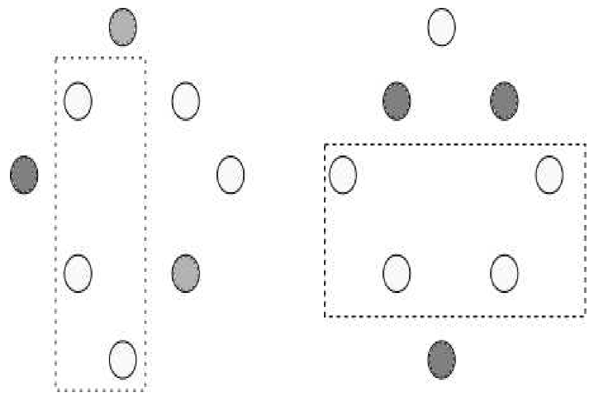


Figure 3: Lower bound for $\epsilon_3^{\mathcal{R}}$

input points in the second row with the least perpendicular distance from V_1 on either side of V_1. Similarly, let c and d be input points in the third row with the least perpendicular distance from V_1 on either side of V_1. We claim that $\{a, b, c, d\}$ is a $\frac{1}{2}$-net.

Let R be an axis-parallel rectangle which contains more than $\frac{n}{2}$ points from P. Clearly, R contains the point p and includes points from at least three rows and three columns. Assume R does not contain any of the points in $\{a, b, c, d\}$. Let V_a, V_b, V_c, V_d be vertical lines passing through the points a, b, c, d respectively. If R includes any point from the first row, then it should be restricted to the region between V_a and V_b and cannot include points from the second row. Similarly if R includes any point from the fourth row, it cannot take points from the third row. Therefore, R can include points from at most two rows, which is a contradiction. Hence, $\epsilon_4^{\mathcal{R}} \leq \frac{1}{2}$.

To prove the lower bound, consider the point set as shown in Figure 4(b), where n points are arranged into seven subsets of equal size inside a grid having three rows and five columns. Let R_{ij} be the subset in the intersection of ith row and jth column where $1 \leq i \leq 3$ and $1 \leq j \leq 5$. A point has to be chosen from R_{23} as part of the $\epsilon_4^{\mathcal{R}}$-net otherwise it forms an axis-parallel rectangle of size $\frac{2n}{7}$ with all other R_{ij}s. At least two points are needed to cover the four axis-parallel rectangles of size $\frac{2n}{7}$ formed by subsets, R_{12}, R_{14}, R_{32} and R_{34}. Assume these two points are chosen from R_{12} and R_{34}. Now there are two disjoint axis-parallel rectangles containing R_{32}, R_{21} and R_{14}, R_{25}. These two rectangles cannot be covered by a single point. Hence, $\epsilon_4^{\mathcal{R}} \geq \frac{2}{7}$.

4 Half spaces

For odd values of i, $\epsilon_i^{\mathcal{H}} = \frac{2}{i+1}$. The lower bound follows from a construction given in [5]. The upper bound follows from a construction given in [6] which reduces the problem of finding an ϵ-net for half spaces in $\mathbb{R}^3$ to the problem of finding such an ϵ-net for points in convex position, by projecting the points on to the convex hull. We can apply the same technique in $\mathbb{R}^2$. For even values of i the best known bounds are $\frac{2}{i+2} \leq \epsilon_i^{\mathcal{H}} \leq \frac{2}{i}$, which follows from the bounds of $\epsilon_{i+1}^{\mathcal{H}}$ and $\epsilon_{i-1}^{\mathcal{H}}$.

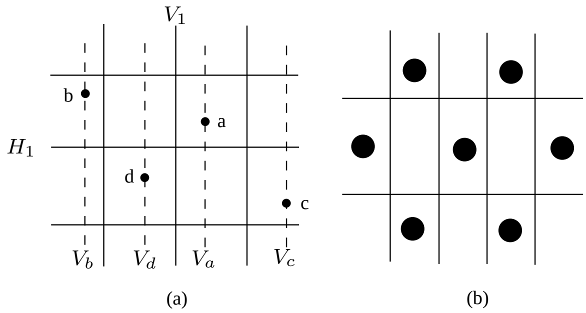


Figure 4: Upper and Lower bounds for $\epsilon_4^{\mathcal{R}}$

Lemma 6 $\frac{5}{9} \leq \epsilon_2^{\mathcal{H}} \leq \frac{2}{3}$

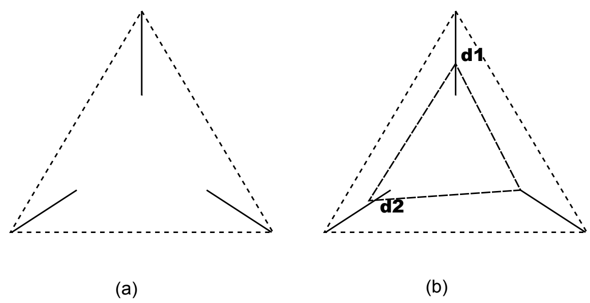


Figure 5: Lower bound for $\epsilon_2^{\mathcal{H}}$

Proof : The upper bound follows from [5]. To show the lower bound, consider the point set as shown in Figure 5(a). The points are arranged as three subsets of equal size near the corners of a triangle along the bold lines. Let a and b be the two points selected. If a and b belong to same subset, then there exists a half space containing all the points from other two subsets i.e, it contains $\frac{2n}{3}$ points. If not, let a and b be the d_1th and d_2th point respectively in their subsets. In this case, there exists a half space that excludes a and b and contains $f(d_1, d_2)$ points, where $f(d_1, d_2) = max(d_1 + d_2, n - (d_1 + d_2), \frac{n}{3} + d_1, \frac{n}{3} + d_2)$. For any d_1, d_2 such that $1 \leq d_1, d_2 \leq \frac{n}{3}$, $f_(d_1, d_2)$ is at least $\frac{5n}{9}$.

5 Intersection of two half spaces

In this section we consider small strong nets for ranges defined by intersection of two half spaces. There are two possibilities. First is the family of strips which are formed by two intersecting half spaces with parallel supporting lines. Second is the family of wedges which are formed by two intersecting half spaces with non-parallel supporting lines.

Lemma 7 *For the family of strips,* $\epsilon_i^{\mathcal{T}} \geq \frac{2}{i+1}$

Proof : Let P be a set of n points arranged uniformly on the boundary of a circle and $\{p_1, p_2, ...p_i\}$ be any subset of P, ordered in clockwise direction. Let d_j be the number of points of P between p_j and p_{j+1} and $d_{max} = \max d_j$.

If d_{max} is unique, a strip that does not contain any of the p_is can contain d_j points,$1 \leq j \leq i$ (See strip A in Figure 6) or $2d_j$ points where $d_j < d_{max}$(See strip B in Figure 6). The maximum number of points that can be present in any of these strips are minimized when all d_j except d_{max} are the same and $2d_j = d_{max}$. Therefore, $d_{max} = \frac{2n}{i+1}$. If d_{max} is not unique, then there exists a strip having at least $\frac{2(n-i)}{i}$ points. Therefore, for sufficiently large n, $\epsilon_i^{\mathcal{T}} \geq \frac{2}{i+1}$.

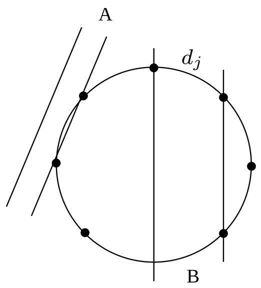


Figure 6: Lower bound for strips

Lemma 8 *For the family of wedges,* $\epsilon_i^{\mathcal{W}} \geq \frac{2}{i}$

Proof : Let P be a set of n points arranged uniformly along the boundary of a circle. Let $N = \{p_1, p_2, ...p_i\} \subset P$ partition P into i intervals. A wedge that does not contain any point from N can still include all the points from any two intervals. Therefore, there exists a wedge that contains $2\frac{n-i}{i}$ points. Therefore, $\epsilon_i^{\mathcal{T}} \geq \frac{2}{i}$.

6 Small Weak Epsilon Nets

In this section, we consider small weak ϵ-nets for disks. We show an improved lower bound of $\epsilon_3^{\mathcal{D}} \geq \frac{2}{7}$. This improves upon the general lower bound of $\frac{1}{4}$.

Lemma 9 $\epsilon_3^{\mathcal{D}} \geq \frac{2}{7}$

Proof : Place n points in a point set P in seven subsets each containing $\frac{n}{7}$ points. Arrange these subsets along the boundary of a circle. Now we claim that for any three points p, q, r in the plane, we can draw a circle which contains at least two subsets and not containing p, q, r.

Consider the line l passing through p and q. l can intersect at most two subsets. So we assume it intersects exactly two subsets. If there exist two or more subsets on both sides of l, we can draw the needed circle on the side not containing r. So, assume that one side of l contains at most one subset and hence the other side of l has at least four subsets and let r lie in that side. In this case, the line pr contains at least two subsets on one of the sides and we can draw the needed circle.

References

[1] B. Aronov, F. Aurenhammer, F. Hurtado, S. Langerman, D. Rappaport, C. Seara, and S. Smorodinsky. Small weak epsilon-nets. *Comput. Geom. Theory Appl.*, 42(5):455–462, 2009.

[2] B. Aronov, E. Ezra, and M. Shair. Small-size ϵ-nets for axis-parallel rectangles and boxes. In *STOC '09: Proceedings of the 41st annual ACM symposium on Theory of computing*, pages 639–648, New York, NY, USA, 2009. ACM.

[3] M. Babazadeh and H. Zarrabi-Zadeh. Small weak epsilon-nets in three dimensions. In *CCCG*, 2006.

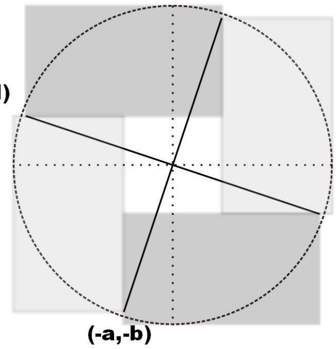
[4] D. Haussler and E. Welzl. Epsilon-nets and simplex range queries. In *SCG '86: Proceedings of the second annual symposium on Computational geometry*, pages 61–71, New York, NY, USA, 1986. ACM.

[5] J. Komlós, J. Pach, and G. Woeginger. Almost tight bounds for ϵ-nets. *Discrete Comput. Geom.*, 7(2):163–173, 1992.

[6] J. Matoušek, R. Seidel, and E. Welzl. How to net a lot with little: small ϵ-nets for disks and halfspaces. In *SCG '90: Proceedings of the sixth annual symposium on Computational geometry*, pages 16–22, New York, NY, USA, 1990. ACM.

[7] E. Pyrga and S. Ray. New existence proofs for ϵ-nets. In *SCG '08: Proceedings of the twenty-fourth annual symposium on Computational geometry*, pages 199–207, New York, NY, USA, 2008. ACM.

[8] S. Ray and N. Mustafa. An optimal generalization of the centerpoint theorem, and its extensions. In *SCG '07: Proceedings of the twenty-third annual symposium on Computational geometry*, pages 138–141, New York, NY, USA, 2007. ACM.



[h]

Figure 7: To prove Lemma 11

A Appendix :Small Weak Epsilon Nets

A.1 General Bound for Convex Objects

Theorem 10 *$\epsilon_i^{\mathcal{C}} \geq \frac{1}{i}$, for $i > 1$.*

The proof is exactly the same as Theorem 1.

A.2 General Bound for axis-parallel rectangles

Lemma 11 *When a circle is divided into i equal sectors, $i \geq 4$, we can draw i axis-parallel rectangles such that each of them contains the corresponding arc from each sector and none of them intersect.*

Proof : We will first prove for $i = 4$.

Assume that the centre of circle is at the origin and the radius of the circle is r. Then the two lines that divide the circle into four equal sectors will pass through (0,0) and will intersect the circle in four different points. Let these points of intersection be $(a, b), (c, -d), (-a, -b), (-c, d)$ where $a, b, c, d \geq 0$.

Without loss of generality, assume that $d \leq b$. Now the four rectangles with the following set of vertices will contain the four arcs (See Figure 7).

- $(-c, d), (-c, r), (a, r), (a, d)$
- $(a, b), (r, b), (r, -d), (a, -d)$
- $(c, -d), (c, r), (-a, r), (-a, -d)$
- $(-a, -b), (r, -b), (r, d), (-a, d)$

Clearly none of these rectangles intersect.

Now the same can be proved for $i > 4$. When a circle is divided into i sectors for $i > 4$, two rectangles that contain two consecutive sectors are contained in two separate bigger rectangles as seen in the previous paragraph. Since the bigger rectangles are already proved to be non-intersecting the smaller rectangles that are contained in them also do not intersect.

Theorem 12 *$\epsilon_i^{\mathcal{R}} \geq \frac{1}{i}$, for $i \geq 4$*

Proof. Let $P = p_1, p_2, ..., p_n$ be a point set of n points arranged along the boundary of a circle, where $n = ik+1$ for some arbitrary k. Consider groups of points, S_i, $1 \leq i \leq n$, of size k in which each group consists of k consecutive points starting with point p_i. Now, by Lemma 11, there exists a family of axis-parallel rectangles $R = \{R_j, 1 \leq j \leq n\}$ such that each R_j contains all the k points of S_j and R_j and R_{j+k} do not have a common intersection. Therefore any point p in the epsilon net can cover at most k rectangles in R, so i points can cover at most ik rectangles. Therefore, wherever you choose i points there exists an axis-parallel rectangle in R which does not contain any of the chosen points. Thus for large k, $\epsilon_i^{\mathcal{R}} \geq \frac{1}{i}$. □