Achieving Sharp Deliveries in Supply Chains through Variance Pool Allocation

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Abstract— **In this paper, our objective is to come up with a sound methodology to design supply chains with outstanding delivery performance. As the first step towards this objective, we consider supply chains with a linear workflow, which we call pipelined supply chains. We define a new index of delivery performance called *delivery sharpness* which measures the precision as well as the accuracy with which products are delivered to the customers. The specific problem we solve is: given the delivery sharpness to be achieved, how can we allocate variability across individual stages of the supply chain in a cost-effective way. We call this the variance pool allocation (VPA) problem. In formulating and solving the VPA problem, we explore interesting relationships among process capability indices C_p, C_{p_k}, and C_{p_m}, and generalize the notion of Motorola six sigma performance. The VPA problem leads to a four step design methodology and the resulting optimization problem is solved using the method of Lagrange multipliers. We present an interesting example of a supply chain in the plastics industry and illustrate the different steps of our methodology.**

Keywords— **Supply chain lead time, Cycle time compression, Delivery Probability (DP), Delivery Sharpness (DS), Process Capability Indices, Variance Pool Allocation (VPA), Generalized Motorola Six Sigma (GMoSS) Concept**

I. Introduction

A supply chain network can be viewed as a network of facilities in which a customer order will flow through internal business processes such as procurement, production, inventory management, and logistics, ultimately resulting in delivery of required products on time to customers. As one can imagine, when the number of resources, operations, and organizations increases, managing the supply chain can become very complex. An entire supply chain could exist within a single company or a supply chain can span multiple enterprises. An important design objective in such networks is to achieve a high probability of delivery of finished products to the customer in a customer specified delivery window. This entails perfect synchronization among supply chain elements and individual business processes embedded within the supply chain process. This in turn requires variability reduction all along the supply chain. Variability reduction in supply chains is the subject of several research papers (several important references are listed in [1] and in [2]).

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A. Variance Pool Allocation Problem

In this paper, we formulate and solve an important problem in the design of synchronized supply chains. We call this problem as the variance pool allocation problem (VPA). We first observe that the supply chain process is a superposition of individual business processes each of which adds value to the product/service desired by the customer. Given a supply chain and the mean and standard deviation of the end-to-end lead time for a certain product mix, the VPA problem seeks to optimally distribute the pool of variance among individual business processes so as to minimize cost and achieve outstanding delivery performance.

In this paper, we describe the delivery performance of a supply chain in terms of two metrics. The first is a traditional metric, *delivery probability* (DP), which is the probability that a typical customer order is delivered during a customer-specified window. We show in the paper that the two popular process capability indices, C_p and C_{p_k} [3], [4] provide an appropriate vehicle for computing the delivery probability. The second metric is a new one that we propose, which we call *delivery sharpness* (DS), which is a measure of how close to the target (most desired) delivery date a customer order is actually delivered. Note that a typical customer-specified window consists upper,lower specification limit and a target value or most desirable value of the delivery time. We are motivated by the Taguchi capability index C_{p_m} [5], [6], [4] in proposing delivery sharpness and in fact in this paper, we use C_{p_m} as a measure of delivery sharpness.

B. Relevant Work

The Motorola six sigma program for design tolerancing is described in [7], [8]. These reports also describe the process capability indices C_p and C_{p_k}. These and other process capability indices are discussed in a comprehensive manner in [6], [4].

An attempt towards synchronizing the internal processes in a supply chain network for better delivery performance has been made by Narahari *et al* [9]. The authors emphasize the use of Motorola Six Sigma approach to analyze and design a given supply chain process for six sigma delivery performance. Two design problems are discussed in this paper: finding nominal pool and finding a variance pool. The present paper is a generalization of the above paper [9].

C. Contributions and Outline

This paper makes the following contributions.

- Describes the delivery performance of a supply chain in terms of two metrics, delivery probability and delivery sharpness, and explains the connection of process capability C_p, C_{p_k}, and C_{p_m} to these two metrics
- Generalizes the notion of Motorola six sigma quality by adding an additional dimension of delivery sharpness
- Formulates and solves the variance pool allocation problem for linear or pipeline supply chains, which we consider an important first step in designing synchronized supply chains; the formulation uses all three process capability indices in an ingenious way
- Illustrates the variance pool allocation problem for a six stage linear pipeline of a real-world plastics supply chain

This paper is organized as follows. Section 2 first describes process capability indices and presents a relationship among them. A new perspective based on yield is presented in respect of the process capability indices. The relation of these indices to delivery probability and delivery sharpness is presented. The section also presents a generalization of the notion of Motorola six sigma quality. In Section 3, we use the findings of Section 2 in formulating the variance pool allocation problem for linear or pipelined supply chains. In Section 4, we present a four-step methodology for variance pool allocation in supply chains. Finally in Section 5, we describe a six stage supply chain in a plastics industry and apply our design methodology. Implications and future work constitute Section 6.

II. Process Capability Indices: A New Perspective

Process capability indices have been applied in a large scale by a variety of industries since early 1980s for the purpose of analysis of capability of manufacturing processes where variability is an inherent effect. This section discusses three important process capability indices $C_p, C_{p_k}, and C_{p_m}$ show that these indices are can be used to define Delivery Probability (DP) and Delivery Sharpness (DS).

TABLE I

Notation Used in the Definitions of PCIs

X	Any general quality characteristic but in our case it is lead time distribution of any business process in the chain
μ	Mean of X
σ	Standard deviation of X
L	Lower specification limit of customer delivery window specified for the X
U	Upper specification limit of customer delivery window specified for the X
τ	Target value for lead time X, specified by customer
T	Tolerance for lead time X, specified by customer
s	$\|\tau - \mu\|$
p	$\min(\|U-\mu\|, \|\mu - L\|)$

A. The Index C_p

The first process capability index, C_p, is defined as

$$C_p = \begin{cases} \frac{USL-LSL}{6\sigma} & \text{Bilateral} \\ \frac{USL-\tau}{3\sigma} & \text{Unilateral with USL known} \\ \frac{\tau-LSL}{3\sigma} & \text{Unilateral with LSL known} \end{cases}$$

From the above definition it is clear that a high value of C_p is desirable and also C_p cannot be negative. It is assumed later in this model that distribution of X is normal and the target value of lead time τ is the mid point of USL and LSL for any business process. Hence C_p can be expressed in following equivalent form.

$$C_p = \frac{T}{3\sigma} \tag{1}$$

where $T = \text{tolerance} = \frac{USL-LSL}{2}$

C_p does not specify anything about relative position of μ (mean of the lead time distribution) and τ. As long as the variability of the distribution σ^2 does not change, the value of C_p would not change. It means C_p measures only the potential of a business process to deliver products within a customer specified delivery window (assuming the process is centered around the target value). This potential can be expressed as:

$$\begin{aligned} \text{potential} &= \Phi\left(\frac{U-\mu}{\sigma}\right) - \Phi\left(\frac{L-\mu}{\sigma}\right) \\ &= 2\left(\Phi\left(\frac{T}{\sigma}\right) - 0.5\right) \\ &= 2\Phi\left(3C_p\right) - 1 \end{aligned} \tag{2}$$

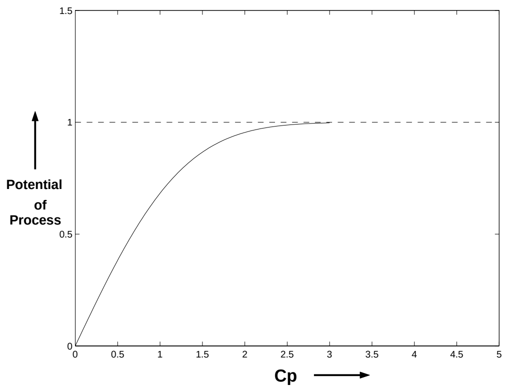

Fig. 1. Potential of the Process vs C_p

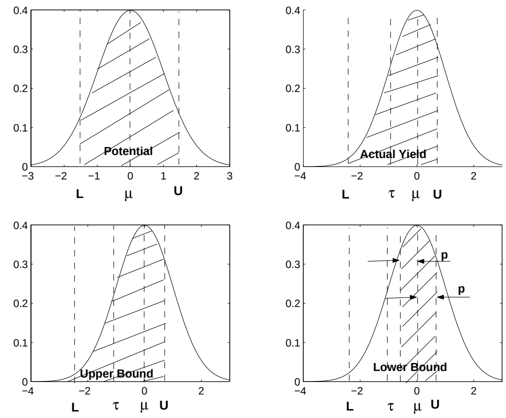

Fig. 2. Potential, Actual Yield, Upper and Lower Bound on Yield of the Process

where $\Phi(Z)$ is cumulative distribution function of standard normal distribution. This function is plotted in Figure 1.

B. The Index C_{p_k}

The index C_p does not reflect the impact that shifting the process mean has on a process's ability to deliver product within specification [5]. For this reason, the C_{p_k} index was developed. C_{p_k} is defined as follows:

$$C_{p_k} = \frac{\min(USL - \mu, \mu - LSL)}{3\sigma} = \left(\frac{p}{3\sigma}\right) \quad (3)$$

Here p stands for the distance of the nearer specification limit from the process mean.

While C_p measures the potential of the process, C_{p_k} helps measure the *actual yield*, i.e., when $\tau \neq \mu$. The potential of the process becomes equal to the actual yield of the process when $\mu = \tau$. So it can be written without any loss of generality:

$$\text{potential of process} \geq \text{actual yield of process} \quad (4)$$

In addition to potential and actual yield, we can define two other quantities: Upper Bound on process yield and Lower Bound on process yield. These bounds depend only on C_{p_k} and the actual yield of the process lies between these two values. These four quantities represent areas of different regions under probability density function, $f_X(x)$, as shown in Figure 2

Table II summarizes the formulae for these four quantities. A proof of these formulae is provided in [10].

C. The Index C_{p_m}

The index C_{p_m} is defined as

TABLE II

Formulae for Potential, Actual Yield, Upper and Lower Bound on Process Yield

Quantity	Formula
Potential	$2\Phi(3C_p) - 1$
Actual Yield	$\Phi(3C_{p_k}) + \Phi(6C_p - 3C_{p_k}) - 1$
Upper Bound	$2\Phi(3C_{p_k})$
Lower Bound	$2\Phi(3C_{p_k}) - 1$

$$C_{p_m} = \frac{\text{USL-LSL}}{6\xi} = \frac{\text{T}}{3\sqrt{\sigma^2 + s^2}} \quad (5)$$

Quantity $E(L) = \sigma^2 + s^2$ is known as "Expected Taguchi Loss" [6], [4]. By observing the definition of C_{p_m}, we see that it will properly model delivery sharpness using the loss function approach. For example, if the lead time variance increases (decreases), the denominator will increase (decrease) and C_{p_m} will decrease (increase). Also, if the mean lead time moves away from (closer to) the target value, the denominator will increase(decrease) and C_{p_m} will decrease (increase).

D. Relationship and Dependencies among C_p, C_{p_k}, C_{p_m}

The following relations can be derived among C_p, C_{p_k}, C_{p_m} [10].

$$C_p \geq C_{p_k} \geq 0 \quad (6)$$

$$C_p \geq C_{p_m} \geq 0 \quad (7)$$

$$C_{p_k} = C_p(1-k) \quad (8)$$

$$\frac{1}{9C_{p_m}^2} = \frac{1}{9C_p^2} + \left(1 - \frac{C_{p_k}}{C_p}\right)^2 \quad (9)$$

where $k = \frac{s}{T}$.

If we look at yield perspective of process capability indices then it can be shown that for a given value of actual yield α (say), there exhibit upper and lower limits for the values of both C_p and C_{p_k}. Table III summarizes such bounds on C_p and C_{p_k}. A detailed proof for all these formulae is discussed in [10].

E. A Generalized View of Motorola Six Sigma Quality

The basic idea of the Motorola Six Sigma concept [7], [8] is that it identifies a "sigma (σ) level" with each value of number of defects per million opportunities (npmo) which is the probability, expressed on a scale of 10^{-6}, that a part is produced with the characteristic X lying outside the specification limits. In other words, a sigma level is attached with each value of actual yield of the process. As the actual yield increases the sigma level also increases.

The Motorola six sigma program assumes that the underlying distribution X is normal with mean μ and standard deviation σ. Also it is assumed that target value of X is midpoint of upper specification limit (USL) and lower specification limit (LSL). It assumes a one-sided 1.5σ shift in the process mean to model the shift and drift in process mean. With this assumption, if USL and LSL coincide with $(\tau+\sigma)$ and $(\tau-\sigma)$ respectively then corresponding *upper bound* on yield will be assigned the 1σ level. Similar case is with 2σ, 3σ and others. if USL and LSL coincide with $(\tau+6\sigma)$ and $(\tau-6\sigma)$ respectively, we obtain six sigma level of performance, which corresponds to a probability of non-conformance of 3.4 parts per million.

It is possible to attach a unique pair of C_p and C_{p_k} with each σ level of performance. For example for 6σ level of performance $C_p = 2$ and $C_{p_k} = 1.5$.

Delivery probability (DP) can be expressed in terms of sigma levels without any difficulty. However, in practice there are many occasions in which even if there are no shifts and drifts in process mean and variance, yet the process mean (μ) is not centered at target value (τ) or in other words there exists some bias (or offset) between μ and τ which we denote by s and is given as:

$$s = |\mu - \tau|$$

In these situations, by six sigma quality we mean the actual yield must be $(1 - 3.4 \times 10^{-6}) \times 100\%$ which is upper bound on process yield for 6σ quality. Our interest is to find out in how many different ways the bias and variance can be adjusted together without disturbing actual yield. This leads to a generalized view of six sigma quality, which we call GMoSS quality.

The concept of GMoSS quality is based on the idea of upper and lower bounds on C_p and C_{p_k} for a given value of actual yield. In order to explain this idea let us start with equation:

$$\text{actual yield} = \Phi\left(3C_{p_k}\right) + \Phi\left(6C_p - 3C_{p_k}\right) - 1$$

If we fix the value of actual yield as α in above equation then there will be two independent variable C_p, C_{p_k}, hence solution set will be unbounded. It can be shown that for a given actual yield α, C_{p_k} and C_{p_k} are bounded in a certain interval. It means the feasible solution set $(C_{p_k}, C_p)$of the equation

$$\alpha = \Phi\left(3C_{p_k}\right) + \Phi\left(6C_p - 3C_{p_k}\right) - 1 \qquad (10)$$

is bounded in following manner:

$$\underline{C^{\alpha}_{p_k}} \le C_{p_k} \le \overline{C^{\alpha}_{p_k}}$$
$$\underline{C^{\alpha}_{p}} \le C_p \le \infty$$

If we substitute $\alpha = (1 - 3.4 \times 10^{-6})$ and plot the curve, then all the points lying on the curve are (C_{p_k}, C_p) pairs that correspond to the 6σ quality level, the points which are lying above the curve correspond to quality levels superior to 6σ and similarly the points which lie below correspond to quality levels inferior to 6σ level.

The idea behind the concept of GMoSS quality is presented in Figure 3 where we have plotted equation 10 on $C_{p_k} - C_p$ plane for $6\sigma, 5\sigma, 4\sigma, 3\sigma$ quality standards.

We can proceed one step further by looking at the connection between delivery probability and delivery sharpness in the light of the GMoSS notion. For this, we consider the plots of σ quality levels on $C_{p_k} - C_p$ plane and then see how C_{p_m} behaves on the same plot. From a process design point of view, it can be said that for a desired level of DS (i.e. C_{p_m}) and DP (i.e. C_p, C_{p_k}); this curve provides a set of 3-tuples (C_p, C_{p_k}, C_{p_m}) which all satisfy these two requirements. The designer has to decide which one of the triples to choose depending upon other limitations.

In order to plot such curves, we consider the identity relation among C_p, C_{p_k} and C_{p_m} (9) in a slightly different form:

$$C_{p_k} = C_p \left(1 - \sqrt{\frac{1}{9C^2_{p_m}} - \frac{1}{9C^2_p}}\right)$$

Now for a given constant value of C_{p_m} (say $C^*_{p_m}$) above equation can be plotted on $C_{p_k} - C_p$ plane which will be a contour of $C_{p_m} = C^*_{p_m}$ on $C_{p_k} - C_p$ plane. It can be shown easily that this equation represents a hyperbola. Figure 3 shows some contours of C_{p_m} on $C_{p_k} - C_p$ plane.

TABLE III

BOUNDS ON PCIs FOR ACTUAL YIELD = α

Bound	Formula	Description
$\underline{C_p^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)$	If process's C_p is less than $\underline{C_p^\alpha}$ then it's actual yield can't be equal to α no matter how large C_{p_k} is.
$\underline{C_{p_k}^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}(\alpha)\right)$	If process's C_{p_k} is less than $\underline{C_{p_k}^\alpha}$ then it's actual yield can't be equal to α no matter how large the C_p is.
$\overline{C_{p_k}^\alpha}$	$\frac{1}{3}\left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)$	If process's C_{p_k} is greater than $\overline{C_{p_k}^\alpha}$ then it's actual yield can't be less than or equal to α no matter how small C_p is
$\overline{C_p^\alpha}$	$\frac{1}{6}\left(3\underline{C_{p_k}^\alpha} + \Phi^{-1}\left(1+\alpha-\Phi\left(3\underline{C_{p_k}^\alpha}\right)\right)\right) = \infty$	For any value of C_p greater than or equal to $\underline{C_p^\alpha}$ it is possible to find a corresponding C_{p_k} such that actual yield of the process is α

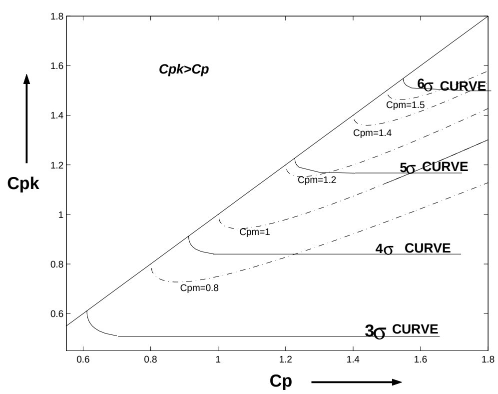

Fig. 3. Contours of C_{p_m} on $C_{p_k} - C_p$ Plane

III. Variance Pool Allocation (VPA) Problem

A. Linear Supply Chains: An Overview of Variation in Lead Time

Let us consider a linear or pipelined supply chain with n processes as shown in Figure 4. In this supply

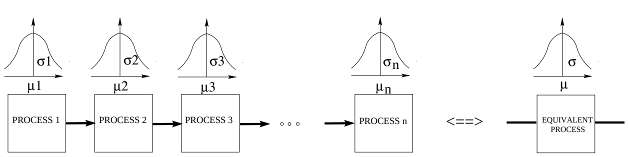

Fig. 4. A Pipelined Supply Chain Architecture

chain, material flows from process 1 to process n and the end product is delivered to the end customer after processing at process n. The end-to-end lead time of each individual process (i.e. $X_i, i = 1, 2, \ldots, n$) is a continuous random variable.

B. Assumptions on Nature of Business Process and Customer Delivery Window

The design methodology proposed in this paper is based upon the following assumptions:
1. End-to-end lead time X_i of each business process i is normally distributed.
2. Each business process i is subjected to delivery requirements on lead time imposed by the downstream process in the chain. This delivery time requirement is called as customer delivery window. A typical customer delivery window for process i is of the form (τ_i, T_i).
3. Individual lead times are mutually independent of one another and also they are under statistical control.
4. There is no time elapsed between transforming the material from process i to process $i+1$. Hence supply chain lead time Y (which is equal to the total sojourn time of material within chain) is equal to the sum of lead times of the individual processes.

$$Y = \sum_{i=1}^{n} X_i \tag{11}$$

It is easy to see that Y will be normally distributed with $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ as it is sum of n independent normally distributed random variables.

C. Problem Description

In this section, we present a list of all known parameters, decision variables, and constraints for the VPA problem.

C.1 Known Parameters

1. Customer delivery window for lead time of each process i.e., $(\tau_i, T_i) \forall i$.

2. Customer delivery window for supply chain lead time i.e., (τ, T).
3. Mean μ_i of random variable X_i $\forall i$.
4. Cost of lead time per unit item produced, denoted by C_i, for each process i. This cost is that part of the total processing cost which is associated with lead time. We take this as a polynomial of order 3 in C_{p_i} where C_{p_i} is the process capability index for lead time of process i.

$$C_i = a_{i0} + a_{i1}C_{p_i} + a_{i2}C_{p_i}^2 + a_{i3}C_{p_i}^3 \qquad (12)$$

here $a_{i0}, a_{i1}, a_{i2}, a_{i3}$ are constants.
In most practical situations, cost of lead time per unit for any process i increases as variance σ_i decreases because μ_i is constant here. On the other hand C_{p_i} is inversely proportional to σ_i. Hence any kind of function $C_i = f(\sigma_i)$ can be expressed in terms of C_{p_i} which means $C_i = f(\sigma_i) = g(C_{p_i})$.
We can invoke Taylor's theorem to approximate the function $g(C_{p_i})$ by a polynomial of order three in C_{p_i}. If we assume that $g, g^{'}, g^{''}, g^{'''}$ are continuous in the interval $[0, C_{p_i}]$ and $g^{'''}$ is differentiable on $(0, C_{p_i})$ then it immediately follows from the Taylor's theorem that cost C_i can be approximated by a third order Taylor polynomial which we have put in the form of equation 12. Conceptual and computational simplicity are the motivation behind choosing a third order Taylor series. Moreover, it is difficult to expect significant effects of greater than third order in situations such as these. In general one can take polynomial of higher orders, if computational simplicity is not a criterion.

C.2 Decision Variables and Constraints

In the VPA problem, the decision variables are variance values σ_i of lead time X_i of each process i in the supply chain.

Following are the constraints in the VPA problem:

1. Standard deviation σ of the supply chain lead time Y depends on $\sigma_i s$. This σ will decide how many customers are expected to receive the delivery within a specified time window (which we have called as delivery probability or DP). Hence DP is directly affected by $\sigma_i s$. Thus the constraint is that the DP for supply chain lead time should be at least at the level of 6σ or any specified σ level.
2. Taguchi loss for supply chain lead time depends upon (τ, μ, σ). But in the VPA problem, τ and μ are fixed so it depends on only on σ which in turn depends on $\sigma_i s$. Therefore, $\sigma_i s$ affect C_{p_m} and hence delivery sharpness (DS) also. Therefore, another constraint is that DS should be at least at the level of $C_{p_m}^*$.

IV. A Design Methodology for VPA Problem

A. Step 1: Problem Formulation

The first point to notice here is that although the decision variables in VPA problem are $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, our design methodology considers $(C_{p_1}, C_{p_2}, \ldots, C_{p_n})$ as decision variables because of three reasons:
- $C_{p_i} = \frac{T_i}{3\sigma_i}$ where T_i is given. Therefore, σ_i can be expressed explicitly in terms of C_{p_i}.
- Cost of processing time at process i (C_i) can be expressed in terms of C_{p_i}.
- It is easy to formulate the constraints of VPA problem in terms of $C_{p_i}s$ rather than $\sigma_i s$.

The VPA problem can now be expressed in the form of a nonlinear optimization problem as follows:
Objective Function:

$$\begin{aligned} \text{Minimize} & & \\ C &= \sum_{i=1}^{n} C_i \\ &= \sum_{i=1}^{n} a_{i0} + \sum_{i=1}^{n} a_{i1}C_{p_i} + \\ & \sum_{i=1}^{n} a_{i2}C_{p_i}^2 + \sum_{i=1}^{n} a_{i3}C_{p_i}^3 \qquad (13) \end{aligned}$$

Constraints:
1. DS for supply chain lead time $\geq C_{p_m}^*$
2. DP for supply chain lead time should be at least at the level of 6σ. This means that value of C_p and C_{p_k} should be chosen in such a way that the point (C_{p_k}, C_p) lies on or above 6σ curve in $C_{p_k} - C_p$ plane.
3. $C_{p_i} \geq 0$ $\forall i$

B. Step 2: Formulation of Constraints in terms of Decision Variables

In step 1, the constraints, except constraint 3, are not expressed in terms of decision variables (i.e. $C_{p_i}^{'}s$). Therefore, the objective of this step is to express the constraints explicitly in terms of decision variables. As per assumptions made in Subsection 4.2, the variance of supply chain lead time Y (i.e. σ) can be expressed in terms of variance (σ_i) of processing time of process X_i. We will just reengineer this expression in order to express the constraints in terms of decision variables. Let us start with expression

$$\begin{aligned} \sigma^2 &= \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 \\ \Rightarrow \frac{T^2}{9C_p^2} &= \frac{p^2}{9C_{p_k}^2} = \frac{T_1^2}{9C_{p_1}^2} + \frac{T_2^2}{9C_{p_2}^2} + \ldots + \frac{T_n^2}{9C_{p_n}^2} \end{aligned} \qquad (14)$$

Equation (14) states that once the pair (C_p, C_{p_k}) is chosen for supply chain lead time Y, the feasible solution set will get automatically fixed and it is the set of all those n-tuples $(C_{p_1}, C_{p_2}, \ldots, C_{p_n})$ which satisfy this equation for the chosen value of C_p and C_{p_k}.

The idea to get such a pair (C_p, C_{p_k}) is to choose the pairs that satisfy both constraints (1) and (2) and use it in equation (14) in order to get the desired constraint in terms of decision variables. But again the problem is that many such pairs may exist. In such a situation, selection of the best pair is an important issue; we consider this below.

C. Step 3: Fixing Values for C_p and C_{p_k}

As pointed out in the last section, our problem is to find a (C_p, C_{p_k}) pair which satisfies constraints (1) and (2) and also satisfies the following relation:

$$\frac{T^2}{9C_p^2} = \frac{p^2}{9C_{p_k}^2} \tag{15}$$

The above condition is because of equation (14). Remember that T and p are known parameters in the VPA problem. This condition forces the desired (C_p, C_{p_k}) pair to lie on the line $C_{p_k} = \frac{p}{T}C_p$ in $C_{p_k} - C_p$ plane. On the other hand, constraint (1) forces it to lie on or above the contour $C_{p_m} = C^*_{p_m}$ and constraint (2) forces it to lie on or above the 6σ curve in the same plane. Therefore, in the $C_{p_k} - C_p$ plane it is possible to find a feasible region such that every point of the region satisfies both constraint 1, 2 and also condition (15). Figure 5 shows different cases of such kinds of feasible regions depending upon relative positions of $C_{p_m} = C^*_{p_m}$ contour (in short C_{p_m} curve) and 6σ curve (in short σ curve). From Figure 5 it is clear that feasible region in each case is the part of the line $C_{p_k} = \frac{p}{T}C_p$ that intersects with the shaded region. We have called it as line 'OP'[1]. In every case, the point 'E' where the line 'OP' enters into shaded region is taken as the final desired (C_p, C_{p_k}) pair. The reason behind choosing this point is that if we choose any other point on the feasible region then the corresponding C_p value will be high which will result in higher value of individual C_{p_i} and hence higher delivery cost. Let us denote this pair as $(C^*_p, C^*_{p_k})$. The point E$(C^*_p, C^*_{p_k})$ can be computed in each case in following way.

Case 1 Observe that in this case there is no point of intersection between C_{p_m} curve and σ curve. In this case point 'E' is the point of intersection of line $C_{p_k} = \left(\frac{p}{T}\right) C_p$ and $C^*_{p_m}$ curve.[2] Therefore solving the corresponding equations results in following expressions for C^*_p and $C^*_{p_k}$.

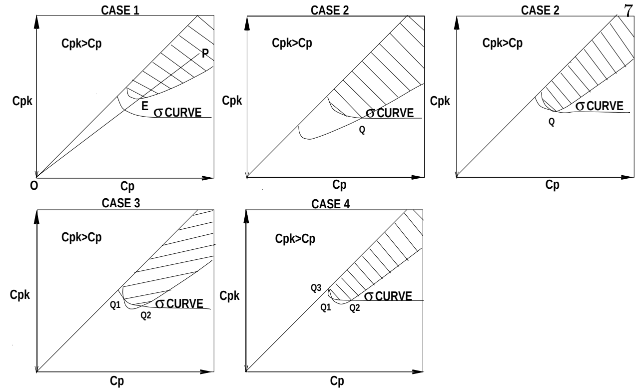

Fig. 5. All the possible configurations of C_{p_m} curve, σ curve and line 'OP' on $C_{p_k} - C_p$ Plane

$$C^*_p = \frac{1}{\sqrt{\frac{1}{(C^*_{p_m})^2} - 9\left(1 - \frac{p}{T}\right)^2}} \tag{16}$$

$$C^*_{p_k} = \left(\frac{p}{T}\right) C^*_p \tag{17}$$

Case 2 Observe that in this case there is only one point of intersection between σ curve and C_{p_m} curve. Let us mark this point as $Q = (C_{p_{k_1}}, C_{p_1})$. It can be easily verified that point E$(C^*_p, C^*_{p_k})$ will be the point of intersection of line OP with C_{p_m} curve or σ curve depending upon whether $\frac{p}{T}$ is less than or greater than $\frac{C_{p_{k_1}}}{C_{p_1}}$ respectively. If these two quantities are equal then point Q will become point E. In any one of these subcases, the point E can be found out by solving appropriate equations.

Case 3 Observe that in this case there are two points of intersection between the C_{p_m} curve and σ curve. Let us mark these points as $Q_1 = \left(C_{p_{k_1}}, C_{p_1}\right)$ and $Q_2 = \left(C_{p_{k_2}}, C_{p_2}\right)$ where $C_{p_1} < C_{p_2}$. The point E can be found out in the same way by comparing the $\frac{p}{T}$ with $\frac{C_{p_{k_1}}}{C_{p_1}}$ and $\frac{C_{p_{k_2}}}{C_{p_2}}$, as discussed in case 2.

Case 4 This case can be handled exactly in the same way as we did for case 3 except that here the points of intersection between the two curves are three.

D. Step 4: Solving the Optimization Problem

Now the optimization problem which we presented in step 1 can be rewritten as: Minimize cost C, given by equation 13, subject to

1.

$$\sum_{i=1}^{n} \frac{T_i^2}{C_{p_i}^2} = \frac{T^2}{{C^*_p}^2} = \frac{p^2}{{C^*_{p_k}}^2} \tag{18}$$

2.

$$C_{p_i} \geq 0 \ \forall i \tag{19}$$

[1] For the sake of clarity of picture, we have not shown the line 'OP' in each case. It is shown only in case 1 but in other cases it is understood

[2] If such a point doesn't exist, then we will stop and declare that the problem doesn't have any feasible solution.

The problem can be solved by using the method of Lagrange multipliers [11]. The method is illustrated in the example in the next section.

V. An Example

We now consider a supply chain for a plastics industry (a certain anonymous firm in the western state of Maharashtra, India) and provide the basis for applying our design methodology. Figure 6 shows the supply chain at an aggregate level. The supply chain has six business processes namely Procurement, Sheet Fabrication, Transportation, Manufacturing, Assembly, and Delivery.

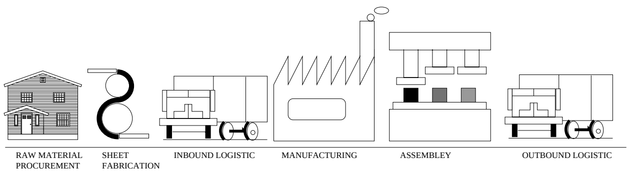

Fig. 6. An Example of a Pipelined Supply Chain: A Typical Plastic Industry Supply Chain

Let all the business processes in the chain satisfy the assumptions mentioned in subsection 4.2. The problem here is to find out standard deviation σ_i of lead time X_i of all the processes in the chain such that a delivery probability of 6σ level is attained and also a designated level of delivery sharpness, say, $C_{p_m} = 1.42782$ (arbitrarily chosen in this case), is achieved. The known parameters are tabulated in Table IV. Let the target value τ of the end-to-end supply chain lead time Y be 82 days and tolerance T be 6.5 days.

Steps 1 and 2: The objective function and constraints can be given as:

Objective Function:

Minimize

$$\begin{aligned} C &= \sum_{i=1}^{6} C_i \\ &= 6 + \sum_{i=1}^{6} C_{p_i} + \sum_{i=1}^{6} C_{p_i}^2 + \sum_{i=1}^{6} C_{p_i}^3 \end{aligned}$$

Constraints:

1.

$$\sum_{i=1}^{6} \frac{T_i^2}{C_{p_i}^2} = \frac{42.25}{C_p^{*^2}} = \frac{30.25}{C_{p_k}^{*^2}} \tag{20}$$

2.

$$C_{p_i} \geq 0 \; \forall \, i = 1, 2, \ldots, 6 \tag{21}$$

Step 3 If we draw the 6σ curve, C_{p_m}=$C_{p_m}^*$=1.42782 curve and line $C_{p_k} = \left(\frac{5.5}{6.5}\right) C_p$ on $C_{p_k} - C_p$ plane then the situation will fall into case 2 of step 3 of our methodology. Therefore point $(C_{p_k}^*, C_p^*)$ can be computed with the help of equations (16) and (17). This point comes out to be:

$$\begin{aligned} C_p^* &= 1.89832236 \\ C_{p_k}^* &= 1.606272774 \end{aligned}$$

Step 4 Substituting the values of $C_p^*, C_{p_k}^*$ in constraint (20), we obtain following constraint to work with while solving optimization problem.

$$\sum_{i=1}^{6} \frac{T_i^2}{C_{p_i}^2} = 11.72429622$$

Now we will apply the Lagrange Multiplier Method in order to solve this optimization problem.

1. Lagrange Function Lagrange function $L(C_{p_1}, \ldots, C_{p_6}, \lambda)$ is given as:

$$\begin{aligned} L(C_{p_1}, \ldots, C_{p_6}, \lambda) &= C + \lambda \left(\sum_{i=1}^{6} \frac{T_i^2}{C_{p_i}^2} \right. \\ &\left. -11.72429622 \right) \end{aligned}$$

2. Necessary Condition for Stationary Points Let point $\mathcal{P}^* = (C_{p_1}^*, \ldots, C_{p_6}^*, \lambda^*)$ correspond to the optimal point then this point must satisfy the following necessary conditions:

$$\begin{aligned} 2\lambda^* &= 3C_{p_1}^{*^5} + 2C_{p_1}^{*^4} + C_{p_1}^{*^3} \\ 18\lambda^* &= 3C_{p_2}^{*^5} + 2C_{p_2}^{*^4} + C_{p_2}^{*^3} \\ 2\lambda^* &= 3C_{p_3}^{*^5} + 2C_{p_3}^{*^4} + C_{p_3}^{*^3} \\ 18\lambda^* &= 3C_{p_4}^{*^5} + 2C_{p_4}^{*^4} + C_{p_4}^{*^3} \\ 8\lambda^* &= 3C_{p_5}^{*^5} + 2C_{p_5}^{*^4} + C_{p_5}^{*^3} \\ 2\lambda^* &= 3C_{p_6}^{*^5} + 2C_{p_6}^{*^4} + C_{p_6}^{*^3} \\ 11.72429622 &= \frac{2}{C_{p_1}^{*^2}} + \frac{18}{C_{p_2}^{*^2}} + \frac{2}{C_{p_3}^{*^2}} + \frac{18}{C_{p_4}^{*^2}} + \frac{8}{C_{p_5}^{*^2}} \\ &+ \frac{2}{C_{p_6}^{*^2}} \end{aligned}$$

Solving this system of equations by standard numerical methods we get only one real solution:

$$\begin{aligned} C_{p_1}^* = C_{p_3}^* = C_{p_6}^* &= 1.005670 \\ C_{p_2}^* = C_{p_4}^* &= 1.645700 \end{aligned}$$

TABLE IV
KNOWN PARAMETERS FOR PLASTIC INDUSTRY SUPPLY CHAIN PROBLEM

Procurement (X_1)	Sheet Fabrication (X_2)	Inbound Logistics (X_3)	Manufacturing (X_4)	Assembly (X_5)	Outbound Logistics (X_6)
$\mu_1 = 7$ days	$\mu_2 = 30$ days	$\mu_3 = 3$ days	$\mu_4 = 30$ days	$\mu_5 = 10$ days	$\mu_6 = 3$ days
$\tau_1 = 6$ days	$\tau_2 = 28$ days	$\tau_3 = 3$ days	$\tau_4 = 28$ days	$\tau_5 = 7$ days	$\tau_6 = 2.5$ days
$T_1 = 1$ days	$T_2 = 3$ days	$T_3 = 1$ days	$T_4 = 3$ days	$T_5 = 2$ days	$T_6 = 1$ days
$a_{10} = 1$	$a_{20} = 1$	$a_{30} = 1$	$a_{40} = 1$	$a_{50} = 1$	$a_{60} = 1$
$a_{11} = 1$	$a_{21} = 1$	$a_{31} = 1$	$a_{41} = 1$	$a_{51} = 1$	$a_{61} = 1$
$a_{12} = 1$	$a_{22} = 1$	$a_{32} = 1$	$a_{42} = 1$	$a_{52} = 1$	$a_{62} = 1$
$a_{13} = 1$	$a_{23} = 1$	$a_{33} = 1$	$a_{43} = 1$	$a_{53} = 1$	$a_{63} = 1$

$$C^*_{p5} = 1.376280$$
$$\lambda^* = 3.074458$$

Under this operating condition cost of delivery is:

$$C^* = 38.601998$$

It can be verified easily by the sufficiency condition that this point corresponds to the point of minimum.

VI. Implications of the Work

In our view, this research has several important implications.

- It provides an elegant characterization for supply chain delivery performance in terms of two metrics, delivery probability and delivery sharpness, and three well known process capability indices.
- The above metrics and capability indices will provide a general framework for developing a design methodology for supply chains.
- The variance allocation problem is an important first step in the design of synchronized supply chains.

The work has some limitations. First of all, only linear or pipelined supply chains have been considered. Only normal distributions are allowed in our analysis and formulations. Also, several assumptions have been made in formulating and solving the variance allocation problem. Many of these limitations can be surmounted in due course of time and provide important directions for further work.

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