Interprocedural analysis: Sharir-Pnueli’s functional approach

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Outline

1. Functional Approach
2. Example
3. Iterative Approach
4. Exercises
We want JOP at $N$.

If transfer functions are distributive, then we can take join over paths at any intermediate point $M$, and then join over paths from $M$ to $N$. 

Equations to capture JOP: why it works
Equation solving: Problems with naive approach

- In non-procedural case, we setup equations to capture JOP assuming distributivity. Least solution to these equations gave us exact/over-approx JOP depending on distributive/monotonic framework.

- Try to set up similar equations for $x_N$ (JVP at program point $N$).

- How do we describe $x_N$ in terms of $x_J$?
Instead try to capture join over **complete** paths first

- Set up equations to capture join over **complete** paths.
- Now set up equations to capture JVP using join over complete path values.
- Root of procedure $p$ is denoted $r_p$.
- Exit (return) of procedure $p$ is denoted $e_p$.
- Sometimes use $r_1$ for $r_{main}$.
- Assume WLOG that main is not called.
Example paths

An example valid path in $IVP(r_1, I)$:

An example valid and complete path in $IVP_0(r_1, D)$:

Path “FGHLFKJMIJ” is valid and complete and is in $IVP_0(r_p, J)$. 
Basic idea: Why join over complete paths help

An IVP path \( \rho \) from \( r_1 \) to \( N \) in procedure \( p \) can be written as \( \delta \cdot \eta \) where \( \delta \) is in IVP\((r_1, r_p)\), and \( \eta \) is in IVP\(_0\)(\(r_p, N\)).

Path \( \eta \) is suffix after last pending call to procedure \( p \) was made.
Valid and complete paths from $r_p$ to $N$

For a procedure $p$ and node $N$ in $p$, define:

$$\phi_{r_p,N} : D \rightarrow D$$

given by

$$\phi_{r_p,N}(d) = \bigsqcup_{\text{paths } \rho \in \text{IVP}_0(r_p,N)} f_{\rho}(d).$$

$\phi_{r_p,N}$ is thus the join of all functions $f_{\rho}$ where $\rho$ is an interprocedurally valid and complete path from $r_p$ to $N$. 
Visualizing $\phi_{r_p,N}$
Using $\phi_{r_p,N}$’s to get JVP values

Assuming distributivity of underlying transfer functions, JVP value at $N$ equals $\phi_{r_p,N}$ applied to JVP value at $r_p$. 
Solving a system of equations using Knaster-Tarski Theorem

- Set up equations \((E_1)\)

\[
\begin{align*}
y_1 &= f_1(y_1, \ldots, y_n) \\
\vdots \\
y_n &= f_n(y_1, \ldots, y_n)
\end{align*}
\]

- Ensure that values come from a complete lattice \((D, \leq)\).
- Ensure that each \(f_i\) is a monotonic function on this lattice: if \(\langle d_1, \ldots, d_n \rangle \leq \langle e_1, \ldots, e_n \rangle\) then \(f_i(d_1, \ldots, d_n) \leq f_i(e_1, \ldots, e_n)\).
- Equivalently, the function \(\overline{F}\) on \((D^n, \leq)\) given by

\[
\overline{F}(\langle d_1, \ldots, d_n \rangle) = \langle f_1(d_1, \ldots, d_n), \ldots, f_n(d_1, \ldots, d_n) \rangle,
\]

is monotonic.

Then, by Knaster-Tarski, the function \(\overline{F}\) on \((D^n, \leq)\) has a LFP, which coincides with the least solution to equations \((E_1)\).
Equations (1) to capture $\phi_{r_p,N}$

\[
\begin{align*}
  y_{r_p,r_p} &= id_D & \text{(root)} \\
  y_{r_p,N} &= f_{MN} \circ y_{r_p,M} & \text{(stmt)} \\
  y_{r_p,N} &= y_{r_q,e_q} \circ y_{r_p,M} & \text{(call)} \\
  y_{r_p,N} &= y_{r_p,L \sqcup y_{r_p,M}} & \text{(join)}
\end{align*}
\]
Example: Available expressions analysis

Lattice for Av-Exp analysis.

- Is \(a \times b\) available at program point \(N\)?

0 (not available)

1 (available)

\(\bot\)
Example: Available expressions analysis

- 0 (not available)
- 1 (available)
- ⊥

Lattice for Av-Exp analysis.

- Is \(a \times b\) available at program point \(N\)?
- No if we consider all paths.

Is \(a \times b\) available at program point \(N\)?

- No if we consider all paths.
Example: Available expressions analysis

Lattice for Av-Exp analysis.

- Is a*b available at program point \( N \)?
- No if we consider all paths.
- Yes if we consider interprocedurally valid paths only.
Functions we will use for example analysis

- \( D = \{ \bot, 1, 0 \} \).
- \( 0 : D \to D \) given by
  
  \[
  \begin{array}{ccc}
  \bot & \mapsto & \bot \\
  0 & \mapsto & 0 \\
  1 & \mapsto & 0 \\
  \end{array}
  \]

- \( 1 : D \to D \) given by
  
  \[
  \begin{array}{ccc}
  \bot & \mapsto & \bot \\
  0 & \mapsto & 1 \\
  1 & \mapsto & 1 \\
  \end{array}
  \]

- \( \text{id} : D \to D \) given by
  
  \[
  \begin{array}{ccc}
  \bot & \mapsto & \bot \\
  0 & \mapsto & 0 \\
  1 & \mapsto & 1 \\
  \end{array}
  \]

- Ordering: \( 1 \leq \text{id} \leq 0 \).
Example: Equations for $\phi$'s

\[
\begin{align*}
y_A,A &= id \\
y_A,B &= 0 \circ y_A,A \\
y_A,C &= 1 \circ y_A,B \\
y_A,P &= y_F,J \circ y_A,C \\
y_A,D &= 1 \circ y_A,P \\
y_A,E &= id \circ y_A,D \\
y_F,F &= id \\
y_F,G &= id \circ y_F,F \\
y_F,K &= id \circ y_F,F \\
y_F,H &= 0 \circ y_F,G \\
y_F,Q &= y_F,J \circ y_F,H \\
y_F,I &= 1 \circ y_F,Q \\
y_F,J &= y_F,I \sqcup y_F,K \\
\end{align*}
\]
Using $\phi_{rp,N}$’s to get JVP values

Assuming distributivity of underlying transfer functions, JVP value at $N$ equals $\phi_{rp,N}$ applied to JVP value at $r_p$. 
Equations (2) to capture JVP

\[ x_1 = d_0 \]
\[ x_{r_p} = \bigcup_{\text{calls } C \text{ to } p} x_C \]
\[ x_N = \phi_{r_p,N}(x_{r_p}) \quad \text{for } N \in \text{ProgPts}(p) - \{r_p\}. \]
**Example: Equations for $x_N$’s (JVP)**

\[
\begin{align*}
  x_A &= 0 \\
  x_B &= \phi_{AB}(x_A) \\
  x_C &= \phi_{AC}(x_A) \\
  x_P &= \phi_{AP}(x_A) \\
  x_D &= \phi_{AD}(x_A) \\
  x_E &= \phi_{AE}(x_A) \\
  x_F &= x_C \sqcup x_H \\
  x_G &= \phi_{FG}(x_F) \\
  x_K &= \phi_{FK}(x_F) \\
  x_H &= \phi_{FH}(x_F) \\
  x_Q &= \phi_{FQ}(x_F) \\
  x_I &= \phi_{FI}(x_F) \\
  x_J &= \phi_{FJ}(x_F).
\end{align*}
\]
Example: Equations for $x_N$’s (JVP)

\[
\begin{align*}
    x_A &= 0 \\
    x_B &= 0(x_A) \\
    x_C &= 1(x_A) \\
    x_P &= 1(x_A) \\
    x_D &= 1(x_A) \\
    x_E &= 1(x_A) \\
    x_F &= x_C \sqcup x_H \\
    x_G &= \text{id}(x_F) \\
    x_K &= \text{id}(x_F) \\
    x_H &= 0(x_F) \\
    x_Q &= 0(x_F) \\
    x_I &= 1(x_F) \\
    x_J &= \text{id}(x_F).
\end{align*}
\]

Fig. shows values of $\phi_{r_p,N}$’s in bold.
Correctness claims

- Consider lattice $(F, \leq)$ of functions from $D$ to $D$, obtained by closing the transfer functions, identity, and $f_\bot : d \mapsto \bot$ under composition and join. (Alternatively, we can take $F$ to be all monotone functions on $D$.)
- Ordering is $f \leq g$ iff $f(d) \leq g(d)$ for each $d \in D$.
- $(F, \leq)$ is also a complete lattice.
- $\bar{f}$ induced by Eq (1) is monotone on complete lattice $(\bar{F}, \leq)$.
  - Sufficient to argue that function composition $\circ$ is monotone when applied to monotone functions.
  - Join operation $\bigvee$ is monotone.
- LFP / least solution (say $y_{r_p,N}^*$'s) exists by Knaster-Tarski.
- Each $y_{r_p,N}^*$ is necessarily monotonic.

Claim

$\phi_{r_p,N}$'s are the least solution to Eq (1) (i.e. $\phi_{r_p,N} = y_{r_p,N}^*$) when $f_{MN}$'s are distributive. Otherwise each $\phi_{r_p,N} \leq y_{r_p,N}^*$.
Using Kildall to compute LFP

- We can use Kildall’s algo to compute the LFP of these equations as follows.
  - Initialize the value at program points with RHS of the constant equations (in this case \( id \) at entry of procedures), and the bottom value (in this case \( f_\bot \)) everywhere else.
  - Mark all values
  - Pick a marked value at point say \( N \), and “propagate” it (i.e. for any node \( M \) in the LHS of an equation in which \( N \) occurs in the RHS, evaluate \( M \) and join it with the existing value at \( M \)). Mark as before in Kildall’s algo.
  - Stop when no more marked values to propagate.

- Kildall’s algo will compute \( y_{r_p,N}^{*} \) if \( D \) is finite. Note that finite height of \((D, \leq)\) is not sufficient for termination.
Consider Eq (2)’:

\[
\begin{align*}
    x_1 &= d_0 \\
    x_{r_p} &= \bigcup \text{calls } C \text{ to } p \cdot x_C \\
    x_N &= y_{r_p,N}^*(x_{r_p}) \quad \text{for } N \in \mathbb{N}_p - \{r_p\}.
\end{align*}
\]

(Recall that \(y_{r_p,N}^*\’s\) are the least solution of Eq (1).)

- \(f\) induced by Eq (2)’ is a monotone function on the complete lattice \((\overline{D}, \leq)\).
- LFP / least solution (say \(x_N^*\’s\)) exists by Knaster-Tarski.

**Claim**

JVP values are the least solution to Eq (2)’ (i.e. \(\text{JVP}_N = x_N^*\)) when \(f_{MN}\’s\) are distributive. Otherwise \(\text{JVP}_N \leq x_N^* \) for each \(N\).

Kleene/Kildall’s algo will compute \(x_N^*\’s\) (assuming \(D\) finite).
Example: Computing $\phi_{r_p,N}$’s ($y_{r_p,N}^*$ to be precise) using Kildall’s algo

\[ y_{A,A} = id \]
\[ y_{A,B} = 0 \circ y_{A,A} \]
\[ y_{A,C} = 1 \circ y_{A,B} \]
\[ y_{A,P} = y_F,J \circ y_{A,C} \]
\[ y_{A,D} = 1 \circ y_{A,P} \]
\[ y_{A,E} = id \circ y_{A,D} \]
\[ y_{F,F} = id \]
\[ y_{F,G} = id \circ y_{F,F} \]
\[ y_{F,K} = id \circ y_{F,F} \]
\[ y_{F,H} = 0 \circ y_{F,G} \]
\[ y_{F,Q} = y_F,J \circ y_{F,H} \]
\[ y_{F,I} = 1 \circ y_{F,Q} \]
\[ y_{F,J} = y_F,I \sqcup y_F,K \]
Example: Computing $\phi_{r_p,N}$'s ($y^*_{r_p,N}$ to be precise) using Kildall's algo

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\begin{align*}
 y_{A,A} & = id \\
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 y_{A,D} & = 1 \circ y_{A,P} \\
 y_{A,E} & = id \circ y_{A,D} \\
 y_{F,F} & = id \\
 y_{F,G} & = id \circ y_{F,F} \\
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\end{align*}
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\end{align*}
\]
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  y_{A,E} &= id \circ y_{A,D} \\
  y_{F,F} &= id \\
  y_{F,G} &= id \circ y_{F,F} \\
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\end{align*}
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Example: Computing $\phi_{r_p,N}$'s ($y_{r_p,N}^*$ to be precise) using Kildall's algo.

\[\begin{align*}
\phi_{r,A} &= id \\
\phi_{r,B} &= 0 \circ \phi_{r,A} \\
\phi_{r,C} &= 1 \circ \phi_{r,B} \\
\phi_{r,P} &= \phi_{F,J} \circ \phi_{r,C} \\
\phi_{r,D} &= 1 \circ \phi_{r,P} \\
\phi_{r,E} &= id \circ \phi_{r,D} \\
\phi_{f,F} &= id \\
\phi_{f,G} &= id \circ \phi_{f,F} \\
\phi_{f,K} &= id \circ \phi_{f,F} \\
\phi_{f,H} &= 0 \circ \phi_{f,G} \\
\phi_{f,Q} &= \phi_{F,J} \circ \phi_{f,H} \\
\phi_{f,I} &= 1 \circ \phi_{f,Q} \\
\phi_{f,J} &= \phi_{f,I} \sqcup \phi_{f,K} \\
\end{align*}\]
Example: Computing $\phi_{r_p,N}$’s ($y_{r_p,N}^*$ to be precise) using Kildall’s algo

$\begin{align*}
y_{A,A} &= \text{id} \\
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y_{A,D} &= 1 \circ y_{A,P} \\
y_{A,E} &= \text{id} \circ y_{A,D} \\
y_{F,F} &= \text{id} \\
y_{F,G} &= \text{id} \circ y_{F,F} \\
y_{F,K} &= \text{id} \circ y_{F,F} \\
y_{F,H} &= 0 \circ y_{F,G} \\
y_{F,Q} &= y_{F,J} \circ y_{F,H} \\
y_{F,I} &= 1 \circ y_{F,Q} \\
y_{F,J} &= y_{F,I} \sqcup y_{F,K} \\
\end{align*}$
Example: Computing $\phi_{r_p,N}$'s ($y_{r_p,N}^*$ to be precise) using Kildall's algo

- $y_{A,A} = id$
- $y_{A,B} = 0 \circ y_{A,A}$
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- $y_{F,F} = id$
- $y_{F,G} = id \circ y_{F,F}$
- $y_{F,K} = id \circ y_{F,F}$
- $y_{F,H} = 0 \circ y_{F,G}$
- $y_{F,Q} = y_{F,J} \circ y_{F,H}$
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Example: Computing $\phi_{r_p,N}$'s ($y^*_{r_p,N}$ to be precise) using Kildall’s algo

$y_{A,A} = \text{id}$
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$y_{F,J} = y_{F,I} \sqcup y_{F,K}$

Diagram of the computation.
Example: Computing $\phi_{rp,N}$’s ($y_{rp,N}^*$ to be precise) using Kildall’s algo

$$
\begin{align*}
y_{A,A} &= id \\
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y_{A,C} &= 1 \circ y_{A,B} \\
y_{A,P} &= y_F, J \circ y_{A,C} \\
y_{A,D} &= 1 \circ y_{A,P} \\
y_{A,E} &= id \circ y_{A,D} \\
y_{F,F} &= id \\
y_{F,G} &= id \circ y_{F,F} \\
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y_{F,J} &= y_F, I \sqcup y_{F,K}
\end{align*}
$$
Example: Computing $\phi_{r_p, N}$’s ($y_{r_p, N}^*$ to be precise) using Kildall’s algo.

\[\begin{align*}
y_{A,A} & = \text{id} \\
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\end{align*}\]
Example: Computing $\phi_{r_p,N}$’s ($y_{r_p,N}^*$ to be precise) using Kildall’s algo

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\begin{align*}
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\end{align*}
\]
Example: Computing $\phi_{r_p,N}$'s ($y_{r_p,N}^*$ to be precise) using Kildall's algo

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    y_{F,J} &= y_{F,I} \uplus y_{F,K} \\
\end{align*}
\]
Example: Computing $\phi_{r_p,N}$'s ($y_{r_p,N}^*$ to be precise) using Kildall's algo

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y_{A,E} &= \text{id} \circ y_{A,D}
\end{align*}
\]

\[
\begin{align*}
y_{F,F} &= \text{id} \\
y_{F,G} &= \text{id} \circ y_{F,F} \\
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\end{align*}
\]
Example: Computing JVP values ($x^*_N$'s to be precise)

\[
\begin{align*}
    x_A &= 0 \\
    x_B &= 0(x_A) \\
    x_C &= 1(x_A) \\
    x_P &= 1(x_A) \\
    x_D &= 1(x_A) \\
    x_E &= 1(x_A) \\
    x_F &= x_C \sqcup x_H \\
    x_G &= id(x_F) \\
    x_K &= id(x_F) \\
    x_H &= 0(x_F) \\
    x_Q &= 0(x_F) \\
    x_I &= 1(x_F) \\
    x_J &= id(x_F).
\end{align*}
\]
Example: Computing JVP values ($x^*_N$’s to be precise)

$x_A = 0$
$x_B = 0(x_A)$
$x_C = 1(x_A)$
$x_D = 1(x_A)$
$x_E = 1(x_A)$

$x_F = x_C \sqcup x_H$
$x_G = id(x_F)$
$x_K = id(x_F)$
$x_H = 0(x_F)$
$x_Q = 0(x_F)$
$x_I = 1(x_F)$
$x_J = id(x_F)$.

Fig shows initial (red) and final (blue) values.
Example: Computing JVP values ($x^*_N$'s to be precise)

\[
\begin{align*}
    x_A &= 0 \\
    x_B &= 0(x_A) \\
    x_C &= 1(x_A) \\
    x_P &= 1(x_A) \\
    x_D &= 1(x_A) \\
    x_E &= 1(x_A) \\
    x_F &= x_C \sqcup x_H \\
    x_G &= id(x_F) \\
    x_K &= id(x_F) \\
    x_H &= 0(x_F) \\
    x_Q &= 0(x_F) \\
    x_I &= 1(x_F) \\
    x_J &= id(x_F).
\end{align*}
\]

Fig shows initial (red) and final (blue) values.
Summary of functional approach

- Uses a two step approach
  1. Compute $\phi_{r_p,N}$’s.
  2. Compute $x_n$’s (JVP’s) at each point.

Summary of conditions: For each property (column heading), the conjunction of the ticked conditions (row headings) are sufficient to ensure the property.

<table>
<thead>
<tr>
<th></th>
<th>Termination</th>
<th>Least Sol of Eq(2) $\geq$ JVP</th>
<th>Least Sol of Eq(2)= JVP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{MN}$'s monotonic</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Finite underlying lattice</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_{MN}$'s distributive</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
Viewing $\phi$ computation as a table

```
read a,b

t := a*b

call p

print t
```

```
a == 0

a := a-1

call p

t := a*b

ret
```
Viewing $\phi$ computation as a table

- Read $a, b$
- $t := a \times b$
- Call $p$
- $t := a \times b$
- Print $t$

- $a == 0$
- $a := a - 1$
- Call $p$
- $t := a \times b$
- Ret
Viewing $\phi$ computation as a table

```
read a, b

A

B

0 1
0 0

C

1 1

D

E

P

Q

a := a - 1

H

I

J

K

L

M

N

O

ret

0 1
```

```
t := a * b

call p

print t
```

```
t := a * b

call p
```

```
a == 0
```

```
a := a - 1
```

```
t := a * b
```

```
ret
```
Viewing $\phi$ computation as a table
Viewing $\phi$ computation as a table
**Iterative/Tabulation Approach**

- **Main idea:** de-couple the propagation of function rows.
- Maintain a **table** of values representing the current value of $\phi_{r_p,N}$ for each program point $N$ in procedure $p$.
- Expand column for data value $d$ in procedure $p$ only if $d$ is reachable at $r_p$.
- Informally, at $N$ in procedure $p$, the table has an entry $d \mapsto d'$ if we have seen
  1. valid paths $\rho$ from $r_1$ to $r_p$ with $\bigcup \rho f_{\rho}(d_0) = d$, and
  2. valid and complete paths $\delta$ from $r_p$ to $N$ with $\bigcup \delta f_{\delta}(d) = d'$. 
Iterative/Tabulation Approach

- Apply Kildall’s algo with initial value of $d_0 \mapsto d_0$ at $r_1$.
- Propagating value $d$ across a call to procedure $p$: (a) begin a column for $d$ at root of $p$ if not already there; Also (b) if $d$ is mapped to $d'$ at the end of $p$, then propagate $d'$ to the return site of the call.
- Propagating across return nodes from procedure $p$: value $d'$ in column for $d$ is propagated to each column at a return site of a call to procedure $p$ that has the value $d$ in the preceding entry.
Example: Computing $\phi$'s iteratively: 1

```
a := a^{-1}
F
G
t := a*b
A
read a,b
t := a*b
print t
D
call p
E
call p
a == 0
B
C
O
L
M
N
6
ret
t := a*b
I
J
K
P
H
Q
```
Example: Computing $\phi$’s iteratively: 2

```
read a, b

A 0 -

B 0 -

C 1 -

D

E

F
g

G

H

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

W

X

Y

Z

a := a - 1

t := a * b

call p

print t

t := a * b

ret
```

```
a == 0

a := a - 1

call p

t := a * b
```

```
F
G
H
I
J
K
L
M
N
O
P
Q
R
S
T
U
V
W
X
Y
Z
```
Example: Computing $\phi$’s iteratively: 3

- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
- $t := a*b$
- $a == 0$
- $a := a-1$
- $t := a*b$
Example: Computing \( \phi \)'s iteratively: 4

read \( a, b \)

\[ t := a \times b \]

call \( p \)

\[ t := a \times b \]

print \( t \)

\[ a := a - 1 \]

call \( p \)

\[ t := a \times b \]

ret

\( a == 0 \)
Example: Computing $\phi$’s iteratively: 5

Example:

```plaintext
read a, b

A => 0

B => 0

C => 1

call p

P

t := a * b

D

print t

E

F

G => 1

call p

H

a := a - 1

K => 1

L

M

N

a \neq 0

O

O

F

G

6

1

ret

t := a * b

I

J

K
```
Example: Computing $\phi$'s iteratively: 6

read a, b

$\\text{t := a} \ast \text{b}$

call p

print t

$\text{f := a} \ast \text{b}$

$\text{ret}$
Example: Computing $\phi$'s iteratively: 7

```
read a, b

$t := a \cdot b$

print t

call p

call p

$a == 0$

$t := a \cdot b$

ret
```

Diagram:

```
A  0 -
B  0 -
C  1 -
P  1 -
D

E  \downarrow

F  6 -
G  1 -
H  0 -
I  1 -
J  1 -
K  1 -
L  1 -
M  1 -
N  1 -
O  1 -
```

Flow:

1. Read $a, b$.
2. $t := a \cdot b$.
3. Print $t$.
5. Call $p$.
6. Check $a == 0$.
7. $t := a \cdot b$.
8. Return.
Example: Computing $\phi$’s iteratively: 8
Example: Computing $\phi$'s iteratively: 9

```
read a, b

A := a - 1
B := a - 1
C := a - 1
D := a - 1

F := a * b
G := a * b
H := a * b
I := a * b
J := a * b
K := a * b
L := a * b
M := a * b
N := a * b

O := a * b
P := a * b
Q := a * b
R := a * b
S := a * b
T := a * b
U := a * b
V := a * b
W := a * b
X := a * b
Y := a * b
Z := a * b

print t
```

Diagram:

- A: 0
- B: 0
- C: 1
- D: 1
- E: 1
- F: 6
- G: 6
- H: 6
- I: 6
- J: 6
- K: 6
- L: 6
- M: 6
- N: 6
- O: 6
- P: 6
- Q: 6
- R: 6
- S: 6
- T: 6
- U: 6
- V: 6
- W: 6
- X: 6
- Y: 6
- Z: 6
Example: Computing $\phi$'s iteratively: 10

```
A: read a, b
B: t := a*b
C: call p
D: t := a*b
E: print t

\[ t := a \cdot b \]
\[ a := a - 1 \]
```

Graph representation of the iterative approach.
Example: Computing $\phi$’s iteratively: 11

\[
a := a - 1
\]
\[
t := a * b
\]
\[
read a, b
\]
\[
print t
\]
\[
call p
\]
\[
a == 0
\]
\[
ret
\]
Example: Computing $\phi$'s iteratively: 12

```
read a, b

A

B

C

D

E

F

G

H

I

J

K

L

M

N

O

P

Q

ret

t := 0

a := a - 1

t := a * b

t := a * b

done
```

Iterative Approach Example

Functional Approach Exercises
Example: Computing $\phi$’s iteratively: 13

```
a := a^{-1}
F
G
t := a*b
A
read a,b
t := a*b
print t
D
call p
E
call p
a == 0
B
C
O
L
M
N
6
ret
t := a*b
I
J
H
K
P
Q
```
Example: Finally compute $x_N$’s from $\phi$ values

At each point $N$ take join of reachable $\phi_{r_p,N}$ values.
Correctness of iterative algo

- Iterative algo terminates provided underlying lattice is finite.
- It computes the $y_{r_p,N}^*$'s (where $y_{r_p,N}^*$'s are the least solution to Eq (1)) “partially”: If it maps $d$ to $d' \neq \bot$ then $y_{r_p,N}^*(d) = d'$.
- The JVP values it gives (say $z_N$'s) are such that

$$JVP_N \leq z_N \leq x^*_N$$

(Where $x^*_N$'s are the solution to Eq (2')).
- If underlying transfer functions are distributive it computes $\phi_{r_p,N}$'s correctly (though partially), and the JVP values correctly.
- It thus computes an overapproximation of JVP for monotonic transfer functions, and exact JVP when transfer functions are distributive.
Exercise 1: Iterative algo

Run the iterative algo to do constant propagation analysis for the program below with initial value $\emptyset$.

\begin{verbatim}
a := 0
\end{verbatim}
\begin{verbatim}
call p
\end{verbatim}
\begin{verbatim}
print a
\end{verbatim}
\begin{verbatim}
a := a + 1
\end{verbatim}
\begin{verbatim}
a := a - 1
\end{verbatim}
\begin{verbatim}
ret
\end{verbatim}
Exercise 2: Functional vs Iterative algo

Run the functional and iterative algos to do constant propagation analysis for the program below with initial value $∅$:
Comparing functional vs iterative approach

- Functional algo can terminate even when underlying lattice is infinite, provided we can represent and compose/join functions “symbolically”.
- Iterative is typically more efficient than functional since it only computes $\phi_{r_p,N}$’s for values reachable at start of procedure.