

Brief Paper

Reliable Stabilization Using a Multi-controller Configuration

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Key Words—Linear systems; reliability; control system synthesis.

Abstract—Given a plant P , we consider the problem of designing a pair of controllers C_1 and C_2 such that their sum stabilizes P , and in addition, each of them also stabilizes P should the other one fail. This is referred to as the reliable stabilization problem. It is shown that every strongly stabilizable plant can be reliably stabilized; moreover, one of the two controllers can be specified arbitrarily, subject only to the constraint that it should be stable. The stabilization technique is extended to reliable regulation.

1. Introduction

SUPPOSE P is a given unstable plant. The problem of determining a controller C such that the feedback system of Fig. 1 is stable has been studied for several years. Recent results provide a characterization of all controllers C that stabilize a given plant P (see Youla *et al.*, 1976; Desoer *et al.*, 1980; Vidyasagar *et al.*, 1982). With the availability of this characterization, interest has been created in the problem of reliable stabilization. In Sacks and Murray (1982) and Vidyasagar and Viswanadham (1982) the object of study is the so-called simultaneous stabilization problem, where one would like to determine whether or not there exists a single controller C that stabilizes each of several given plants P_0, P_1, \dots, P_n . The motivation for the problem formulation is that P_0 represents the model of the plant in its normal mode, while P_1, \dots, P_n represent the same plant under various structural perturbations, such as sensor/actuator failures, changes in the operating point, etc. Thus, if the simultaneous stabilization problem has a solution, then not only does C stabilize the nominal plant P_0 , but this stabilization is reliable against a prespecified set of structural changes in the plant.

The problem studied in this paper is in a sense the dual of the simultaneous stabilization problem. Consider the system shown in Fig. 2, where P is a given plant, and C_1, C_2 are controllers to be determined. The objective is to select C_1, C_2 (if possible) such that the system of Fig. 2 is stable as shown, as well as when either C_1 or C_2 is set equal to zero. The structure shown in Fig. 2 is called a

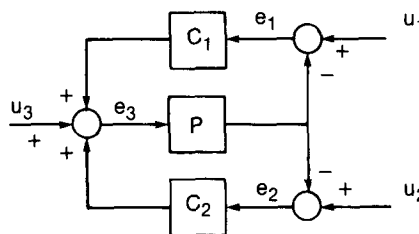


FIG. 2. A multi-controller configuration with passive redundancy.

multi-controller configuration, and the above requirements on C_1, C_2 mean that the two controllers together stabilize P , and in addition, each controller individually stabilizes P . This problem was introduced by Siljak (1978,1980), who calls one of the controllers as a "hot spare". The motivation for the problem formulation is as follows: in the normal mode, both controllers C_1 and C_2 are operational and the system is stable. Should either controller fail (which is modeled by setting $C_i = 0$ for $i = 1$ or 2), the system is still stable, even though other properties such as sensitivity might be affected adversely. Thus, if one can find controllers C_1 and C_2 satisfying the above requirements, then the stabilization scheme of Fig. 2 is reliable against a single controller failure.

It should be emphasized that the reliable stabilization scheme proposed in Fig. 2 is quite distinct from the standard technique of having redundancy in key controllers (see e.g. Willsky, 1976). The redundancy scheme can be represented as in Fig. 3, where the backup controller is switched in once the failure of the main controller is detected. Thus only one controller is connected to P at any one time. In contrast, in the normal mode of operation of the system shown in Fig. 2, both controllers are connected to P at all times. One could refer to the scheme of Fig. 2 as passive redundancy and to that of Fig. 3 as active redundancy. Each scheme has its own place in the design of reliable control systems. The active redundancy scheme is to be used when one wishes to guard against controllers failing due to component malfunction and the like; it presupposes that the failure of controllers can be detected swiftly and accurately, and that this knowledge can be transmitted reliably to the backup controller. The passive redundancy scheme is to be used when the detection mechanism is itself a possible source of failure, or increases the weight of the overall control system beyond acceptable limits. Further, in the so-called problem of reconfigurable controllers, it is natural to suppose that the two controllers are at physically different

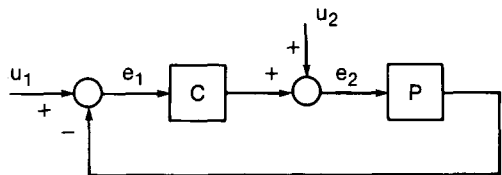


FIG. 1. System to be stabilized with controller C given unstable plant, P .

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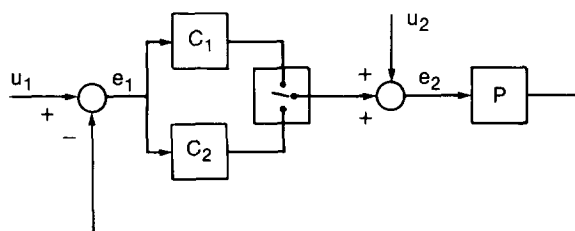


FIG. 3. A multi-controller configuration with active redundancy.

locations, and that they have access to different plant inputs and outputs.

Finally, a word is in order concerning the model of controller failure used here. It is assumed that, when a controller fails, it is to be replaced by the zero controller. Physically, this assumption is most natural in the case of hardover failure, when the output of a controller gets "stuck" at a constant value, irrespective of the input value. In such a case, it is possible to include internal detection mechanisms simply to "turn off" the controller when its output does not appear to be changing; alternatively, one can model hardover failure by replacing the dynamic element C_i by a zero transfer matrix plus a constant (i.e. step) disturbance at the output. In either case, for the purposes of analyzing the stability of the system following a failure, the controller that has failed is to be replaced by the zero transfer matrix.

The objective of the paper is to present conditions on P that ensure the existence of controllers C_1 and C_2 that achieve reliable stabilization. The problem is of course trivial if a controller C can be found that stabilizes P in such a way that the feedback system has a gain margin greater than two; in such a case, one simply chooses $C_1 = C_2 = C$. If P is a minimum phase plant, the results of Kwakernaak and Sivan (1972) and Doyle and Stein (1981) imply that one can actually find a stabilizing controller with infinite gain margin. See also Maeda and Vidyasagar (1985) for other results concerning the existence of controllers with infinite gain margin. However, if P is not minimum phase, these results do not apply, and the existence of appropriate controllers C_1 and C_2 is still open. The main result of this paper states that suitable C_1 and C_2 exist if P is strongly stabilizable, i.e. P can be stabilized using a stable controller; moreover, one of the two controllers can be selected arbitrarily, subject only to the constraint that it should be stable. This result carries over with very little modification to the problem of reliable robust regulation.

The present results considerably extend those of Siljak, which present some sufficient conditions for the reliable stabilization problem to have a solution, based on the idea of overlapping decompositions. In contrast, the present approach is based on simultaneous stabilization, and leads to more complete results. The restriction to strongly stabilizable plants is not too restrictive, since generically a multivariable plant has this property (see Vidyasagar and Viswanadham, 1982).

2. Problem statement and main result

Let $\mathbb{R}(s)$ denote the field of rational functions with real coefficients, and let \mathbf{S} denote the subset of $\mathbb{R}(s)$ consisting of proper stable rational functions; in other words, \mathbf{S} consists of functions in $\mathbb{R}(s)$ that do not have poles in the closed right half-plane nor at infinity. Let $\mathbf{M}(\mathbb{R}(s))$ denote the set of matrices, of whatever order, whose elements all belong to $\mathbb{R}(s)$, and define $\mathbf{M}(\mathbf{S})$ analogously.

Consider now the system in Fig. 1, and suppose $P, C \in \mathbf{M}(\mathbb{R}(s))$ have compatible dimensions. Then the relationship between the vectors $e = [e_1 \ e_2]'$ and $u = [u_1 \ u_2]'$ is given by

$$e = H(P, C)u, \quad (1)$$

where

$$H(P, C) = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix}. \quad (2)$$

We say that the system of Fig. 1 is stable, and that C stabilizes P if $H(P, C) \in \mathbf{M}(\mathbf{S})$. Given a plant P , we denote by $S(P)$ the set of all controllers that stabilize P ; i.e. $S(P)$ consists of all C such that $H(P, C) \in \mathbf{M}(\mathbf{S})$. We say that P is strongly stabilizable if it can be stabilized by a stable controller, i.e. if the intersection $S(P) \cap \mathbf{M}(\mathbf{S})$ is nonempty; this definition is due to Youla *et al.* (1974).

The *Reliable Stabilization Problem (RSP)* studied in this paper can now be stated precisely. Given a plant $P \in \mathbf{M}(\mathbb{R}(s))$, find, if possible, two controllers C_1 and C_2 such that

- (i) $C_1 \in S(P)$,
- (ii) $C_2 \in S(P)$,
- (iii) $C_1 + C_2 \in S(P)$, and
- (iv) if we express C_1, C_2 as $C_i = \bar{D}_{ci}^{-1} \bar{N}_{ci}$ where $\bar{D}_{ci}, \bar{N}_{ci}$ are left-coprime for each i , then $\bar{D}_{c1}, \bar{D}_{c2}$ are right-coprime.

The first three conditions state that each controller acting alone stabilizes P , and that the combination of the two controllers also stabilizes P . The last condition states that the parallel combination of C_1 and C_2 does not have any internal duplication of poles in the closed right half-plane, i.e. that the parallel combination is stabilizable and detectable. Note that, since coprimeness over the ring \mathbf{S} does not rule out the possibility of common zeros in the open left half-plane, it is possible that the parallel combination of C_1 and C_2 is not minimal even if condition (iv) is satisfied; however, the combination will nevertheless be internally stable, since any common poles between the two controllers will be in the stable region.

In what follows, we study a problem that is slightly different from the above, termed the *Modified Reliable Stabilization Problem (MRSP)*.

Given a plant $P \in \mathbf{M}(\mathbb{R}(s))$, find if possible a pair of controllers C_1 and C_2 such that

- (i') $C_1 \in S(P)$,
- (ii') C_2 stabilizes both P and the plant $P_1 = P(I + C_1 P)^{-1}$.

By the so-called decomposition principle (Zames, 1981; Callier and Desoer, 1982), the fact that C_1 stabilizes P and C_2 stabilizes $P(I + C_1 P)^{-1}$ implies that $C_1 + C_2$ stabilizes the plant P . Hence the satisfaction of (i') and (ii') above implies the satisfaction of (i)–(iii) above. However, (iv) is not mentioned above. Note that, unlike the RSP, the modified RSP is not symmetric in the two controllers C_1 and C_2 , since it is not required that C_1 should also stabilize the plant $P(I + C_2 P)^{-1}$.

The results of this section can be summarized as follows: a complete solution is given for the modified RSP, and it is shown that if the MRSP has a solution then the RSP also has a solution. In the scalar case, it is shown that the converse is also true. The one remaining open area is the possibility that, in the multivariable case, the RSP has a solution whereas the MRSP does not.

Theorem 1. Given a plant $P \in \mathbf{M}(\mathbb{R}(s))$, the MRSP has a solution if and only if the plant P is strongly stabilizable. Moreover, if P is strongly stabilizable, then for any $C \in S(P) \cap \mathbf{M}(\mathbf{S})$, there exists a corresponding C_2 such that (i') and (ii') hold.

Proof. First, suppose P is strongly stabilizable, and let C_1 be any strongly stabilizing controller for P . Define $P_1 = P(I + C_1 P)^{-1}$. Then $P = P_1(I - C_1 P_1)^{-1}$; moreover, $(P_1, I - C_1 P_1)$ is a right-coprime factorization (r.c.f.) for P , since

$$C_1 \cdot P_1 + I \cdot (I - C_1 P_1) = I. \quad (3)$$

Similarly, $P = (I - P_1 C_1)^{-1} P_1$, and $I - P_1 C_1, P_1$ are left-coprime.

To show that the MRSP has a solution, we show that the two plants P and P_1 are simultaneously stabilizable. Since P_1 is stable, by Vidyasagar and Viswanadham (1982), this is the case if and only if the plant $P - P_1$ is strongly stabilizable. Now

$$P - P_1 = P_1(I - C_1 P_1)^{-1} - P_1 = P_1 C_1 P_1 (I - C_1 P_1)^{-1}. \quad (4)$$

To show that this auxiliary plant is strongly stabilizable, we proceed as follows. Let k be any integer larger than $\|P_1 C_1\|$, where the norm of a matrix in $\mathbf{M}(\mathbf{S})$ is defined in the usual manner, i.e.

$$\|F\| = \sup_{\omega} \bar{\sigma}(F(j\omega)) \quad (5)$$

Then $I - C_1 P_1/k$ is a unit of $\mathbf{M}(\mathbf{S})$, since $\|C_1 P_1/k\| < 1$. Hence $(I - C_1 P_1/k)^k$ is also a unit. However, by the binomial expansion,

$$\left\{ I - \frac{C_1 P_1}{k} \right\}^k = I + C_1 P_1 + \sum_{i=2}^k r_i (C_1 P_1)^i, \quad (6)$$

where r_i are appropriate binomial coefficients. Now let

$$R = C_1 \sum_{i=2}^k r_i (C_1 P_1)^{i-2}. \quad (7)$$

Then clearly $I - C_1 P_1 + R P_1 C_1 P_1 = (I - C_1 P_1/k)^k$ is a unit. This shows that the stable controller R stabilizes the plant

$P_1 C_1 P_1 (I - C_1 P_1)^{-1}$, whence this plant is strongly stabilizable. If we now define

$$C_2 = R(I - P_1 R)^{-1}, \quad (8)$$

then, as shown in Vidyasagar and Viswanadham (1982), C_2 stabilizes both P and P_1 . Thus C_1 and C_2 together solve the MRSP.

Next, it is shown that if P is not strongly stabilizable, then the MRSP does not have a solution. Let (N, D) , (\tilde{D}, \tilde{N}) be any r.c.f and any l.c.f. of P , and select matrices $X, Y \in \mathbf{M}(\mathbf{S})$ that satisfy the Bezout identity $XN + YD = I$. Then, as shown in Desoer *et al.* (1980), Vidyasagar *et al.* (1982),

$$S(P) = \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}) : R \in \mathbf{M}(\mathbf{S})\}. \quad (9)$$

Now the point of the theorem is that, irrespective of how C_1 is chosen from the above set, the resulting plants P_1 and P are not simultaneously stabilizable. Routine calculations show that

$$P_1 = P(I + C_1 P)^{-1} = N(Y - R\tilde{N}). \quad (10)$$

As before, P and P_1 are simultaneously stabilizable if and only if the difference $P - P_1$ is strongly stabilizable. Now

$$\begin{aligned} P - P_1 &= ND^{-1} - N(Y - R\tilde{N}) \\ &= N(I - (Y - R\tilde{N})D)^{-1}. \end{aligned} \quad (11)$$

Now, by the results of Youla *et al.* (1974), Vidyasagar and Viswanadham (1982), this plant is strongly stabilizable if and only if the quantity $|D|$ has the same sign at all real RHP blocking zeros of the "numerator" $N(I - (Y - R\tilde{N})D)$. Since this quantity is a multiple of N , every blocking zero of N (i.e. of P) is also a blocking zero of this quantity as well (of course there may also be others). However, since P is not strongly stabilizable, $|D|$ does not have the same sign at all real RHP blocking zeros of N . Hence this auxiliary plant is also not strongly stabilizable, and as a consequence P and P_1 are not simultaneously stabilizable. This shows that the MRSP does not have a solution if P is not strongly stabilizable.

The above proof provides a recipe for constructing a C_2 given a C_1 . However, in practice this controller tends to be of very high order. If P is a multivariable plant, there exist techniques based on a genericity arguments for constructing low order strongly stabilizing controllers; these will be reported elsewhere.

One question not resolved by Theorem 1 is the following: given a strongly stabilizable plant P which is stabilized by an unstable controller C_1 , does there exist a C_2 that solves the MRSP? It is now shown that if the MRSP has a solution then the RSP also has a solution. Actually this is quite obvious. If the MRSP has a solution, then P is strongly stabilizable, and as shown above, one can choose C_1 to be stable, though C_2 of course need not be stable. Now, since the denominator of a stable matrix is the identity matrix, it is automatically coprime with the denominator of C_2 , whatever it may be.

Lemma 2. Suppose the plant p is scalar, and suppose c_1 and c_2 solve the RSP. Then p is strongly stabilizable.

This result shows that, in the case of scalar plants, if the RSP has a solution, then by Theorem 1 above, that the MRSP also has a solution. However, the case of multivariable plants is still open.

Proof. Suppose c_1, c_2 solve the RSP. Factor p, c_1, c_2 in the form

$$p = \frac{n}{d}, \quad c_1 = \frac{x}{y}, \quad c_2 = \frac{a}{b}, \quad (12)$$

and suppose without loss of generality that $xn + yd = 1$, and that $a = x + rd, b = y - rn$ for some $r \in \mathbf{S}$. Then

$$c_1 + c_2 = \frac{xb + ya}{yb}, \quad (13)$$

and since y, b are coprime it is not difficult to show that $xb + ya, yb$ are also coprime. Since $c_1 + c_2$ stabilizes p by assumption, it follows that the return difference

$$u = (xb + ya)n + ybd \quad (14)$$

is a unit of \mathbf{S} . Using the identity $xn + yd = 1$ shows that $u = b + yan = y - rn + yan$. Since $y + (ya - r)n$ is a unit, we see that the plant n/y is strongly stabilizable, since it is stabilized by the stable controller $ya - r$. This means that y has the same sign at all real RHP blocking zeros of n . Now, since $yd = 1$ whenever $n = 0$, it follows that d also has the same sign at all real RHP blocking zeros of n , i.e. that p is strongly stabilizable.

3. Robust regulation

The results of Section 2 extend readily to the problem of reliably stabilizing a plant while at the same time tracking a given reference signal, or rejecting a disturbance. In order to present this extension, a few facts are recalled from Francis and Vidyasagar (1983).

Given a plant $P \in \mathbf{M}(\mathbb{R}(s))$, a basic neighborhood of P is a set $\mathbf{N}(P)$ in $\mathbf{M}(\mathbb{R}(s))$ of the form

$$\mathbf{N}(P) = \left\{ N_1 D_1^{-1} : \left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\| < \varepsilon \right\}, \quad (15)$$

where (N, D) is an r.c.f. of P . A property such as stability, tracking, or disturbance rejection, is said to be robust against perturbations in P if there is a basic neighborhood $\mathbf{N}(P)$ such that the property holds for all plants in $\mathbf{N}(P)$.

Consider first the problem of robust tracking, as depicted in Fig. 4. The reference signal r is the output of an unstable system, $R = \tilde{D}_r^{-1} \tilde{N}_r$, where \tilde{D}_r, \tilde{N}_r are left-coprime. The controller C solves the robust tracking problem if

- (i) $C \in S(P)$,
- (ii) $(I + PC)^{-1} \tilde{D}_r^{-1} \tilde{N}_r \in \mathbf{M}(\mathbf{S})$,
- (iii) both (i) and (ii) are robust against perturbations in P .

The following result is proved in Francis and Vidyasagar (1983).

Lemma 3. Let (\tilde{D}, \tilde{N}) be any l.c.f. of P , and let α_r denote the largest invariant factor of \tilde{D}_r . Then the robust tracking problem has a solution if and only if \tilde{N} and $\alpha_r I$ are right-coprime. Suppose $C \in S(P)$, and let (N_c, D_c) be any r.c.f. of C . Then C solves the robust tracking problem if and only if α_r divides every element of D_c .

A ready consequence of Lemma 3 is the following.

Lemma 4. With all symbols as in Lemma 3, suppose $\alpha_r I, \tilde{N}$ are right-coprime. Then C solves the robust tracking problem if and only if $\alpha_r C$ stabilizes P/α_r . Thus the set of all solutions to the robust tracking problem is given by $\alpha_r^{-1} S(P/\alpha_r)$.

Proof. The coprimeness of $\alpha_r I, \tilde{N}$ implies that $(\alpha_r \tilde{D}, \tilde{N})$ is an l.c.f. of P/α_r .

"if": Suppose $\alpha_r C$ stabilizes P/α_r , and let $C_1 = \alpha_r C$. Then it follows that C_1 has an r.c.f. (B, A) such that

$$\alpha_r \tilde{D} A + \tilde{N} B = I, \quad (16)$$

or equivalently,

$$\tilde{D} \alpha_r A + \tilde{N} B = I. \quad (17)$$

Now (17) implies that $C = B(A\alpha_r)^{-1} = C_1/\alpha_r$ stabilizes P . Moreover, since $\alpha_r A$ and B are clearly right-coprime, it follows from Lemma 3 that C solves the robust tracking problem.

"only if": Suppose C solves the robust tracking problem, and let (N_c, D_c) be any r.c.f. of C such that

$$\tilde{D} D_c + \tilde{N} N_c = I. \quad (18)$$

By Lemma 3, α_r divides every element of D_c . Accordingly, suppose $D_c = \alpha_r M$. Then (18) implies that:

$$\alpha_r \tilde{D} M + \tilde{N} N_c = I. \quad (19)$$

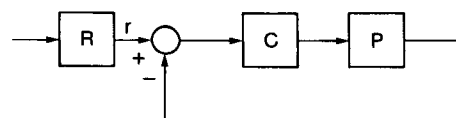


FIG. 4. System for robust tracking.

