Design of Six Sigma Supply Chains

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***Abstract*—**

***Index Terms*—Supply chain lead time, Cycle time compression, Delivery Probability (DP), Delivery Sharpness (DS), Process Capability Indices, Variance Pool Allocation (VPA), Generalized Motorola Six Sigma (GMoSS) Concept**

I. Introduction

SUPPLY chains provide the backbone for manufacturing, service, and E-business companies. The supply chain process is a complex, composite business process comprising a hierarchy of different levels of value-delivering business processes. Achieving outstanding delivery performance is the primary objective of any industry supply chain. Electronic or web-enabled supply chains hold the promise of accelerating the delivery of products to customers but also entail high levels of synchronization among all business processes from sourcing to delivery. Designing supply chains with superior levels of delivery performance is thus an important but at the same time a very challenging problem.

A. Motivation

Lead time of individual business process and amount of inventory maintained at various stages are two prime factors in deciding the quality of the end delivery process in any given supply chain. As one can imagine, when the number of resources, operations, and organizations increases, managing the supply chain and achieving outstanding delivery performance becomes more complex. Given the size and complexity of these supply chains, a common problem for managers is not knowing how to quantify the delivery performance of the supply chain and the trade-off between delivery quality and the investment in inventory required to support that quality level. The problem is made more difficult because real world supply chains are highly dynamic: uncertainty in customer demands, variability in processing time (lead time) at each stage of supply chain, multiple dimensions for customer satisfaction, finite resources, etc.

In this paper, we study the combined effect of lead time variability, demand uncertainty, and inventory levels on the delivery performance of a supply chain and come up with a sound methodology to design supply chains for outstanding delivery performance. We are motivated by variability reduction which is a key idea in areas such as statistical process control, mechanical design tolerancing, and cycle time compression.

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The objective of this paper is two fold. First, we recognize the key role of variability reduction in achieving outstanding delivery performance in a given supply chain and explore the connection between statistical design tolerancing and lead time compression. Using this analogy, in this paper, we propose *six sigma supply chains* as a notion to describe supply chains with superior delivery performance. We then investigate the design of six sigma supply chains through mathematical programming techniques.

Second, we describe a representative supply chain model, with four stages: supplier, inbound logistics, manufacturers, and outbound logistics [1], that is appropriate for an asset manager to use in quantitatively assessing the inventory-service level trade off. We investigate this supply chain to explore the connection between design of six sigma supply chains and supply chain inventory optimization.

B. Relevant Work

The subject matter of this paper falls in the intersection of several areas of current interest. These include: (1) variability reduction and lead time compression techniques for business processes, (2) statistical design tolerancing, and in particular, the Motorola six sigma program, and (3) inventory optimization in supply chains.

Lead time compression in business processes is the subject matter of a large number of papers in the last decade. See for example, the papers by Hopp, Spearman, and Woodruff [2]; Adler, Mandelbaum, Nguyen, and Schwerer [3]; and Narahari, Viswanadham, and Kiran Kumar [4]. Variability reduction is a key strategy used in the above papers and other related papers. Hopp and Spearman, in their book [5], have brought out this key role played by variability reduction. Lead time compression in supply chains is the subject of several recent papers, see for example, Narahari, Viswanadham, and Rajarshi [6].

Statistical design tolerancing is a mature subject in the design community. The key ideas in statistical design tolerancing which provide the core inputs to this paper are: (1) theory of process capability indices [7], [8], [9]; (2) tolerance analysis and tolerance synthesis techniques [10], [11], [12]; (3) Motorola six sigma program [13], [14]; (4) Taguchi methods [15], [16]; and (5) design for tolerancing [17], [18], [19].

Inventory optimization in supply chains is the topic of numerous papers in the past decade. Important ones of relevance here are on multiechelon supply chains [20], [21], [22] and bullwhip effect [23]. Variability reduction is a central theme in many of these papers. Recent work by L.B. Schwartz and Z. Kevin Weng [1] is particularly relevant here. This paper discusses the joint effect of lead time variability and demand uncertainty, as well as the effect of "fair-shares" allocation, on safety stocks in a four-link JIT supply chain. The formulation by Masters [20] is similar to what we have in mind here, although his decision variables are different from those identified here. The supply chain network model developed by Markus [21] resembles our model in several ways. This model takes lead times, the demand and cost data, and the required customer service level as input. In return, the model generates the base stock level at each store-the stocking location for a part or an end-product, so as to minimize the overall inventory capital throughout the network and to guarantee the customer service requirement. Other notable contributions in this direction are due to Schwarz [24], Song [25], and Eppen [26].

The salient feature of our model which makes it attractive and distinguishes it from all the above discussed models, is the notion of six sigma quality for end delivery process. Existing models in the literature consider either the availability of product to the customer as a criterion for customer service level or probability of delivering the product to the customer within a window as a measure of customer's service level. Away from these classical measurements of customer service levels in the inventory optimization problem, we propose an entirely different and novel approach for customer service level: namely accurate and precise deliveries which is primary objective of any modern electronic or web-enabled supply chain.

C. Outline of the Paper

Section 2 of this paper presents a review of work on process capability indices C_p, C_{pk}, C_{pm}, and a few interesting findings about them. An interesting contribution of this section is to interpret these indices for lead times in supply chains and define two novel metrics, *delivery probability (DP)* and *delivery sharpness (DS)*, for the purpose of measuring the delivery performance of the supply chains. The conceptual contributions of this paper is to generalize the notion of Motorola six sigma quality and define the notion of *six sigma supply chain* based on that. A comprehensive material is devoted to this contribution in Section 3. The findings of Section 2 and Section 3 are used in Section 4 where a general mathematical programming problem is formulated for design of six sigma supply chains. A few representative design problems, based on the formulation, are also addressed in this section. In Section 5, we describe a representative four stage supply chain and formulate the design problem for it. We show that the design problem becomes a nonlinear optimization problem with equality and inequality constraints. We solve this problem, after relaxation of inequality constraints, through the Lagrange Multiplier method. The solution provides insights into inventory tradeoffs in six sigma supply chains. A counterintuitive result here is that even a completely make-to-order, inventoryless system could provide the best option. Implications and future work constitute Section 6.

II. A Review of Work on Process Capability Indices

A. Introduction

The process capability indices C_p, C_{pk}, and C_{pm} [7] are popular in the areas of design tolerancing and statistical process

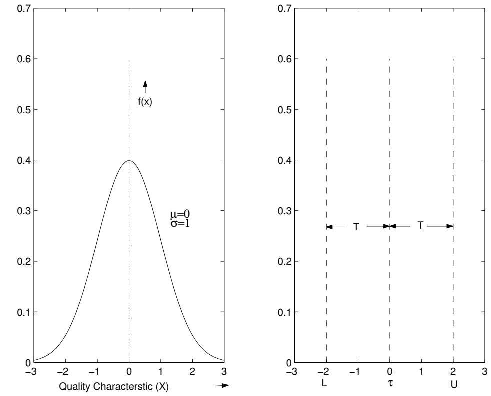

Fig. 1. Process variability and customer delivery window

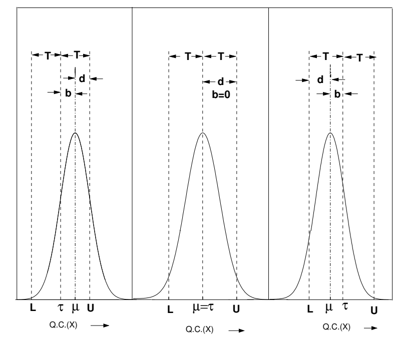

Fig. 2. Process characterization

control. Whether it is a manufacturing process which is producing some parts with given design dimensions, a service process delivering a specified level of service or a business process delivering the products in a certain time interval, the variation is an inherent feature. By the laws of physics, it is known that variation is ever-present in the universe, resulting in the impossibility of ever making two products alike. However, it is possible to describe probabilistically the *chances* of any given part produced within specifications or performance goal is met within specification. In such a situation, the capability of the process is dependent on both customer specification (qualitative or quantitative, which describes what customer 'wants') as well as process variation.

Let us consider the situation depicted by Figure 1 in order to describe an idea of how capability of a process, where variability is an inherent effect, can be measured. The notation used in this figure is listed in Table I. In this figure, variability of the process is characterized by the probability density of the quality characteristic X produced by the process, and customer specifications are characterized by a delivery window which consists of tolerance T and target value τ. Normal distribution is a popular and common choice for X because it is the basis for the theory of process capability indices. The target value τ can be any value between L and U but we have assumed it as the mid point of two limits because for the sake of convenience. Figure 2 explains different possible geometries of the probability density curve and customer delivery window when superimposed.

From these figures it is quite intuitive that the capability of

TABLE I

NOTATION USED IN THE DEFINITIONS OF PCIS

X	Lead time or any general quality characteristic X
μ	Mean of X
σ	Standard deviation of X
L	Lower specification limit of customer delivery window
U	Upper specification limit of customer delivery window
τ	Target value for X, specified by customer
T	Tolerance for X, specified by customer
b	Bias $\|\tau - \mu\|$
d	$\min(\|U-\mu\|, \|\mu - L\|)$

a process can be computed by comparing the distribution of the process output within product tolerances. To measure the capability of the process, Juran *et al* [27]first introduced the concept of process capability indices. Juran defined the first process capability index, C_p, and the others were developed to provide additional information about the process. In next section we describe the three most popular indices C_p, C_{pk}, and C_{pm} along with their implication.

B. The indices C_p, C_{pk}, C_{pm}

1) The Index C_p: The process capability index, C_p, is defined as

$$C_p = \frac{U-L}{6\sigma}$$

Because it is assumed here that the distribution of X is normal and the target value of lead time τ is the mid point of U and L for any business process. Hence C_p can be expressed in following equivalent form.

$$C_p = \frac{T}{3\sigma} \tag{1}$$

where $T = \text{tolerance} = \frac{U-L}{2}$

C_p measures only the potential of a process to produce acceptable products. It does not bother about actual yield of the process where potential and actual yield of any process are defined in following manner.

Actual Yield: The probability of producing a part within specification limits.

Potential: The probability of producing a part within specification limits, if process distribution is centered at the target value i.e. $\mu = \tau$.

It is easy to see [28] that the potential of the process is equal to the area under the probability density function taken from $X = L$ to $X = U$ when $\mu = \tau$ and it can be expressed by following relation:

$$\text{Potential} = 2\Phi(3C_p) - 1 \tag{2}$$

where $\Phi(Z)$ is the cumulative distribution function of standard normal distribution.

2) The Index C_{pk}: Index C_p does not reflect the impact that shifting the process mean or target has on a process's ability to produce a product within specification [8]. For this reason, the C_{pk} index was developed. C_{pk} is defined as follows:

$$C_{pk} = \frac{\min(U-\mu, \mu - L)}{3\sigma} = \left(\frac{d}{3\sigma}\right) \tag{3}$$

C_{pk} alone is not enough to measure actual yield of the process. However, when used with C_p, it can measure the actual yield of the process. The formula for actual yield can be given as below. A proof for this is provided in [28].

$$\text{Actual Yield} = \Phi(3C_{pk}) + \Phi(6C_p - 3C_{pk}) - 1 \tag{4}$$

3) The Index C_{pm}: Actual yield of the process is related to the fraction of the total number of units produced by the process which are defective, called as fraction defective. It is common to measure the quality of the process in terms of fraction defective. Although commonly used, this measure of quality is often incomplete and misleading when used alone. Fraction defective is an indicator for process precision and it does not take accuracy of the process into account. Accuracy of the process is something which analyzes the pattern in which value X is distributed within tolerance limits. It investigates whether more parts or less parts are having an X value nearer to target. In order to include the notion of accuracy along with precision, Hasiang and Taguchi defined independently the index C_{pm} [16]. Later it was defined formally by Chen *et al* [29] as follows

$$C_{pm} = \frac{\text{U-L}}{6\xi} = \frac{\text{T}}{3\sqrt{\sigma^2 + s^2}} \tag{5}$$

Quantity $E(L) = \sigma^2 + s^2$ is known as "Expected Taguchi Loss" [15], [9].

C. Relationship and Dependencies among C_p, C_{pk}, C_{pm}

The following relations can be derived among C_p, C_{pk}, C_{pm} [28].

$$C_p \geq C_{pk} \geq 0\,; C_p \geq C_{pm} \geq 0 \tag{6}$$

$$C_{pk} = C_p(1-k) \text{ where } k = \frac{b}{T} \tag{7}$$

$$\frac{1}{9C_{pm}^2} = \frac{1}{9C_p^2} + \left(1 - \frac{C_{pk}}{C_p}\right)^2 \tag{8}$$

Besides mutual relationships among themselves, indices have a tight coupling with process yield also. It is easy to show [28] that for a given value of actual yield α (say), there exist lower and upper bounds for the values of both C_p and C_{pk}. We denote these lower and upper bounds by $\underline{C_p^\alpha}, \underline{C_{pk}^\alpha}, \overline{C_p^\alpha}$ and $\overline{C_{pk}^\alpha}$ respectively. A crisp idea behind the intent of these bounds is as follows.

- If process's C_p (C_{pk}) is less than $\underline{C_p^\alpha}$ $\left(\underline{C_{pk}^\alpha}\right)$ then its actual yield cannot be equal to α, no matter how large C_{pk} (C_p) is.
- If process's C_{pk} is greater than or equal to $\overline{C_{pk}^\alpha}$ then its actual yield cannot be less than α, no matter how small C_p is.

TABLE II

BOUNDS ON PCIS FOR ACTUAL YIELD = α

Bound	Formula
$\underline{C_p^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)$
$\underline{C_{pk}^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}(\alpha)\right)$
$\overline{C_{pk}^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)$
$\overline{C_p^\alpha}$	$\frac{1}{6}\left(3\underline{C_{pk}^\alpha} + \Phi^{-1}\left(1+\alpha-\Phi\left(3\underline{C_{pk}^\alpha}\right)\right)\right) = \infty$

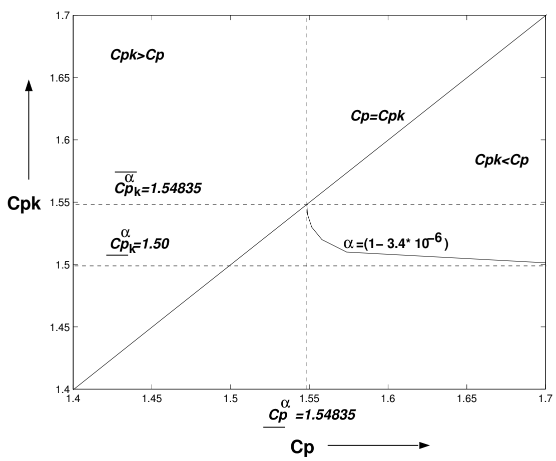

Fig. 3. Typical variation of C_{pk} with respect to C_p for constant actual yield

- The case with $\overline{C_p^\alpha}$ is a little different. For any value of C_p greater than or equal to $\overline{C_p^\alpha}$ it is possible to find a corresponding C_{pk} such that the actual yield of the process is α.

Table II summarizes such bounds on C_p and C_{pk}. Figure 3 shows a typical variation of C_{pk} with respect to C_p when the actual yield α is constant. It immediately follows from this curve that for a given pair (C_p, C_{pk}), the value of actual yield is fixed. But for a given actual yield value, there exist infinite such (C_p, C_{pk}) pairs.

D. Delivery Probability and Delivery Sharpness

In a typical manufacturing process, size, shape, strength, and color of the product are usual quality characteristics X_i and customers provide specification for each one of them to the supplier. Similarly, for every business process which is a part of modern electronic or web enabled supply chain, delivery time of product or service is an important quality characteristic. Variability in lead time is inherent to almost all the business processes, therefore, it will be apt to apply the notion of process capability indices to measure the delivery capability or delivery quality of any business process [30].

It is easy to see from the relations presented in the last section that for a given business process and for a given value of actual yield, there exist infinite pairs (C_p, C_{pk}) such that each one results in the same given actual yield. Similarly, for a given value of C_{pm}, there exists an infinite number of feasible (C_p, C_{pk}) pairs. Nevertheless, it can be shown that for a given pair (Actual yield, C_{pm}) there exists a unique feasible pair (C_p, C_{pk}). It suggests that the 3-tuple (C_p, C_{pk}, C_{pm}) is sufficient to measure the delivery quality of any business process in a given supply chain. This 3-tuple (C_p, C_{pk}, C_{pm}) can be substituted by the pair (Actual yield, C_{pm}) to measure the delivery quality. Being an indicator for precision and accuracy of the deliveries, we prefer to call actual yield of the process as *Delivery Probability (DP)* and C_{pm} as *Delivery Sharpness (DS)*. In the present paper, we use these two indices to measure the quality of any delivery process in a given supply chain. Rather than expressing the DP in terms of numerical values, we prefer to express it in terms of $\theta\sigma$ levels where $\theta \in \Re^+$. Each $\theta\sigma$ level corresponds to a unique number in the interval $[0, 1]$.

We are motivated by the Motorola six sigma (MSS) program in using the idea of $\theta\sigma$ levels. In the MSS program, a unique σ level is attached to a unique number in the interval $[0, 1]$. In this program these numbers correspond to upper bounds on the yield of the process but here we assume that these numbers correspond to the actual yield of the process. For example, according to the MSS program, in the presence of process mean shifts and drifts, if upper bound on yield of the process is equal to $1 - 3.4 \times 10^{-6}$ then its quality is 6σ quality. In the framework of DP and DS, we call a process DP is 6σ iff its actual yield is $1 - 3.4 \times 10^{-6}$. Moreover, in the framework of DP and DS no shifts and drifts are allowed in process mean, only bias is allowed between process mean and target value.

III. NOTION OF SIX SIGMA SUPPLY CHAINS

A. Motorola Six Sigma Quality Program

The six sigma concept, proposed by Harry in 1987 at Motorola Inc. [14], [13], is a way to measure fractional defectives in a lot. Six sigma quality is the benchmark of excellence for product and process quality. The idea behind this concept is as follows. A unique σ level is attached with each value of *number of defects per million opportunities (npmo)*. The npmo is the probability, expressed on a scale of 10^{-6}, that a part is produced with quality characteristic X lying outside the specification limits. Here it is assumed that X is normally distributed. Also, it is assumed that target value τ is the midpoint of upper specification limit (U) and lower specification limit (L).

It is common in all manufacturing processes that as the machine tool begins to wear and other independent variables such as room temperature, material hardness etc. come into play, μ begins to drift away from the nominal value of engineering specification. It is assumed that the variance in process mean is zero and also there is one sided 1.5σ shift (bias) in the mean

The idea behind fixing the value of σ levels in this case is as follows. If U and L coincide with $(\tau + \sigma)$ and $(\tau - \sigma)$ respectively, which is different than $(\mu + \sigma)$ and $(\mu - \sigma)$ (see Figure 4), then corresponding upper bound on yield is assigned as 1σ level. Similar is the case with 2σ, 3σ, and others. In this case, instead of actual yield, upper bound on yield is used to attach value with σ levels because for normal distribution the tail area

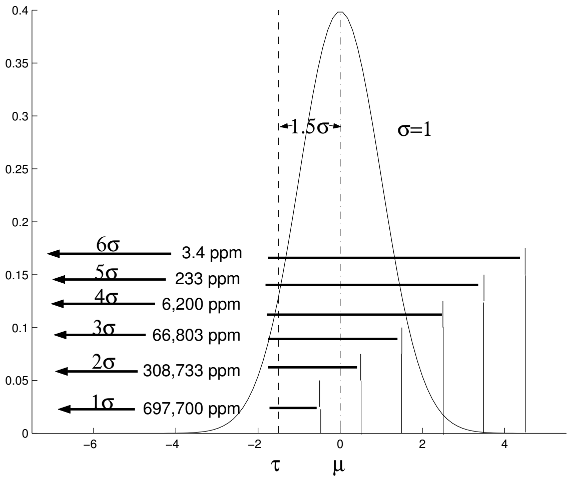

Fig. 4. MSS quality in the presence of shifts and drifts in process mean

is too small and there is no appreciable difference between the actual yield and upper bound for 6σ and higher quality levels.

B. Motorola Six Sigma Quality Program: A Generalized View

When we say that our goal is to achieve $\theta\sigma$ quality, it seems more realistic and logical to have some target value for actual yield instead for upper bound. In this setting, we define $\theta\sigma$ quality as the actual yield equal to upper bound given by MSS program for the same quality level. For example, we call DP of the process is 6σ iff actual yield of the process is $(1 - 3.4 \times 10^{-6})$ which is the upper bound for 6σ quality according to MSS program.

For a given (C_p, C_{pk}) pair, the value of actual yield is fixed. But for a given actual yield value, there exist infinite such (C_p, C_{pk}) pairs. Hence DP can be completely determined by knowing C_p and C_{pk}. However, there are numerous (in fact, infinitely many) ways in which we can choose the pair (C_p, C_{pk}) to achieve a given value of DP. This leads to a generalized view of six sigma quality. MSS is a special case of this in which bias is fixed i.e. 1.5σ. In order to explain this idea let us start with the equation:

$$\text{actual yield} = \Phi\left(3C_{pk}\right) + \Phi\left(6C_p - 3C_{pk}\right) - 1$$

If we fix the value of actual yield as α in the above equation then there will be two independent variables C_p, C_{pk}, hence the solution set will be unbounded. But we have earlier shown that for a given actual yield α, C_p and C_{pk} are bounded within certain range. Hence the solution is bounded by $\underline{C_{pk}^{\alpha}} \leq C_{pk} \leq \overline{C_{pk}^{\alpha}}$; $\underline{C_p^{\alpha}} \leq C_p \leq \infty$. If we substitute $\alpha = (1-3.4\times10^{-6})$ and plot a graph, then all points lying on the curve give (C_p, C_{pk}) pairs that correspond to the 6σ quality level. This equation can be generalized for any $\theta\sigma$ level by expressing α in terms of θ. It is easy to see from Figure 4 that the upper bound in the MSS program for $\theta\sigma$ level is $\Phi(\theta - 1.5)$. Equating this to the actual yield of the process we get the following equation for $\theta\sigma$ quality curve on the $C_p - C_{pk}$ plane.

$$\Phi(\theta - 1.5) = \Phi\left(3C_{pk}\right) + \Phi\left(6C_p - 3C_{pk}\right) - 1$$

Some of these curves are plotted in Figure 5. We can proceed one step further by looking at the connection between delivery probability and delivery sharpness in the light of our generalized notion of six sigma quality. For this, we consider the plots of σ quality levels on $C_p - C_{pk}$ plane and then see how C_{pm} behaves on the same plot. To see this we use the identity relation (8) among C_p, C_{pk}, and C_{pm} and plot this relation for a constant value of C_{pm} (say C_{pm}^*). The plot comes out to be a section of a hyperbola. From a process design point of view, it can be said that for a desired level of DS (i.e. C_{pm}) and DP (i.e. C_p, C_{pk}), this curve provides a set of 3-tuples (C_p, C_{pk}, C_{pm}) which all satisfy these two requirements. The designer has to decide which one of the triples to choose depending upon the requirements. Figure 5 shows some C_{pm} curves on the $C_p - C_{pk}$ plane.

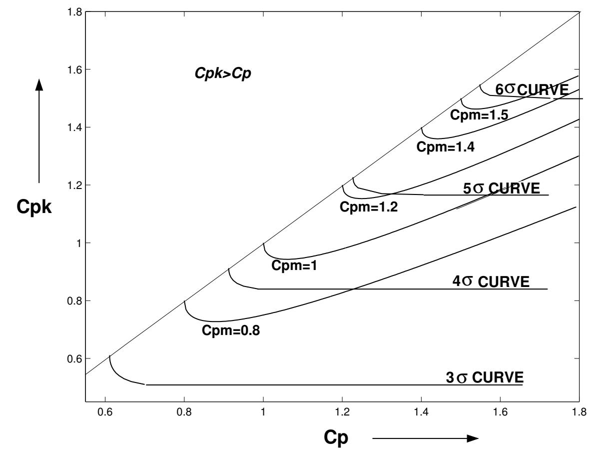

Fig. 5. $\theta\sigma$ curves and C_{pm} curves on $C_{pk} - C_p$ plane

C. Notion of Six Sigma Supply Chains

Motivated by the discussion in the previous section, we seek to define the notion of six sigma supply chains, to characterize a supply chain with superior delivery performance.We define a six sigma supply chain as a network of supply chain elements which, given the customer specified window and the target delivery date, results in defective deliveries (i.e. DP) not more than 3.4 ppm. All triples (C_p, C_{pk}, C_{pm}) that guarantee an actual yield of at least 3.4 ppm (or DP=6σ) would correspond to a six sigma supply chain.

Table III provides sample values of process capability indices that achieve six sigma delivery performance. It is important to note that in order to achieve DP=6σ, the delivery sharpness needs to assume appropriately high values. In a given setting, however, there may be a need for extremely sharp deliveries (highly accurate deliveries) implying that the C_{pm} index is required to be very high. This can be specified as an additional requirement of the designer.

IV. Design of Six Sigma Supply Chains

A. Formulation of the Design Problem

We can say that the design objective in supply chain networks is to deliver the finished products to the customers within a time

TABLE III

SAMPLE VALUES OF 3-TUPLES (C_p, C_{pk}, C_{pm}) WHICH ACHIEVE SIX SIGMA DELIVERY PERFORMANCE

C_p	C_{pk}	C_{pm}
1.548350	1.548350	1.548350
1.548900	1.540000	1.548348
1.551535	1.530000	1.548307
1.557998	1.520000	1.547972
1.573665	1.510000	1.545724
1.721814	1.500010	1.433445
1.726667	1.5000000001	1.427826

as close to the target delivery date as possible, with as few defective deliveries as possible at the minimum cost. To give an idea of how the design problem of a complex supply chain network can be formulated, let us consider a supply chain with n business processes such that each of them contributes to the order-to-delivery cycle of customer desired products. Let X_i be the cycle time of process i. It is realistic to assume that each X_i is a continuous random variable with mean μ_i and standard deviation σ_i. The order-to-delivery time Y can then be considered as a deterministic function of X_i's:

$$Y = f(X_1, \ldots, X_n)$$

If we assume that the cost of delivering the products depends only on the first two moments of these random variables, the total cost of the process can be described as:

$$Z = g(\mu_1, \sigma_1, \ldots, \mu_n, \sigma_n)$$

where g is some deterministic function.

The customer specifies a lower specification limit L, an upper specification limit U, and a target value τ for this order-to-delivery lead time. With respect to this customer specification, we are required to choose the parameters of $X_1, \ldots X_n$ so as to minimize the total cost involved in reaching the products to the customers, achieving a six sigma level of delivery performance.

Thus the design problem can be stated as the following mathematical programming problem:

Minimize $Z = g(\mu_1, \sigma_1, \ldots, \mu_n, \sigma_n)$
subject to

$$\begin{aligned} \text{DS for order-to-delivery time} &\geq C^*_{pm} \\ \text{DP for order-to-delivery time} &\geq 6\sigma \\ \mu_i &> 0\ \forall i \\ \sigma_i &> 0\ \forall i \end{aligned}$$

where C^*_{pm} is a required lowerbound on delivery sharpness. The objective function Z of this formulation captures the total cost involved in taking the product to the customer, going through the individual business processes. We have assumed that this cost is determined by the first two moments of lead times of the individual business processes. One can define Z in a more general way if necessary. The decision variables in this formulation are means and/or standard deviations of individual processes. The constraints of this formulation guarantee a minimum level of delivery sharpness (C^*_{pm} is the minimum level of delivery sharpness required) and at least a six sigma level of delivery probability.

While solving the design problem, an important step is to express the constraints in terms of the decision variables. This will be elaborated upon in the next section.

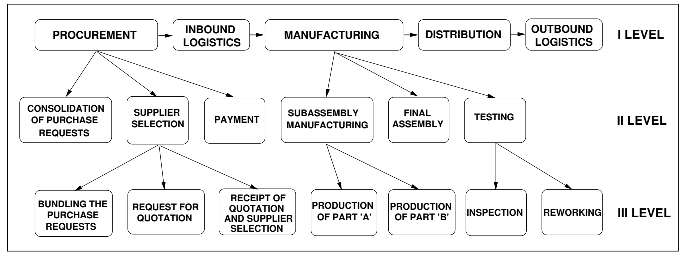

Fig. 6. Idea of hierarchical design

B. Concept of Hierarchical Design

The supply chain process is a complex, composite business process comprising a hierarchy of different levels of value-delivering business processes. At the first level of this hierarchy, we have processes such as procurement, inbound logistics, manufacturing, distribution, outbound logistics, etc. Each of these value-delivering processes can be further decomposed resulting in a second level of the hierarchy. Each of the second level processes can be further decomposed into third level processes and so on. See Figure 6. One can exploit the above hierarchy and natural decomposition to come up with a hierarchical design methodology for six sigma supply chains. The idea is to first formulate the design problem at the highest level of abstraction and obtain the optimal values of decision variables at that level. Use these optimal values as input parameters to a design problem at the second level of abstraction and obtain optimal values for the problem at the second level of detail. Now formulate the design problem at the third level and so on.

C. Representative Design Problems

Depending on the nature of the objective function and decision variables chosen, the six sigma supply chain design problem assumes interesting forms. We consider some problems below under two categories: (1) generic design problems and (2) concrete design problems.

1) Generic Design Problems:
- Optimal allocation of process means
- Optimal allocation of process variances
- Optimal allocation of customer windows

2) Concrete Design Problems:
- Due date setting
- Choice of customers
- Inventory allocation
- Capacity planning
- Vendor selection
- Choice of logistics
- Choice of manufacturing control policies

These problems can arise at any level of the hierarchical design. Thus inorder to develop a complete suite for designing a complex supply chain network for six sigma delivery performance through the hierarchical design scheme, we need to address all such sub problems beforehand. In the next section, we consider one such subproblem, optimal allocation of inventory in a multistage six sigma supply chain, and develop a methodology for this problem.

V. Inventory Optimization in a Multistage Supply Chain

In this section, we describe a representative supply chain model, with four stages: supplier, inbound logistics, manufacturer, and outbound logistics [1]. We formulate the six sigma design problem, based on the theory developed in earlier sections, for this supply chain and explore the connection between delivery performance of the supply chains and supply chain inventory optimization.

A. A Four Stage Supply Chain Model with Demand and Lead Time Uncertainty

1) Model Description: Consider N geographically dispersed distribution centers (DCs) supplying retailer demand for some product as shown in Figure 7. The product belongs to a category which does not make it profitable for the distribution center to maintain any inventory. An immediate example is a distributor who supplies trucks laden with bottled Liquid-Petroleum-Gas (LPG) cylinders (call these as LPG trucks or finished product now onward) to retail outlets and industrial customers. In a situation like this, as soon as a demand for a LPG truck arrives at any DC, the DC immediately places an order for one unit of product (in this case, an LPG truck) to a major regional depot (RD). The RD maintains an inventory of LPG trucks and after receiving the order, if on-hand inventory of LPG trucks is positive then an LPG truck is sent to the DC via outbound logistics. On the other hand, if on-hand inventory is zero, the order gets backordered at the RD. At the RD, the processing involves unloading the LPG from LPG tankers into LPG reservoirs, filling the LPG into cylinders, bottling the cylinders and finally loading the cylinders onto trucks.

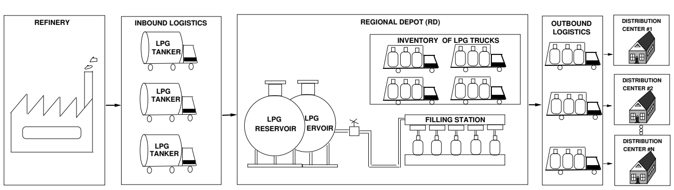

Fig. 7. A four link linear supply chain model

The inventory at RD is replenished as follows. The RD starts with on-hand inventory R and every time an order is received, it places an order to the supplier for one LPG tanker (called as semi-finished product now onward) which is sufficient to produce one LPG truck. In this case, the supplier corresponds to a refinery which will produce LPG tankers. In the literature such a replenishment model is known as the $< Q, R >$ model [31] with $Q = 1$. In such a model, the inventory position (on-hand plus on-order minus backorders) is always constant and is equal to R.

It is assumed that raw material (crude oil or naphtha) required for producing an LPG tanker is always available with the refinery, but the refinery needs to do some processing of this raw material to transform it to LPG and load it onto a tanker. Therefore, as soon as the refinery receives an order from the RD, it starts processing the raw material and sends an LPG tanker via inbound logistics to the RD.

We shall generalize the LPG supply chain to a four stage supply chain network. Description in parentheses corresponds to the LPG example. Let us call these stages as

1) Procurement or Supplier (refinery)
2) Inbound Logistics (transportation of LPG tankers from refinery to RD)
3) Manufacturing (RD)
4) Outbound Logistics (customer order processing and transportation of LPG trucks from RD to a DC)

The distribution centers in the LPG example correspond to the customer in the general setting of four stage supply chain network. In the next section, we articulate all the assumptions we have made regarding behavioral and operational characteristics of this model. We believe that these assumptions are reasonable enough to make the model realistic and at the same time the mathematical model that is formulated out of it is tractable.

2) Assumptions :

1) A customer places her order for only one unit of finished product (called as item) to the manufacturer i.e., batch orders are not permitted from customers.
2) The orders arrive at the manufacturer in Poisson fashion from each customer. The Poisson arrival streams of orders are independent across the customers.
3) Each customer specifies a delivery window while placing an order. This window is assumed to be the same for all the customers. Also, in this window the date which customer targets for delivery of the item has equal offset from the upper specification limit and lower specification limit.
4) If the item is not on-hand with the manufacturer, then the customer's order gets backordered there. All such backorders are fulfilled in FIFO manner by the manufacturer because items for different orders are indistinguishable.
5) Lead time for an item at each stage of the supply chain is a *normal* random variable. Lead times of the four stages are mutually independent.
6) The supplier can be viewed as comprising infinite servers, which implies that as soon as it receives an order from the manufacturer, processing commences on the corresponding raw material. Thus, there is no queue in front of the supplier node. The lead time of these servers are *iid* random variables.
7) Inbound and outbound logistics facilities are always available. Therefore, as soon as an item finishes its processing at the supplier, its shipment starts via the inbound logistics. Similarly, as soon as an order of a customer is received by the manufacturer, the shipment of an item, if available, commences using the outbound logistics. Otherwise the shipment commences as soon as it becomes

available at the manufacturing node (following an FIFO policy). Inbound logistics lead times are iid random variables. Similarly outbound logistics lead times are also *iid* random variables.

8) The manufacturing node has infinite processing capacity. This means that any number of items can get their processing done at the same time. Therefore, an item that arrives from the supplier does not wait in queue at manufacturer for getting it processed. Lead time for items at the manufacturing node are *iid* random variables.
9) The processing cost per item at each stage depends only on the mean and variance of the lead time of the stage.
10) Costs related to maintenance of inventory at the manufacturing node are fixed. Such costs include order placing cost, inventory carrying cost, cost of raw material of an item, fixed cost against backorder of an item, etc. But the variable cost of backorder is a function of time for which the order is backordered.

3) System Parameters: This section presents the notation used for various system parameters.

Lead Time Parameters:

$$X_1 \sim N\left(\mu_1, {\sigma_1}^2\right) = \text{Procurement lead time of an item}$$
$$X_2 \sim N\left(\mu_2, {\sigma_2}^2\right) = \text{Inbound logistics lead time of an item}$$
$$X_3 \sim N\left(\mu_3, {\sigma_3}^2\right) = \text{Manufacturing lead time of an item}$$
$$X_4 \sim N\left(\mu_4, {\sigma_4}^2\right) = \text{Outbound logistics lead time of an item}$$
$$L_m = \text{End-to-end lead time of manufacturer's order}$$
$$L_f = \text{Time required, after placing the order by manufacturer, to get the product ready with manufacturer in finished form}$$
$$L_c = \text{End-to-end lead time of customer's order}$$
$$\overline{L_c} = \text{An upper bound on } L_c$$
$$\left(\mu_m, {\sigma_m}^2\right) = \text{Mean and variance of } L_m$$
$$\left(\mu_f, {\sigma_f}^2\right) = \text{Mean and variance of } L_f$$
$$\left(\mu_c, {\sigma_c}^2\right) = \text{Mean and variance of } L_c$$
$$\left(\bar{\mu_c}, \bar{\sigma_c}^2\right) = \text{Mean and variance of } \overline{L_c}$$

Demand Process Parameters:

$$\lambda_i = \text{Order arrival rate from } i^{\text{th}} \text{ customer (item/year)}$$
$$\lambda = \sum_{i}^{N} \lambda_i = \text{Poisson arrival rate of orders at the manufacturer}$$
$$R = \text{Inventory level at the manufacturing node}$$
$$Q = 1 = \text{Reorder quantity of the manufacturer}$$
$$M_o = \text{Stockout probability at the manufacturing node}$$
$$E = \text{Average number of backorders per unit time at the manufacturing node (item/time)}$$
$$B = \text{Expected number of backorders with the manufacturer at arbitrary time } t \text{ (item)}$$
$$D = \text{Expected number of onhand inventory with the manufacturer at arbitrary time } t \text{ (item)}$$
$$\psi_m(x) = \text{Steady state probability that the manufacturer has net inventory equal to } x$$
$$p\left(x;\lambda t\right) = \frac{\exp(-\lambda t)(\lambda t)^x}{x!}$$
$$P(r;\lambda t) = \sum_{x=r}^{\infty} p\left(x;\lambda t\right)$$

Cost Parameters:

$$\mathcal{K}_1 = f_1\left(\mu_1,\sigma_1\right) = \text{Procurement cost (\$/item)}$$
$$\mathcal{K}_2 = f_2\left(\mu_2,\sigma_2\right) = \text{Inbound logistics cost (\$/item)}$$
$$\mathcal{K}_3 = f_3\left(\mu_3,\sigma_3\right) = \text{Manufacturing cost (\$/item)}$$
$$\mathcal{K}_4 = f_4\left(\mu_4,\sigma_4\right) = \text{Outbound logistics cost (\$/item)}$$
$$A = \text{Order placing cost for manufacturer (\$/order)}$$
$$\Pi = \text{Fixed part of backorder cost (\$/item)}$$
$$\widehat{\Pi} = \text{Variable part of backorder cost (\$/item-time)}$$
$$I = \text{Inventory carrying cost (\$/time-\$ invested)}$$
$$C = \text{Cost of raw material (\$/item)}$$
$$C_m = \text{Capital tied up with each item ready to be shipped via outbound logistics (\$/item)}$$

Delivery Quality Parameters:

$$C_p, C_{pk}, C_{pm} = \text{Supply chain process capability indices for end-to-end lead time of customer order}$$
$$(\tau, T) = \text{Delivery window specified by customer}$$
$$U = \tau + T = \text{Upper limit of delivery window}$$
$$L = \tau - T = \text{Lower limit of delivery window}$$
$$b = |\tau - \mu_c| = \text{Bias for } L_c$$
$$\bar{b} = |\tau - \bar{\mu_c}| = \text{Bias for } \overline{L_c}$$
$$d = \min(U - \mu_c, \mu_c - L)$$
$$\bar{d} = \min(U - \bar{\mu_c}, \bar{\mu_c} - L)$$

B. System Analysis

1) Lead Time Analysis of Delivery Process: In this section we study the dynamics of flow of material in the chain triggered by an end customer order as well as manufacturer order and calculate the related end-to-end lead times. First, observe that end-to-end lead time experienced by manufacturer after placing an order to supplier (i.e. L_m) is the sum of X_1 and X_2. Therefore,

$$L_m = \sum_{i=1}^{2} X_i \text{ with } \mu_m = \sum_{i=1}^{2} \mu_i \text{ and } \sigma_m^2 = \sum_{i=1}^{2} \sigma_i^2$$

Similarly, it is a straightforward calculation that the time taken to get finished product ready, after manufacturer places the corresponding order for semi-finished product to supplier (i.e. L_f) is the sum of X_1, X_2, and X_3. Therefore,

$$L_f = \sum_{i=1}^{3} X_i \text{ with } \mu_f = \sum_{i=1}^{3} \mu_i \text{ and } \sigma_f^2 = \sum_{i=1}^{3} \sigma_i^2$$

The lemma below provides an upper bound on end-to-end lead time experienced by a typical customer.

Lemma V-B.1: An upper bound on end-to-end lead time (L_c) experienced by an end customer is

$$\overline{L_c} = X_4 + M_o\,(X_1 + X_2 + X_3)$$

where M_o is the stockout probability at the manufacturer.
Proof: Recall that the arrival of an end customer order triggers an order request placed by the manufacturer to the supplier. Also, the item is shipped to the end customer immediately if it is available in stock at the manufacturer, otherwise the order gets backordered at the manufacturer. In the first case the lead time experienced by the end customer will be same as the outbound logistics time i.e. X_4. For the second case, assume that the customer order is the i^{th} backorder at the manufacturer where $i = 1, 2, \ldots, \infty$. By virtue of the inventory replenishment policy followed by the manufacturer, there will be $(R + i)$ outstanding orders of semi-finished products immediately after arrival of this backorder. Remember that it is an underlying assumption of the model that X_1, X_2, and X_3 are independent across items also. Hence L_f is independent across all these $(R + i)$ orders. It is a direct consequence of this result that the orders placed by the manufacturer can cross each other which means that a product for which supply chain activities were started later may be ready in finished form earlier than the product for which activities were started earlier. A comprehensive idea of this phenomenon is presented in p. 200-212 of [31].

In view of the crossing of finished products at the manufacturer, it can be said that the finished product which is allocated to some backorder may not be the one which results from the corresponding order placed by the manufacturer to the supplier on arrival of this backorder. If it were so then the time taken to serve a backorder by the manufacturer would have been no more than $X_1 + X_2 + X_3$. But in the presence of crossing and assumption of indistinguishable products as well as FIFO policy for serving the backorders, the time is definitely less than $X_1 + X_2 + X_3$. Hence it can be said that random variable $X_1 + X_2 + X_3 + X_4$ gives an upper bound on the time taken to get the finished product by end customer in the second case.

If we consider the first and second case together in the light of theory of total probability then it is easy to prove that an upper bound on lead time experienced by end customer is $(1 - M_o)\,X_4 + M_o\,(X_1 + X_2 + X_3 + X_4)$ which comes out to be $X_4 + M_o\,(X_1 + X_2 + X_3)$. ∎

It is easy to perceive that if there is no crossover at all, then the end-to-end lead time, in the second case, will be exactly equal to $X_1+X_2+X_3+X_4$ which means that $\overline{L_c}$ will represent the end-to-end lead time L_c rather than an upper bound on this. Not only this, as the variability in lead time or demand process reduces the tendency of crossover also reduces which in turn brings $\overline{L_c}$ closer to L_c. More formally it can be said that the difference $\triangle = \overline{L_c} - L_c$ is a monotonically increasing function of crossover probability p_c.

2) Analysis of Inventory at the Manufacturing Node: Observe that the manufacturer follows a $< Q, R >$ policy, with $Q = 1$, for replenishing the finished product inventory. There is a well-known theorem (after Tackács 1956) [31] for $< Q, R >$ models with $Q = 1$ which says that:

Theorem V-B.1: Let the $< Q, R >$ policy with $Q = 1$ is followed for controlling the inventory of a given item at a single location where the demand is Poisson distributed with rate λ, and the replenishment lead times are nonnegative independent random variables (i.e., orders can cross) with density $g(t)$ and mean μ. The steady state probability of having net inventory (on hand inventory minus backorders) x by such a system can be given by:

$$\psi(x) = \frac{\exp(-\lambda\mu)(\lambda\mu)^x}{x!}$$

In other words, the state probabilities are independent of the nature of the replenishment lead time distribution if the lead times are nonnegative and independent.
In the context of the four stage supply chain, the replenishment lead time for manufacturer is L_f which has already been shown to be independent over finished products (i.e. finished products can cross each other). However, L_f is a normal random variable which is not nonnegative. Therefore, the above theorem cannot be applied to finished product inventory directly.

Nevertheless, it is safe to assume that the probability of X_1, X_2, and X_3 taking negative values is small enough that the above theorem can be applied for lead time L_f without significant error. For example, if $\mu_f \geq 6\sigma_f$ then, the negative area of the PDF of L_f is no more than 10^{-6} which can be ignored for all practical purpose and L_f can be assumed as virtually nonnegative. In view of this argument, the steady state probability of having a net inventory x of finished products with the manufacturer can be given as follows.

$$\psi_m(x) = \frac{\exp(-\lambda\,\mu_f)(\lambda\,\mu_f)^x}{x!} \quad \forall\, x = R, R-1, \ldots, 0, -1, -2, \ldots$$

Now it is no more difficult [31] to derive the expressions for the stockout probability (M_o), the average number of backorders per unit time (E), the expected number of backorders at any random instant (B), and the expected number of onhand inventory at any random instant (D). These expressions are listed below.

$$\begin{aligned} M_o &= P(R; \lambda\,\mu_f) = \sum_{k=R}^{\infty} \frac{\exp(-\lambda\,\mu_f)(\lambda\,\mu_f)^k}{k!} \\ &= 1 - \exp(-\lambda\,\mu_f) \sum_{k=0}^{R-1} \frac{(\lambda\,\mu_f)^k}{k!} \qquad (9) \\ E &= \lambda M_o \qquad (10) \\ B &= \lambda\,\mu_f P(R-1; \lambda\,\mu_f) - R M_o \\ &= M_o\,(\lambda\,\mu_f - R) + \frac{\exp(-\lambda\,\mu_f)(\lambda\,\mu_f)^R}{(R-1)!} \qquad (11) \\ D &= R - \lambda\,\mu_f + B \qquad (12) \end{aligned}$$

The expression for M_o serves in deriving an important conclusion about upper bound on end-to-end lead time for customer (i.e. $\overline{L_c}$) which is summarized in the form of Lemma V-B.2.

Lemma V-B.2: The upper bound on end-to-end lead time experienced by an end customer (i.e. $\overline{L_c}$) is a normal random variable with mean $\bar{\mu_c}$ and variance $\bar{\sigma_c}^2$ given as follows:

$$\bar{\mu_c} = \mu_4 + M_o(\mu_1 + \mu_2 + \mu_3) \quad (13)$$

$$\bar{\sigma_c}^2 = \sigma_4^2 + M_o^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (14)$$

where M_o is given by Equation (9).

Proof: Observe in Equation (9) that M_o depends only on R, λ, μ_1, μ_2, and μ_3. Hence M_o is a scalar quantity for given values of system parameters. Therefore, according to Lemma V-B.1, $\overline{L_c}$ is also normal random variable with mean and variance given above. ∎

C. Formulation of IOPT

The objective of the study here is to find out how variability should be allocated to the lead times of the individual stages and what should be the optimal value of inventory level R, such that specified levels of DP and DS are achieved in the steady state condition for the customer lead time, in a cost effective manner. We call this problem as the *Inventory Optimization* (IOPT) problem in six sigma supply chains.

It is easy to see that an increase in the value of R results in high inventory carrying cost, and improved quality of deliveries. Similarly, variance reduction of lead time at any stage(s) of the supply chain results in a high processing cost and improved quality of deliveries. This means that a specified level of quality for the delivery process can be achieved either by increasing the value of R or by reducing the variance of lead time for one or more stages or both. The problem here is to determine a judicious balance between these two such that the cost is minimized.

Depending upon whether $R > 0$ or $R = 0$, there is a slight change in the formulation of the problem. Therefore, we formulate two separate IOPT problems for the cases $R > 0$ (which we call "with stock") and $R = 0$ (which we call "with zero stock"). Since in both cases, we use a make-to-order policy to pull the products, we can more completely describe these two policies as *MTOS Policy* (Make To Order with Stock) and *MTOZS Policy* (Make To Order with Zero Stock), respectively. The input parameters and decision variables are the same for both MTOS Policy and MTOZS Policy. However, the objective function as well as the constraints are different for these two policies. The input parameters, decision variables, objective function, and constraints in the IOPT problem are as follows.

1) Input Parameters: The input parameters to the IOPT problem are: Mean μ_i of random variable X_i, for $i = 1, 2, 3, 4$, Arrival rate λ for customer orders, Customer delivery window (τ, T), Desired levels of DP $(\theta\sigma)$ and DS (C_{pm}^*) for customer lead time, and Coefficients of the first three terms in Taylor series expansion of processing costs $\mathcal{K}_i$ for $i = 1, 2, 3, 4$.

It is assumed in Section V-A.2 that $\mathcal{K}_i$ is a function of μ_i and σ_i. But μ_i's are known in the IOPT problem. Therefore, $\mathcal{K}_i$ is now a function of σ_i only. Hence the first three terms in the Taylor series expansion of $\mathcal{K}_i$ can be given as follows:

$$\mathcal{K}_i = A_{i0} + A_{i1}\sigma_i + A_{i2}\sigma_i^2 \quad (15)$$

2) Decision Variables: The decision variables in IOPT are standard deviation σ_i of each individual stage i $(i = 1, 2, 3, 4)$, and the inventory position R.

3) The Objective Function: Although the objective function for both MTOS and MTOZS policies is the average annual operating cost (COST) of the supply chain, the expression for COST is different under the two policies. For each policy, we consider only those costs which are influenced by system parameters. An expression for COST for each policy is developed as follows.

MTOS Policy: We identify the following costs as significant costs for this policy.

- Average Annual Outbound Logistics Cost= $\lambda\mathcal{K}_4$ \$/year
- Average Annual Inbound Logistics Cost= $\lambda\mathcal{K}_2$ \$/year
- Average Annual Manufacturing Cost= $\lambda\mathcal{K}_3$ \$/year
- Average Annual Processing Cost for Supplier= $\lambda\mathcal{K}_1$ \$/year
- Average Annual Order Placing Cost= λA \$/year
- Average Annual Backorder Cost= $\Pi E + \widehat{\Pi} B$ \$/year
- Average Annual Inventory Carrying Cost= IDC_m \$/year
- Average Annual Cost of Raw Material = λC \$/year

The sum of all the above mentioned costs gives the COST for the MTOS Policy. This comes out to be

$$\mathcal{K} = \lambda \sum_{i=1}^{4} \mathcal{K}_i + A\lambda + \Pi E + \widehat{\Pi} B + IDC_m + \lambda C \quad (16)$$

It is easy to see that

$$C_m = C + A + \sum_{i=1}^{3} \mathcal{K}_i \quad (17)$$

This results in the following expression for COST.

$$\mathcal{K} = (\lambda + ID)\sum_{i=1}^{3}\mathcal{K}_i + \lambda(\mathcal{K}_4 + A + C) + \Pi E + \widehat{\Pi} B + ID(A + C) \quad (18)$$

Because $\lambda, B, D, E, C, A, \Pi, \widehat{\Pi}, I$ are all known parameters in the IOPT problem, all the constant terms in the above expression can be combined into one single constant α and the equation reduces to

$$\mathcal{K} = (\lambda + ID)(\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3) + \lambda\mathcal{K}_4 + \alpha \quad (19)$$

where $\alpha = \lambda(A + C) + \Pi E + \widehat{\Pi} B + ID(A + C)$. If we use Equation (15) to express $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ then the above expression becomes

$$\mathcal{K} = (\lambda + ID)\left[\sum_{i=1}^{3}\left(A_{i0} + A_{i1}\sigma_i + A_{i2}\sigma_i^2\right)\right] + \lambda\left[A_{40} + A_{41}\sigma_4 + A_{42}\sigma_4^2\right] + \alpha \quad (20)$$

MTOZS Policy: This policy is a special case of the previous one in which the inventory carrying cost need not to be considered. The processing costs for each stage in this policy are the same as those of the MTOS Policy because these costs have no relation to the finished product inventory at the manufacturer. Also, raw material cost and order placing cost are the same as in the previous one. But the expression for the cost of backorders is a little different. Under this policy, not every order of the customer which arrives at the manufacturer finds the finished product and therefore may get backordered there. This implies $M_o = 1$ so the values of E and B become λ and $\lambda\mu_f$, respectively. In view of this, the annual backorder cost becomes $\Pi\lambda + \widehat{\Pi}\lambda\mu_f$. Summing up all the significant costs, we get the following expression for COST in this policy.

$$\mathcal{K} \;=\; \lambda \sum_{i=1}^{4} \mathcal{K}_i + A\lambda + \Pi\lambda + \widehat{\Pi}\lambda\,\mu_f + \lambda C \qquad (21)$$

4) Constraints :

MTOS Policy: Observe that $\overline{L_c}$ is an upper bound on end customer lead time L_c, so if we specify the constraints which assure to attain the specified levels of DP and DS for $\overline{L_c}$, it will automatically imply that L_c attains the same or even better levels of DP and DS than specified. These constraints can be written down as follows.

$$\text{DS for } \overline{L_c} \;\geq\; C^*_{pm} \qquad (22)$$

$$\text{DP for } \overline{L_c} \;\geq\; \theta\sigma \qquad (23)$$

To express these constraints in terms of decision variables σ_i's consider the Lemma V-B.2 which provides the relation of variance $\bar{\sigma_c}^2$ with variances of individual stages. $\bar{\sigma_c}^2$ can be expressed in terms of C_p and C_{pk} of $\overline{L_c}$ in the following manner:

$$\bar{\sigma_c}^2 \;=\; \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2} \qquad (24)$$

where T, the tolerance of customer delivery window, is a known parameter in the IOPT problem and $\bar{d}$ is given as follows.

$$\bar{d} = \min(U - \bar{\mu_c}, \bar{\mu_c} - L) \qquad (25)$$

Substituting the value of $\bar{\mu_c}$, from Lemma V-B.2 in the above relation, we get

$$\bar{d} = \min\left(\left[U - \mu_4 - M_o \sum_{i=1}^{3} \mu_i\right], \left[\mu_4 + M_o \sum_{i=1}^{3} \mu_i - L\right]\right)$$

In the above equation, $U, L, \mu_1, \mu_2, \mu_3, \mu_4$ are all known parameters. Also, M_o, according to Equation (9), depends only on λ, R, and μ_f. Therefore, for a given value of R, $\bar{d}$ is a known parameter. The only unknown quantities in Equation (24) are C_p and C_{pk}. Substituting the value of Equation (24) in Equation (14) we get the following relation which is the crux of the problem of converting constraints in terms of decision variables.

$$\sigma_4^2 + M_o^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \;=\; \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2} \qquad (26)$$

The unknown pair (C_p, C_{pk}) in the above equation is chosen in a way that it satisfies both the constraints (22) and (23). The idea behind getting such a pair is as follows. The relation (26) forces the desired (C_p, C_{pk}) pair to lie on the line $C_{pk} = \frac{\bar{d}}{T}C_p$ in the $C_p - C_{pk}$ plane. Also, it is easy to see that the Constraint (22) forces the desired pair to lie on or above the curve $C_{pm} = C^*_{pm}$ in $C_p - C_{pk}$ plane. Similarly, Constraint (23) forces it to lie on or above the 6σ curve in the same plane. All these result in a feasible region in the $C_p - C_{pk}$ plane. Figure 8 shows all possible geometries for such a feasible region, depending upon the relative position of $C_{pm} = C^*_{pm}$ curve and the $\theta\sigma$ curve. From Figure 8, it is clear that the feasible region

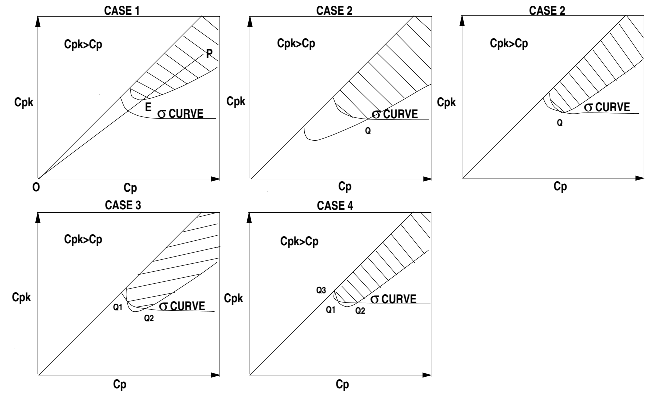

Fig. 8. Possible geometric shapes of feasible region for C_P and C_{pk} of $\overline{L_c}$

in each case is the part of the line $C_{pk} = \frac{\bar{d}}{T}C_p$, denoted by EP, which intersects the shaded region. For the sake of clarity, we have shown the line OP only in Case 1. In all other cases it is understood. Each point of the feasible region satisfies both Constraints (22) and (23) and therefore can be used as a design point in Equation (26). The concern here is which point should be selected as design point. Before we investigate further in this direction, let us consider a few interesting findings about such a (C_p, C_{pk}) pair.

Lemma V-C.1:

For given values of T and $\bar{d}$, there is an upper bound on Delivery Sharpness which can be achieved for $\overline{L_c}$. This is given by

$$\overline{C_{pm}} \;=\; \frac{T}{3\left(T - \bar{d}\right)}$$

Proof: Observe from Equation (26) that, for a given value of T and $\bar{d}$, C_p and C_{pk} of the process $\overline{L_c}$ must satisfy the following relation which is a straight line when plotted on the $C_p - C_{pk}$ plane.

$$C_{pk} \;=\; \left(\frac{\bar{d}}{T}\right) C_p \qquad (27)$$

If we take any point on this line, it represents a unique combination of C_p, C_{pk}, and C_{pm}. Hence if we choose this point as design point then the DS for $\overline{L_c}$ gets fixed. Now consider the following equation for a typical C_{pm} curve on the $C_p - C_{pk}$ plane.

$$\frac{1}{{C_{pm}}^2} \;=\; \frac{1}{C_p^2} + 9\left(1 - \frac{C_{pk}}{C_p}\right)^2$$

It can be verified that this equation represents a hyperbola. It is quite possible that the line given by Equation (27) becomes an asymptote of such a hyperbola. Such a hyperbola is the curve of $\overline{C_{pm}}$ because it is clear from the geometry of the figure that this line cannot intersect any other C_{pm} curve which is more than $\overline{C_{pm}}$. Hence it is not possible to achieve the C_{pm} value (or DS) higher than $\overline{C_{pm}}$ for process $\overline{L_c}$.

It is easy to show that the slope of asymptotes of $\overline{C_{pm}}$ curve is $\left(1 \pm \frac{1}{(3\overline{C_{pm}})}\right)$. Equating these to the slope of the line (27) we get

$$\overline{C_{pm}} = \frac{T}{3(T - \bar{d})}$$

. ■

Lemma V-C.2:
For a given value of T and $\bar{d}$, a unique value of DP gets fixed automatically for $\overline{L_c}$ whenever it is attempted to fix a unique DS for $\overline{L_c}$ and also vice versa. Moreover, these DP and DS have positive correlation.
Proof: Earlier we said that (C_p, C_{pk}) pair is chosen for $\overline{L_c}$ in a way that apart from satisfying both the Constraints (22) and (23), the pair must lie on the line (27).

It is easy to verify that a unique C_{pm} curve and a unique $\theta\sigma$ curve pass through a unique point of the line (27). These $\theta\sigma$ values and C_{pm} values are final DP and DS respectively which are achieved for $\overline{L_c}$ if this particular point is chosen as design point. Hence, it can be concluded that once a value is chosen for DP of $\overline{L_c}$, it will automatically decide the corresponding value of DS and also vice versa. To prove the other statement of the lemma, observe that as we move from point $C_p = 0$ to point $C_p = \infty$ on the line (27), the values of both C_{pm} curve and σ curve which pass through that point increase. Therefore, DP increases (or decreases) as DS increases (or decreases) for given values of T and $\bar{d}$. ■

The implication of Lemma V-C.1 is as follows. If desired C^*_{pm} is greater than $\overline{C_{pm}}$ for given values of T and $\bar{d}$ then the problem is infeasible. In such situation we need not proceed any further. Lemma V-C.2 also has a key implication on the problem of fixing the values of C_p and C_{pk} for $\overline{L_c}$. According to Lemma V-C.2, DP and DS of $\overline{L_c}$ get fixed immediately as soon as a feasible point from line (27) is chosen as design point. It is easy to see that each point on the $C_{pk} - C_p$ plane is unique on its own because it has a unique combination of DP and DS. Therefore, it is quite possible the point which we have chosen results in either higher DP or higher DS than required for the end-to-end delivery process. Hence it can be claimed that it is not always true that the DP and DS obtained for $\overline{L_c}$ from design are exactly same as given in Constraints (22) and (23).

In view of the above findings, the problem of fixing the values of C_p and C_{pk} can be addressed as follows. First step towards this is to test the feasibility of the problem through Lemma V-C.1. If the problem turns out to be feasible then each point in the feasible region is allowed to be chosen as design point. However, depending upon the point which is chosen as design point, the final cost $\mathcal{K}^*$ (which we get out of solving the optimization problem) may vary. At this point, we cannot say which feasible point will result in minimum cost. Hence, the problem is handled in an indirect manner. The proposed scheme is like this. First solve the optimization problem without any constraint and get the optimal variance $\bar{\sigma}^g$ for $\overline{L_c}$. It will result in global minimum cost. Now use this variance $\bar{\sigma}^g$ to get C^g_p and C^g_{pk} for $\overline{L_c}$ which result in minimum cost. If the point (C^g_p, C^g_{pk}) falls in the feasible region then this point is used as a design point (C^*_p, C^*_{pk}), otherwise the point E where the line OP enters into the shaded region is taken as the final desired (C^*_p, C^*_{pk}) pair. The reason behind choosing point E as design point is as follows. The values DP and DS which result from point E are minimum possible values satisfying both the Constraints (22) and (23). If we choose any other feasible point then even though the resulting DP and DS for $\overline{L_c}$ will satisfy the Constraints (22) and (23), yet their values will be a bit high and this will lead to higher cost. In this way we convert the constraints in terms of decision variables for a given value of R.

An important point to note here is that Equation (14) holds true only when the negative area of L_f is negligibly small. In order for this condition to hold, it is necessary that the following constraint must also be satisfied along with Constraint (26).

$$\mu_f \geq 6\sigma_f \tag{28}$$

$$\Rightarrow \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \leq \frac{(\mu_1 + \mu_2 + \mu_3)^2}{36} \tag{29}$$

The following equality and inequality constraints are now ready for the IOPT problem under the MTOS Policy.

$$\sigma_4^2 + M_o^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2} \tag{30}$$

$$\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \leq \frac{(\mu_1 + \mu_2 + \mu_3)^2}{36} \tag{31}$$

$$\sigma_i > 0\,\forall\, i = 1,2,3,4 \tag{32}$$

MTOZS Policy: This policy is a special case of the MTOS Policy with $R = 0$. Under this policy every order of the customer which arrives at the manufacturer will get backordered there. It implies that in this case $M_o = 1$. Therefore, the constraint (30) remains the same except $M_o = 1$. However, the constraint (31) is no more needed because in this case $M_o = 1$ irrespective of whether $\overline{L_c}$ is nonnegative or not. Thus the constraints for the IOPT problem under MTOZS Policy can be given as follows.

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 = \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2} \tag{33}$$

$$\sigma_i > 0\,\forall\, i = 1,2,3,4 \tag{34}$$

5) Optimization Problem: The IOPT problem for each policy can be formulated as follows.
MTOS Policy:
Minimize

$$\mathcal{K} = (\lambda + ID)\left[\sum_{i=1}^{3}\left(A_{i0} + A_{i1}\sigma_i + A_{i2}\sigma_i^2\right)\right] + \lambda\left[A_{40} + A_{41}\sigma_4 + A_{42}\sigma_4^2\right] + \alpha \tag{35}$$

subject to

$$\sigma_4^2 + M_o^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2} \tag{36}$$

$$\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \leq \frac{(\mu_1 + \mu_2 + \mu_3)^2}{36} \tag{37}$$

$$\sigma_i > 0\,\forall\, i = 1,2,3,4 \tag{38}$$

MTOZS Policy:
Minimize

$$\mathcal{K} = \lambda \sum_{i=1}^{4} \mathcal{K}_i + A\lambda + \Pi\lambda + \widehat{\Pi}\lambda\,\mu_f + \lambda C$$

subject to

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 = \frac{T^2}{9C_p^2} = \frac{\bar{d}^2}{9C_{pk}^2}$$
$$\sigma_i > 0 \,\forall\, i = 1,2,3,4$$

Following are some insights about the optimization problem under MTOS policy, which will be used in the next section while developing a scheme to solve this problem.

- Because the formulated constraints ensure the provided levels of DP and DS for $\overline{L_c}$ rather than for L_c, the minimum cost which we get after solving this problem is actually greater than or equal to what is actually required to achieve the specified levels of DP and DS for L_c. In other words, it can be said that the COST which we get here is an optimal upper bound on the COST for achieving specified levels of DP and DS on L_c.
- If one looks behind the purpose of constraint (37) then it is easy to see that it enforces nonnegativity of L_f without which it is not possible to use formula (9) for M_o. We imposed condition $\mu_f \geq 6\sigma_f$ to ensure this nonnegativity and because of that only we got this constraint. We could have as well chosen $\mu_f \geq 5\sigma_f$ but in that case error involved in computing the M_o with the help of formula (9) would have been higher.

D. Solution of IOPT

Observe that the objective function $\mathcal{K}$ for each policy is a function of σ_i's and B, D, E which are all functions of R. Theoretically R can take any value from set of natural numbers and σ_i's can take any positive real value. It makes the optimization problem under both the policies as mixed integer nonlinear optimization problem. Fortunately R cannot take any arbitrarily large value. For example, a seasoned asset manager who is engaged in managing the inventory can tell by his experience that R can never exceed a certain value. Also, often times, there is a constraint on storage space, or capital tied up with inventory, etc. which further limits the value of R. This feature of R is deployed to come up with a scheme to solve this problem. The procedure to solve this problem involves the following steps.

1) Fix the value of R under the MTOS policy and solve the resulting subproblem for σ_i's and achieve the optimal upper bound on COST.
2) Repeat this procedure for all possible values of R.
3) Solve the problem for MTOZS Policy.
4) Find out the minimum among all such optimal upper bounds on COST computed above for a given DP and DS. The corresponding R is the optimal inventory level.

In the next section, we present a numerical example for the LPG supply chain and solve the IOPT problem to explain the theory developed so far.

E. Numerical Example

Let us consider the LPG supply chain once again and give reasonably realistic values to all the system parameters. We have chosen following values for the typical known parameters of the IOPT problem in the context of the LPG supply chain.

Lead Time Parameters:
$\mu_1 = 1$ day, $\mu_2 = 3$ days, $\mu_3 = 2$ days, $\mu_4 = 7$ days
Demand Process Parameters:
$\lambda = 1500$ trucks/year
Cost Parameters:
$\mathcal{K}_1 = 10\left(1 + \exp\left(\frac{1}{\sigma_1}\right) - \frac{\mu_1}{200}\right)$ \$/truck
$\mathcal{K}_2 = 100\left(1 + \exp\left(\frac{1}{\sigma_2}\right) - \frac{\mu_2}{200}\right)$ \$/truck
$\mathcal{K}_3 = 10\left(1 + \exp\left(\frac{1}{\sigma_3}\right) - \frac{\mu_3}{200}\right)$ \$/truck
$\mathcal{K}_4 = 100\left(1 + \exp\left(\frac{1}{\sigma_4}\right) - \frac{\mu_4}{200}\right)$ \$/truck
$A = 5$ \$/order
$\Pi = 0$ \$/truck
$\widehat{\Pi} = 500$ \$/truck-year
$I = 0.2$ \$/year-\$invested
$C = 1000$ \$/truck
Delivery Quality Parameters
$\tau = 10$ days
$T = 10$ days

For the sake of numerical experimentations we consider following four different sets of constraints and solve the problem under each case.

1) DP=3σ and DS=0.7 for L_c
2) DP=4σ and DS=0.8 for L_c
3) DP=5σ and DS=0.9 for L_c
4) DP=6σ and DS=1.0 for L_c

Assume that it is not possible for the RD to keep more than 40 LPG trucks ready at any given point of time.

We first describe Step 1 of the procedure to solve IOPT, discussed in last section, for this numerical example. Let us choose Constraints set DP=3σ and DS=0.7 to work with. Step 2 can be carried out in the same manner for all the other values of R. Step 3 and Step 4 are also trivial. The same procedure can be repeated for other constraints sets also.

To start with, let us fix $R = 10$. It is straightforward to compute the following parameters for the given numerical values.
$\mu_f = 6$ days
$M_o = 0.999722639663766$
$E = 1499.583959$ trucks/year
$B = 14.657947016303742$ trucks
$D = 0.000412769728402651$ trucks
$\alpha = 4.983672090063828 \times 10^6$
$\bar{d} = 7.001664162017404$ days

Substitution of these values in Equation (35) and (36) results in following optimization problem.
Minimize

$$\mathcal{K} = 1500.000083 \sum_{i=1}^{3}\left(A_{i0} + A_{i1}\sigma_i + A_{i2}\sigma_i^2\right) + 1500\left[A_{40} + A_{41}\sigma_4 + A_{42}\sigma_4^2\right] + 4.983672 \times 10^6$$

$$\left(\text{say}, \mathcal{K} = a + \sum_{i=1}^{4} a_{i1}\sigma_i + a_{i2}\sigma_i^2\right) \tag{39}$$

where
$a_{11} = 1500.000083 A_{11}$, $a_{12} = 1500.000083 A_{12}$
$a_{21} = 1500.000083 A_{21}$, $a_{22} = 1500.000083 A_{22}$
$a_{31} = 1500.000083 A_{31}$, $a_{32} = 1500.000083 A_{32}$
$a_{41} = 1500 A_{41}$, $a_{42} = 1500 A_{42}$
$a = 1500.000083 \sum_{i=1}^{3} A_{i0} + 1500 A_{40} + 4.983672 \times 10^6$
subject to
$\sigma_4^2 + 0.999445\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \frac{100}{9C_p^2} = \frac{49.0233}{9C_{pk}^2}$
$\sigma_i > 0 \,\forall\, i = 1, 2, 3, 4$

The constants A_{ij} can be determined with the help of Taylor expansion of the cost functions $\mathcal{K}_i$. We have expanded all the cost functions at $\sigma_i = 1$ and used the corresponding coefficients as the constants A_{ij}. The immediate problem is to find out values of C_p and C_{pk}. As a first step toward this, it is required to check the feasibility of the problem as per guidelines provided in Lemma V-C.1. Note the upper bound on DS for this case is

$$\overline{C_{pm}} = \frac{T}{3(T - \bar{d})} = 1.111727809$$

Hence, as far as feasibility is concerned, there is no problem because all the desired values of DS are within permissible range. As a next step we findout the pair C_p^g and C_{pk}^g that results in global minimum and test whether it belongs to the feasible region or not. For this let us assume $S = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4) : \sigma_i \in \Re^+ \;\forall\, i = 1, 2, 3, 4\}$. It immediately follows from this definition of S that $\mathcal{K} : S \rightarrow E_1$ where S is a nonempty open convex set. To test the convexity of objective function $\mathcal{K}$, we compute gradient vector $\nabla\mathcal{K}(\bar{\mathbf{X}})$ and Hessian matrix $H(\bar{\mathbf{X}})$ for function $\mathcal{K}$ at point $\bar{\mathbf{X}} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)^T$. These matrices come out to be

$$\nabla\mathcal{K}(\bar{\mathbf{X}}) = \begin{bmatrix} \frac{\partial\mathcal{K}(\bar{\mathbf{X}})}{\partial\sigma_1} \\ \frac{\partial\mathcal{K}(\bar{\mathbf{X}})}{\partial\sigma_2} \\ \frac{\partial\mathcal{K}(\bar{\mathbf{X}})}{\partial\sigma_3} \\ \frac{\partial\mathcal{K}(\bar{\mathbf{X}})}{\partial\sigma_4} \end{bmatrix} = \begin{bmatrix} a_{11} + 2a_{12}\sigma_1 \\ a_{21} + 2a_{22}\sigma_1 \\ a_{31} + 2a_{32}\sigma_1 \\ a_{41} + 2a_{42}\sigma_1 \end{bmatrix}$$

$$H(\bar{\mathbf{X}}) = \begin{bmatrix} 2a_{12} & 0 & 0 & 0 \\ 0 & 2a_{22} & 0 & 0 \\ 0 & 0 & 2a_{32} & 0 \\ 0 & 0 & 0 & 2a_{42} \end{bmatrix}$$

Observe that the gradient vector and Hessian matrix exist for each $\bar{\mathbf{X}} \in S$. It directly follows that function $\mathcal{K}$ is twice differentiable over S. Moreover, Hessian matrix is independent of $\bar{\mathbf{X}}$. Therefore, it is sufficient that we test the Positive Definiteness (PD) or Positive Semi Definitiveness (PSD) of the Hessian matrix at any one point of S instead of testing it all over S.

It is easy to see that all the diagonal elements of Hessian matrix are positive real numbers because A_{i2} are positive. Therefore, Hessian matrix is PD and function $\mathcal{K}$ is strictly convex which implies that a local optimal solution of unconstrained problem is the unique global optimal solution. This can be obtained by equating $\nabla\mathcal{K}(\bar{\mathbf{X}})$ to 0. For the present numerical example it results in

$$\sigma_1^g = \sigma_2^g = \sigma_3^g = \sigma_4^g = 1.333 \text{ days}$$

These σ_i^g can be used to find out σ^g which comes out to be 2.66544 days. Indices C_p^g and C_{pk}^g can be computed by using σ^g. For the present example these indices are $C_p^g = 1.25057$ and $C_{pk}^g = 0.875609$. These C_p^g and C_{pk}^g can further be utilized to findout the value of DP and DS at the global minimum point which come out to be 4.12678σ and 0.83088 respectively. These quality levels are more than what is desired. Hence, we use C_p^g and C_{pk}^g as design values. If these quality levels come out to be less than what is specified in constraints then it is required to use the scheme suggested earlier in Section V-C.4. Substituting the σ_i^g's in objective function (39) gives optimal upper bound on COST (2.5921 million \$) of supply chain with $R = 10$.

Fortunately, in the present situation the global minimum point becomes a design point so we need not proceed for any further calculation. But if it is not so then we solve the underlying optimization problem by *Lagrange multiplier method* and get stationary points which satisfy the necessary conditions. This is explained below.

Method of Lagrange Multipliers:

Lagrange Function:

The Lagrange function $L(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \nu)$ is given as:

$$L(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \nu) = \mathcal{K} + \nu\left(\sigma_4^2 + 0.999445\left(\sum_{i=1}^{3}\sigma_i^2\right) - \theta\right)$$

where $\theta = \frac{100}{9C_p^2} = \frac{49.0233}{9C_{pk}^2}$ and $\mathcal{K}$ is given by Equation (39).

Necessary Condition for Stationary Points:

Let point $\mathcal{P}^* = (\sigma_1^*, \sigma_2^*, \sigma_3^*, \sigma_4^*, \nu^*)$ correspond to a local optimal point, then this point must satisfy the following necessary conditions for being a stationary point.

$$\left.\frac{\partial L}{\partial\sigma_1}\right|_{\mathcal{P}^*} = \left.\frac{\partial L}{\partial\sigma_2}\right|_{\mathcal{P}^*} = \left.\frac{\partial L}{\partial\sigma_3}\right|_{\mathcal{P}^*} = \left.\frac{\partial L}{\partial\sigma_4}\right|_{\mathcal{P}^*} = \left.\frac{\partial L}{\partial\nu}\right|_{\mathcal{P}^*} = 0$$

These necessary conditions result in the following relations.

$$\sigma_1 = \frac{-a_{11}}{2\left(a_{12} + 0.999445\nu\right)}$$
$$\sigma_2 = \frac{-a_{21}}{2\left(a_{22} + 0.999445\nu\right)}$$
$$\sigma_3 = \frac{-a_{31}}{2\left(a_{32} + 0.999445\nu\right)}$$
$$\sigma_4 = \frac{-a_{41}}{2\left(a_{42} + \nu\right)}$$
$$\theta = \frac{0.999445a_{11}^2}{4\left(a_{12} + 0.999445\nu\right)^2} + \frac{0.999445a_{21}^2}{4\left(a_{22} + 0.999445\nu\right)^2} + \frac{0.999445a_{31}^2}{4\left(a_{32} + 0.999445\nu\right)^2} + \frac{a_{41}^2}{4\left(a_{42} + \nu\right)^2}$$

Solving the above system of equations, by some numerical technique, will give the desired stationary points. First of all those stationary points are discarded which are either imaginary or for which the non-negativity condition does not hold. After this, we apply second order conditions to determine whether the point is a maxima or a minima. Among all the minima points, the one which yields minimum COST is considered as the solution of the problem and we call it as the optimal upper bound on COST for $R = 10$ yielding at least DP=5σ, and DS=0.7.

F. Numerical Results

The results of the above problem are summarized in the following four curves each for each constraints set. Each curve represents the variation of optimal upper bound on COST ($/year) with inventory level R. For each of the above curves,

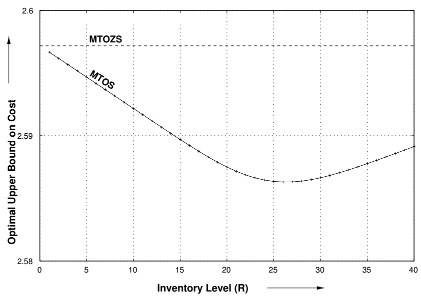

Fig. 9. Optimal inventory level R for DP=3σ and DS=0.7

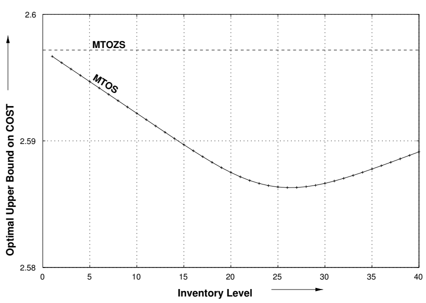

Fig. 10. Optimal inventory level R for DP=4σ and DS=0.8

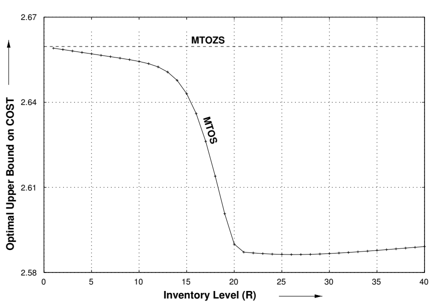

Fig. 11. Optimal inventory level R for DP=5σ and DS=0.9

we find the point of minima. If it is lying on the $R = 0$ curve, then the regional depot should not maintain any inventory. Otherwise, the value of R corresponding to the point of minima is the optimal R for the RD. Notice that optimal value of inventory is $R^* = 26$ for all the four constraints. This is peculiar for this

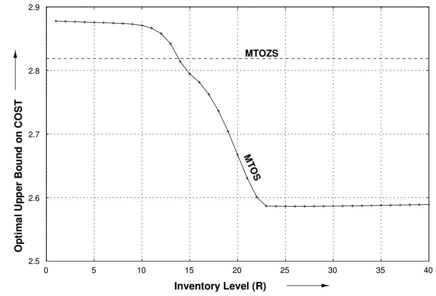

Fig. 12. Optimal inventory level R for DP=6σ and DS=1.0

example and an immediate observation we wish to make is that R^* in general will be different for different quality criterion. Even though there is a little change in the value of R^* for four different quality levels yet R^* increases as the desired quality level increases. This can be verified by observing the trends of the plots. Notice that for lower values of DP and DS, keeping inventory is always profitable. It implies that MTOS policy outperforms the MTOZS policy at every R for lower values of DP and DS. However, at higher values of DP and DS, the MTOS policy is outperformed by MTOZS policy at lower values of R. If desired DP and DS levels are high, the variabilities of the individual processes must be low enough to afford the luxury of having very less inventory. It results in higher cost. In such a situation, higher inventory levels can only allow us to have luxury of high DP and DS.

VI. Summary and Future Work

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