Data-flow Analysis / Abstract Interpretation

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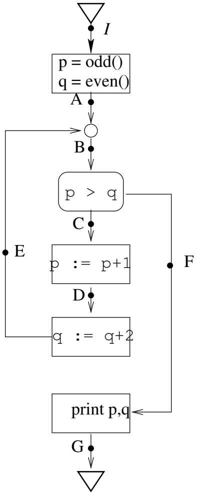
IISc

What is data-flow analysis

- "Computing 'safe' approximations to the set of values / behaviours arising dynamically at run time, statically or at compile time."
- Typically used by compiler writers to optimize running time of compiled code.
  - Constant propagation: Is the value of a variable constant at a particular program location.
  - Replace `x := y + z` by `x := 17` during compilation.
- More recently, used for verifying properties of programs.

Collecting semantics of a program = set of (concrete) states occurring at each program point.



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Collecting semantics – example

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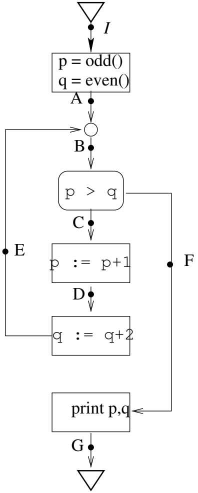

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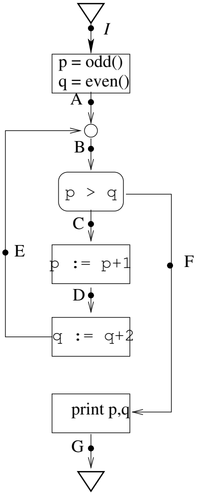

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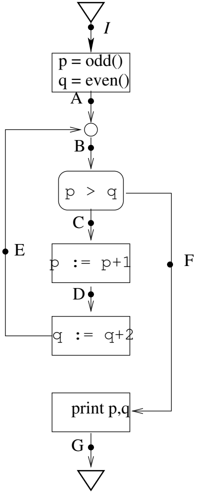

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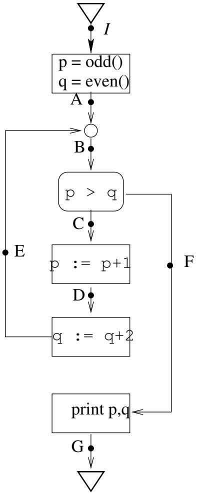

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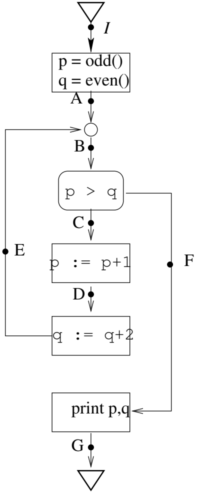

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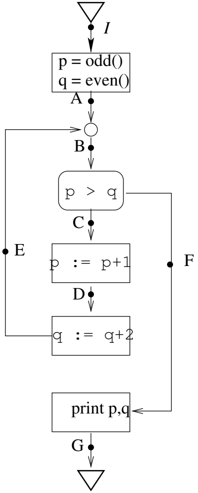

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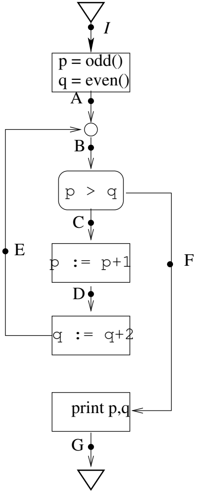

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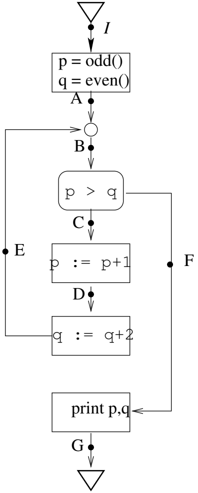

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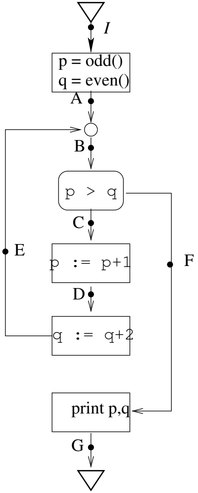

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Therefore, collecting semantics:

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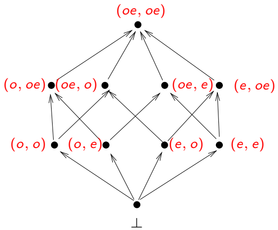
C	$\{(i,j) \mid j$ even, $i > j\}$
D	$\{(i,j) \mid j$ even, $i > j+1\}$
E	$\{(i,j) \mid j$ even, $i \geq j\}$
F	$\{(i,j) \mid i$ odd, j even, $i < j\} \cup \{(i,j) \mid i$ even, j even, $i = j\}$

Components of an abstract interpretation:

- Set of abstract states D, forming a complete lattice.
- "Concretization" function $\gamma : D \rightarrow 2^{State}$, which associates a set of concrete states with each abstract state.
- Transfer function $f_n : D \rightarrow D$ for each type of node n, which "interprets" each program statement using the abstract states.

- Abstract lattice D



- Transfer functions for an assignment node n of the form `p := exp`

$$f_n(s) = \begin{cases} \bot & \text{if } s \text{ is } \bot \\ (o, s[2]) & \text{if exp evaluates to } o \text{ in state } s, \\ & \text{where } s[2] \text{ is the 2nd component} \\ & \text{of the pair } s \\ (e, s[2]) & \text{if exp evaluates to } e \text{ in state } s \\ (oe, s[2]) & \text{if exp evaluates to } oe \text{ in state } s \end{cases}$$

- The concretization function γ
  - $\gamma((oe, oe)) =$

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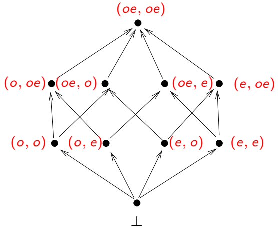

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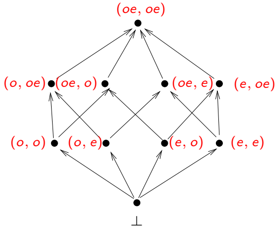

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Abstract interpretation – example

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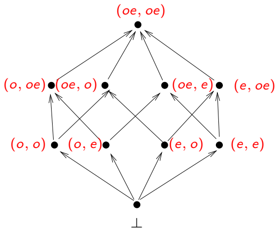

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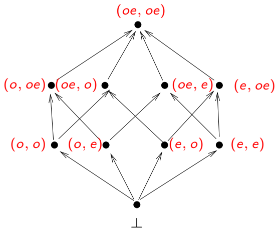

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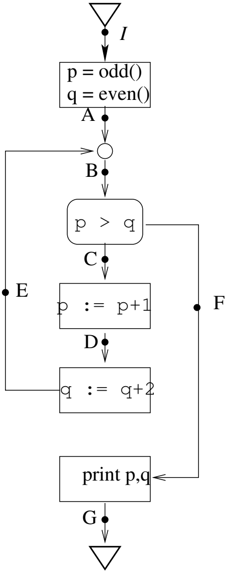

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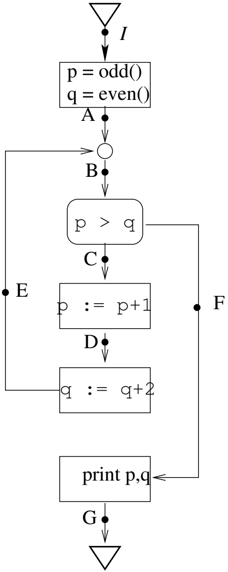

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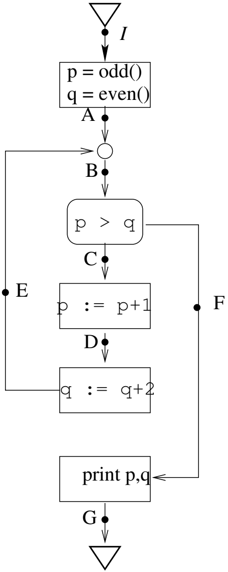

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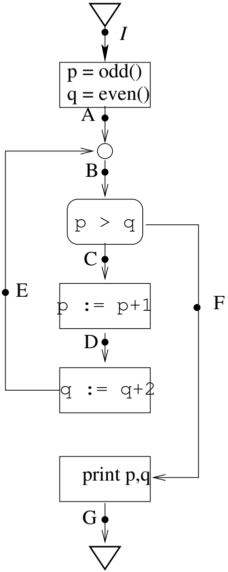

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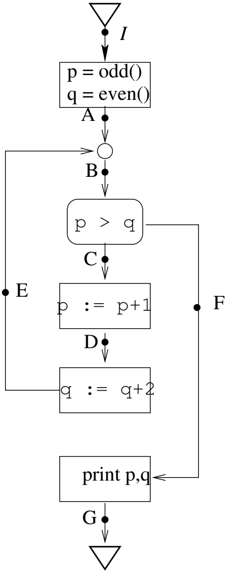

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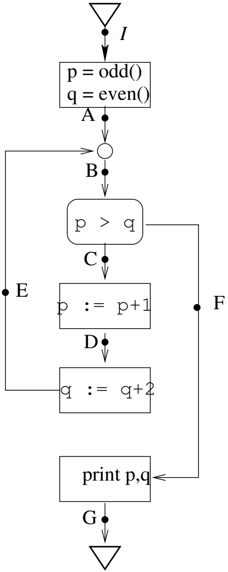

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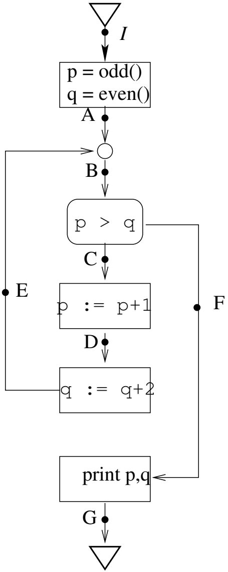

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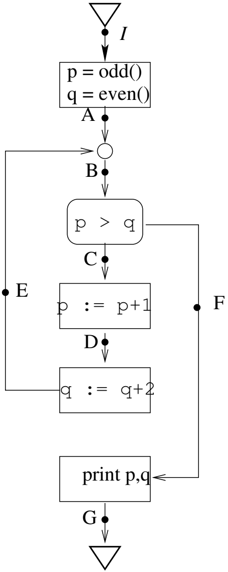

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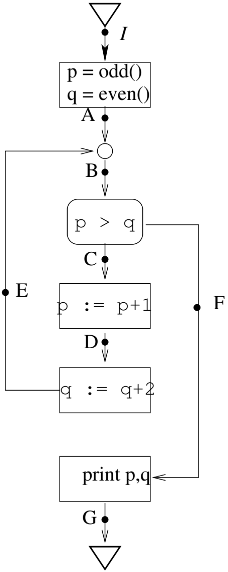

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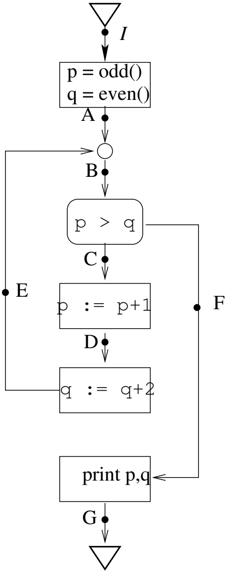

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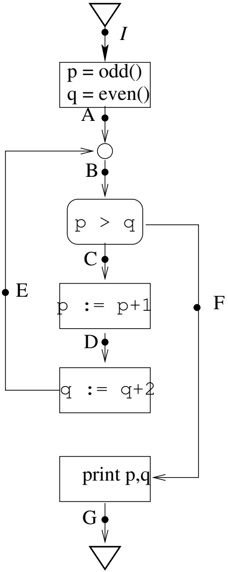

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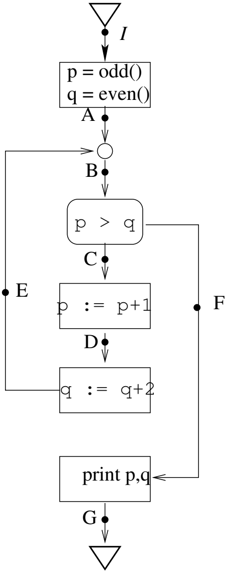

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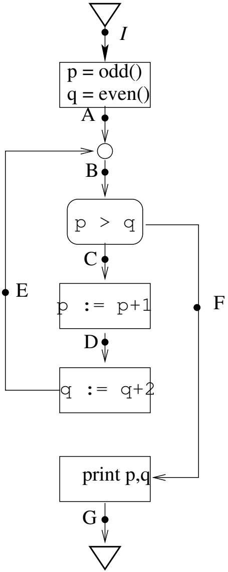

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Therefore, joining all abstract values at each point:

A	(o, e)		
B	$(o, e) \sqcup (e, e)$	$=$	(oe, e)
C	$(o, e) \sqcup (e, e)$	$=$	(oe, e)
D	$(e, e) \sqcup (o, e)$	$=$	(oe, e)
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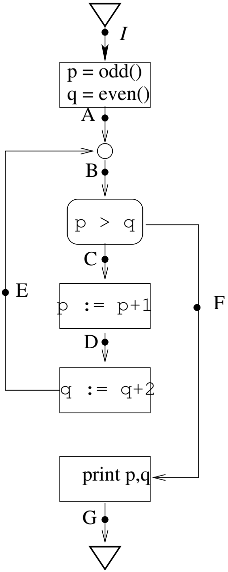

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This is *abstract join-over-all-paths* (JOP) solution.

Comparison of abstract JOP states and collecting states

Abstract JOP:

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Collecting states:

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Comparison of abstract JOP states and collecting states

Abstract JOP:		Collecting states:	
A	(o, e)	A	$\{(i,j) \mid i \text{ odd}, j \text{ even}\}$
B	(oe, e)	B	$\{(i,j) \mid i \text{ odd}, j \text{ even}\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i \geq j\}$
C	(oe, e)	C	$\{(i,j) \mid j \text{ even}, i > j\}$
D	(oe, e)	D	$\{(i,j) \mid j \text{ even}, i > j+1\}$
E	(oe, e)	E	$\{(i,j) \mid j \text{ even}, i \geq j\}$
F	(oe, e)	F	$\{(i,j) \mid i \text{ odd}, j \text{ even}, i < j\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i = j\}$

Note that at each point γ image of abstract solution is over-approximation of collecting states.

A given abstract interpretation is said to be *correct* if, for all abstract states $d_0 \in D$, for all programs P and for all program points p in P,

γ image of join of all abstract states arising at p (i.e., abstract JOP solution at p), with d_0 as the initial abstract value at P's entry

$$\supseteq$$

collecting semantics at p, with $\gamma(d_0)$ as the initial set of concrete states at P's entry

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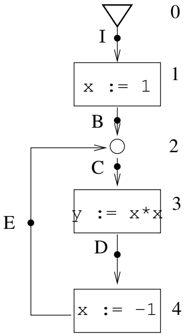
We will study later certain sufficient conditions for a given abstract interpretation to be correct.

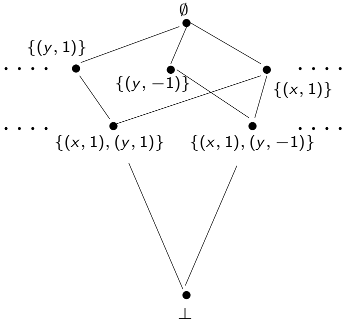


Path	Characterization of concrete states
I	*true* (given)
IB	x = 1
IBC	x = 1
IBCD	x = 1 $\wedge$ y = 1
IBCDE	x = -1 $\wedge$ y = 1
IBCDEC	x = -1 $\wedge$ y = 1
IBCDECD	x = -1 $\wedge$ y = 1
...	x = -1 $\wedge$ y = 1

Therefore, collecting semantics:

Point	Characterization of concrete states
I	*true*
B	x = 1
C	(x = 1) $\vee$ (x = -1 $\wedge$ y = 1)
D	(y = 1) $\wedge$ (x = -1 $\vee$ x = 1)
E	x = -1 $\wedge$ y = 1

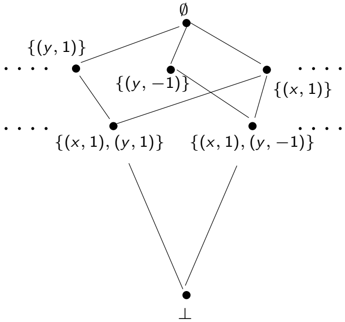
- Abstract lattice D



- Concretization function: What is $\gamma(d)$?

Abstract interpretation for constant propagation

- Abstract lattice D



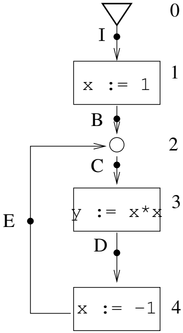
- Concretization function: What is $\gamma(d)$?

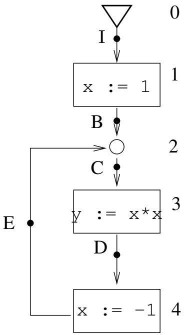
$$
\begin{array}{lcl}
\perp & \mapsto & \{\} \\
\emptyset & \mapsto & State \\
\{(v,c)\} & \mapsto & \{s \mid s \in State \wedge s[v] = c\} \\
\{(x,c),(y,d)\} & \mapsto & \{(c,d)\}
\end{array}
$$

Transfer function for assignment node n of the form $x := exp$.

$$
\begin{array}{lcll}
f_n(P) & = & \bot, & \text{if } P \text{ is } \bot \\
& = & \{(y,c) \in P \mid y \neq x\} \cup \{(x,d)\}, & \text{else if } [exp]_P = d \\
& = & \{(y,c) \in P \mid y \neq x\}, & \text{otherwise}
\end{array}
$$

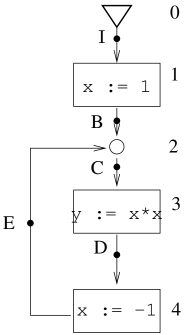
Path	Abstract value at end of path
I	$\emptyset$





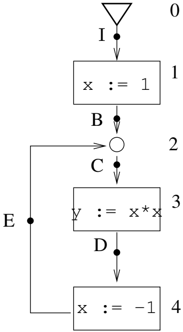
Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$

JOP using abstract lattice



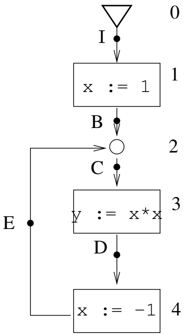
Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$

JOP using abstract lattice



Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$

JOP using abstract lattice



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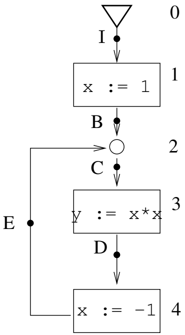
JOP using abstract lattice



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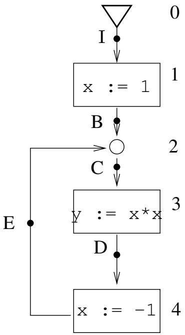

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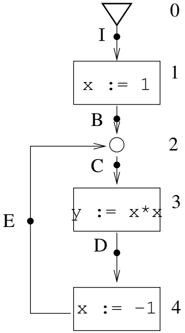

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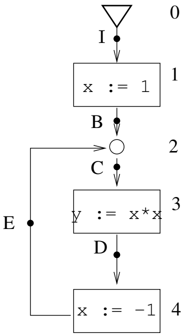
JOP using abstract lattice



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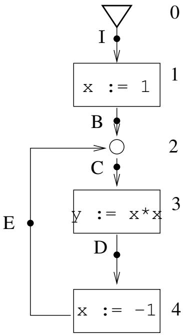
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	$\emptyset$

JOP using abstract lattice



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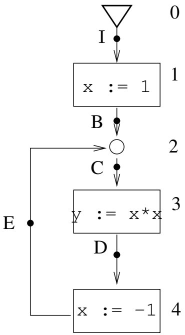
Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$

JOP using abstract lattice



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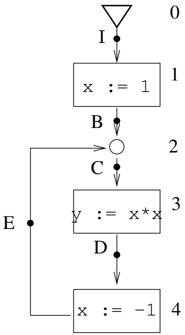
Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$
C	$\emptyset$



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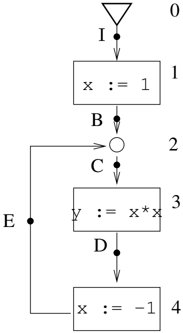
Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$
C	$\emptyset$
D	$\{(y, 1)\}$



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Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$
C	$\emptyset$
D	$\{(y, 1)\}$
E	$\{(x, -1), (y, 1)\}$

Correctness in previous example

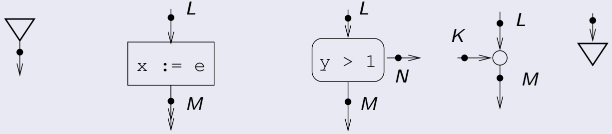
Verify that

- at points I, B and E
  γ(abstract JOP value) $=$ collecting semantics.
- at points C and D
  γ(abstract JOP value) $\supset$ collecting semantics.
- the abstract transfer functions given are the *best* possible for the given lattice L. That is, imprecision is due to the lattice, not the transfer functions.

Formal definition of control-flow graphs

Programs are finite directed graphs with following nodes (statements):

Nodes or statements in a program



- Expressions:

$$e ::= c \mid x \mid e + e \mid e - e \mid e * e.$$

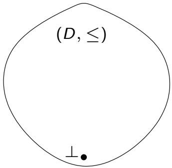
- Boolean expressions:

$$be ::= tt \mid ff \mid e \leq e \mid e = e \mid \neg be \mid be \vee be \mid be \wedge be.$$

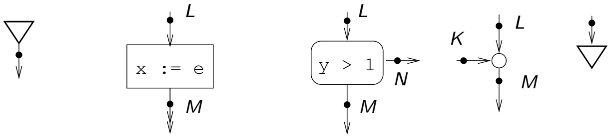
- Assume unique initial program point I.

Formal definition of an abstract interpretation

- Complete join semi-lattice $(D, \leq)$, with a least element $\perp$.
- Concretization function $\gamma : D \to 2^{State}$
- $\perp \in D$ represents unreachability of the program point (i.e., $\gamma(\perp) = \emptyset$)



- Transfer function $f_{LM} : D \to D$ for each node n and incoming edge L into n and outgoing edge M from n.



- Junction nodes have identity transfer function.

What we want to compute for a given program

- Path in a program: Sequence of connected edges or program points.
- Transfer functions extend to paths in program:

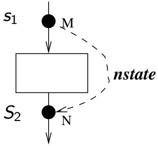
$$f_{ABCD} = f_{CD} \circ f_{BC} \circ f_{AB}.$$

  where $(f_a \circ f_b)(x)$ is defined as $f_a(f_b(x))$.
- f_p is $\lambda d.\bot \;\Rightarrow\;$ path p is *infeasible*.
- Join over all paths (JOP) definition: For each program point N

$$d_N = \bigsqcup_{\text{paths } p \text{ from } I \text{ to } N} f_p(d_0).$$

  where d_0 is a given initial abstract value at entry node.

- Let *Val* be the set of all *concrete* values; e.g., $Integer \cup Boolean$.
- *State* is normally the domain $Var \rightarrow Val$. *However, in general, it can be any semantic domain.*
- Program semantics is given by the functions $nstate_{MN} : State \rightarrow 2^{State}$



- These induce the functions $nstate' : 2^{State} \rightarrow 2^{State}$

$$nstate'_{MN}(S_1 \in 2^{State}) = \bigcup_{s_1 \in S_1} nstate_{MN}(s_1)$$

Formalization of collecting semantics – contd.

- Collecting semantics SS is a map $\mathit{ProgramPoints} \rightarrow 2^{\mathit{State}}$
- At each program point N,

$$\mathrm{SS}(N) = \bigcup_{p \text{ path from } I \text{ to } N} \mathit{nstate}'_p(S_0).$$

where I is entry point of CFG, S_0 is the given initial set of states, and nstate'_p is composition of nstate' functions of edges that constitute p.