

Data-flow Analysis / Abstract Interpretation

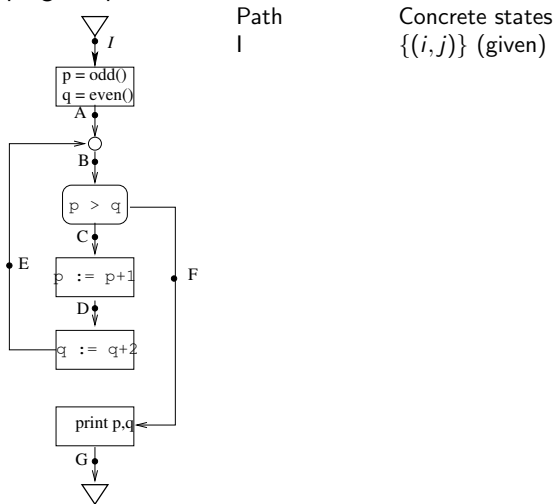
Deepak D'Souza and K. V. Raghavan

IISc

- “Computing ‘safe’ approximations to the set of values / behaviours arising dynamically at run time, statically or at compile time.”
- Typically used by compiler writers to optimize running time of compiled code.
 - Constant propagation: Is the value of a variable constant at a particular program location.
 - Replace $x := y + z$ by $x := 17$ during compilation.
- More recently, used for verifying properties of programs.

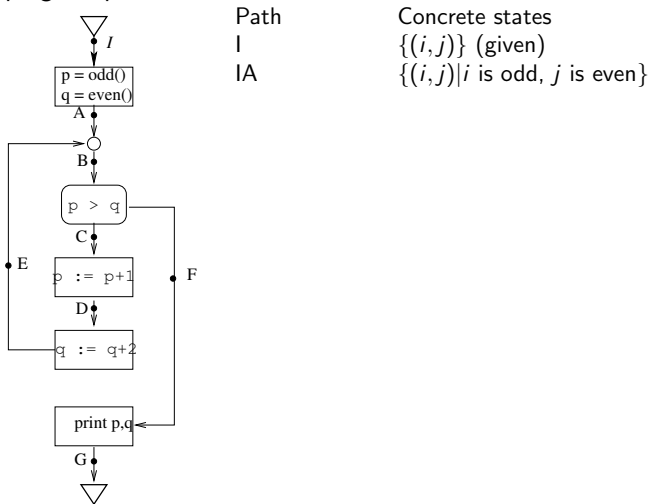
Collecting semantics – example

Collecting semantics of a program = set of (concrete) states occurring at each program point.



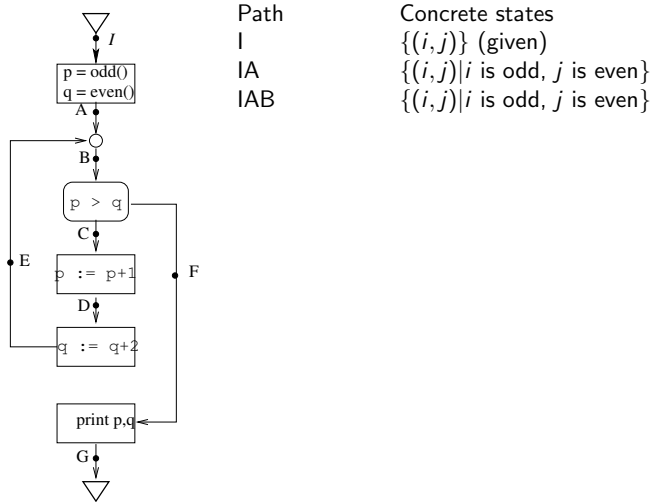
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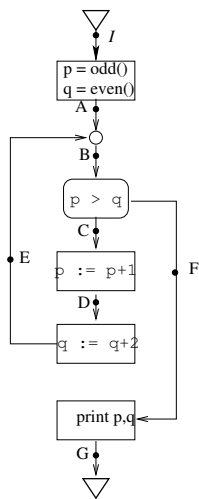
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Collecting semantics – example

Collecting semantics of a program = set of (concrete) states occurring at each program point.



Path

I

Concrete states

$\{(i, j)\}$ (given)

IA

$\{(i, j) \mid i \text{ is odd, } j \text{ is even}\}$

IAB

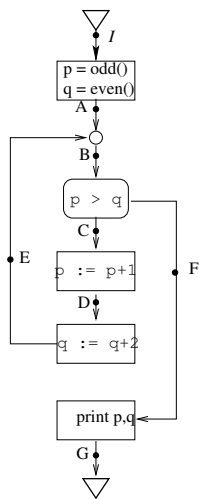
$\{(i, j) \mid i \text{ is odd, } j \text{ is even}\}$

IABC

$\{(i, j) \mid i \text{ is odd, } j \text{ is even, } i > j\}$

Collecting semantics – example

Collecting semantics of a program = set of (concrete) states occurring at each program point.



Path

I

IA

IAB

IABCD

IABCD

Concrete states

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$\{(i, j) \mid i \text{ is odd, } j \text{ is even}\}$

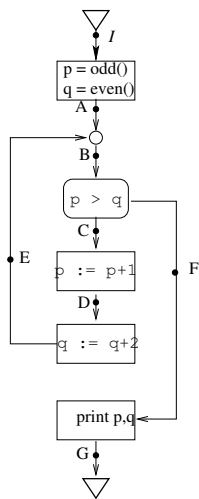
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Collecting semantics – example

Collecting semantics of a program = set of (concrete) states occurring at each program point.



Path

I

IA

IAB

IABCD

IABCE

IABCDE

Concrete states

$\{(i, j)\}$ (given)

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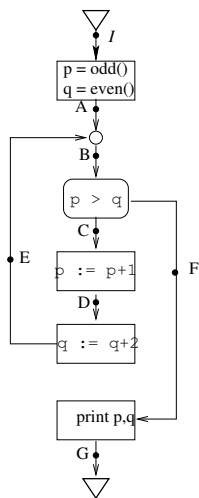
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Collecting semantics – example

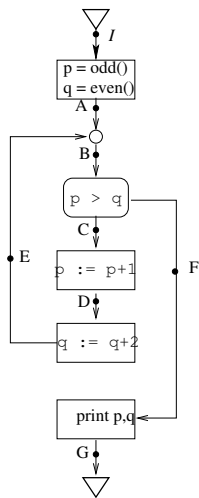
Collecting semantics of a program = set of (concrete) states occurring at each program point.



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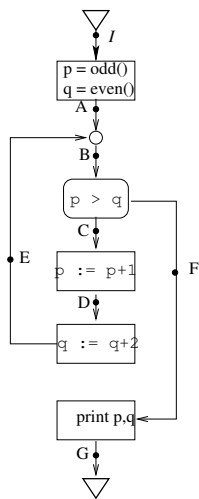
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Collecting semantics – example

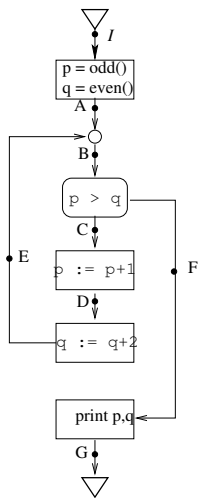
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...	

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...

Therefore, collecting semantics:

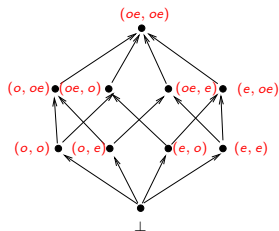
A	$\{(i, j) \mid i \text{ odd, } j \text{ even}\}$
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F	$\{(i, j) \mid i \text{ odd, } j \text{ even, } i < j\} \cup \{(i, j) \mid i \text{ even, } j \text{ even, } i = j\}$

Components of an abstract interpretation:

- Set of **abstract states** D , forming a complete lattice.
- “**Concretization**” function $\gamma : D \rightarrow 2^{State}$, which associates a set of concrete states with each abstract state.
- **Transfer function** $f_n : D \rightarrow D$ for each type of node n , which “interprets” each program statement using the abstract states.

Abstract interpretation – example

- Abstract lattice D



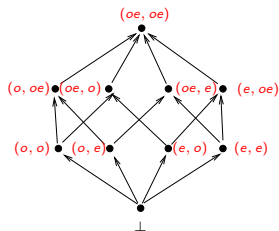
- Transfer function for an assignment node $n: p := p+q$

$$f_n(s) = \begin{cases} \perp & \text{if } s \text{ is } \perp \\ (o, s[q]) & \text{if } s[p] \text{ is } o \text{ and } s[q] \text{ is } e, \\ & \text{or } s[p] \text{ is } e \text{ and } s[q] \text{ is } o \\ (e, s[q]) & \text{if both } s[p] \text{ and } s[q] \text{ are } o \\ & \text{or both } s[p] \text{ and } s[q] \text{ are } e \\ (oe, s[q]) & \text{otherwise} \end{cases}$$

- The concretization function γ
 - $\gamma((oe, oe)) =$

Abstract interpretation – example

- Abstract lattice D



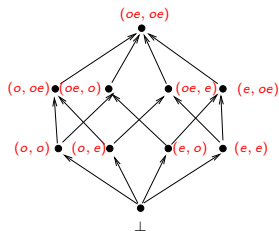
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 - $\gamma((oe, oe)) = State$, $\gamma(\perp) =$

Abstract interpretation – example

- Abstract lattice D



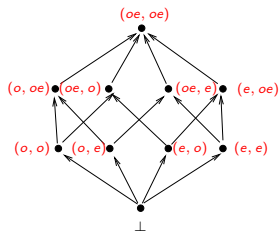
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Abstract interpretation – example

- Abstract lattice D



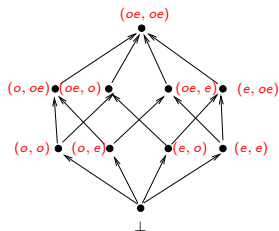
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 - $\gamma((oe, oe)) = State$, $\gamma(\perp) = \emptyset$, $\gamma((o, oe)) = \{(m, n) \mid m \text{ is odd}\}$
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Abstract interpretation – example

- Abstract lattice D

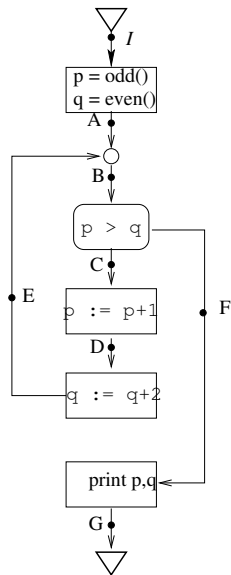


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 - $\gamma((oe, oe)) = State$, $\gamma(\perp) = \emptyset$, $\gamma((o, oe)) = \{(m, n) \mid m \text{ is odd}\}$
 - $\gamma((o, e)) = \{(m, n) \mid m \text{ is odd and } n \text{ is even}\}, \dots$

Collecting abstract values – example



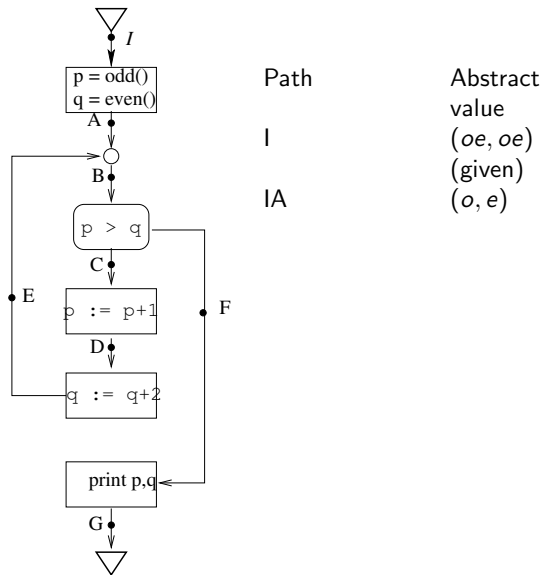
Path

I

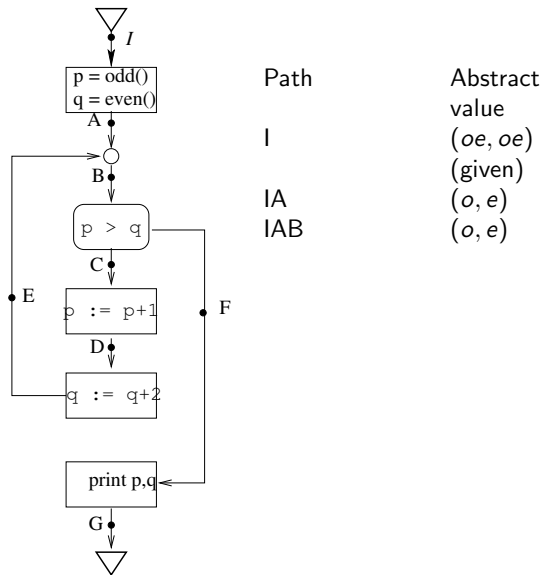
Abstract
value

(oe, oe)
(given)

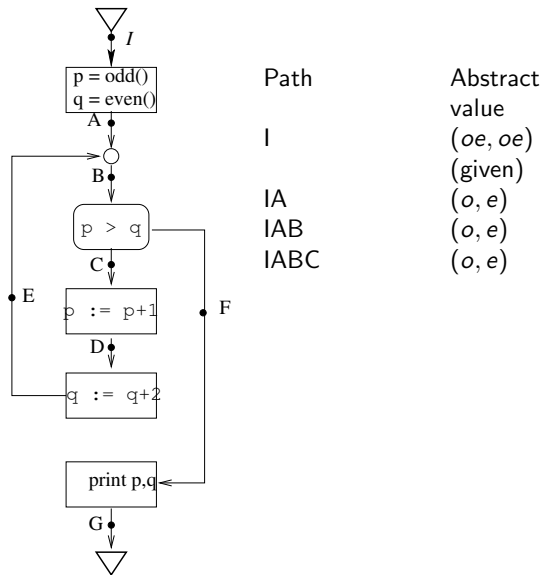
Collecting abstract values – example



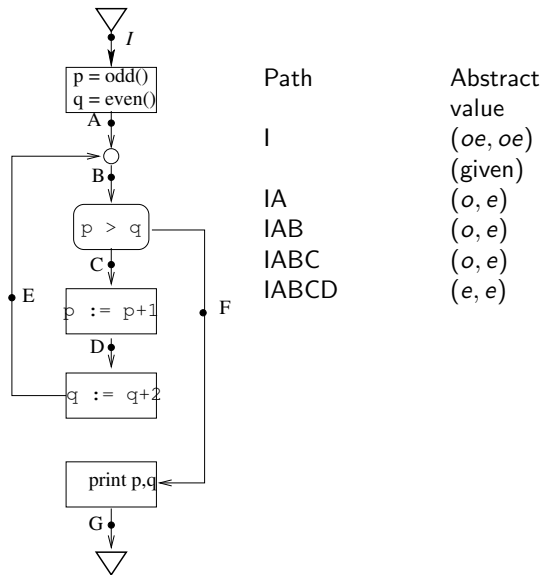
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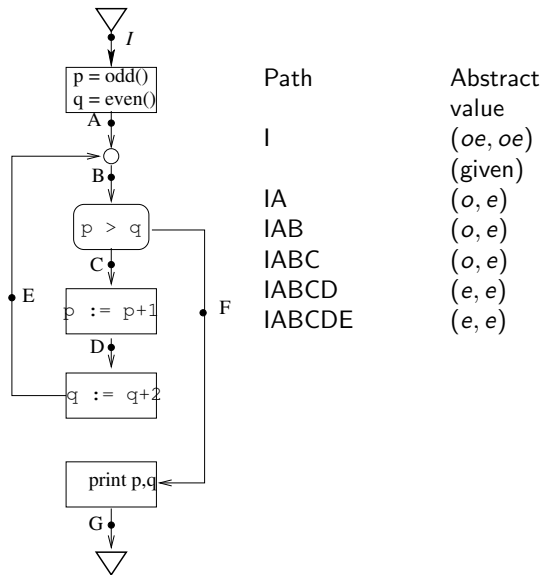
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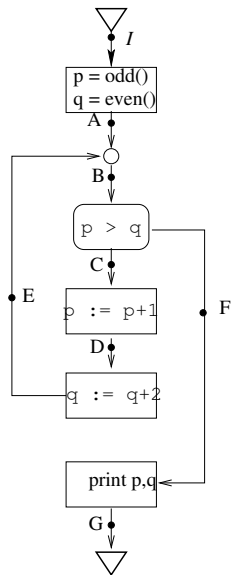
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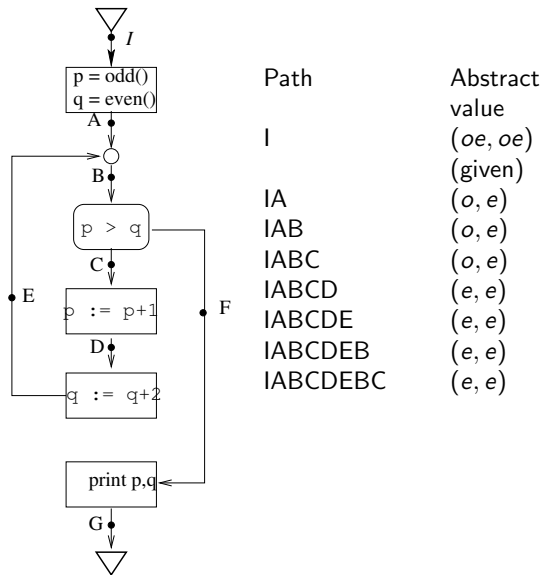


Collecting abstract values – example

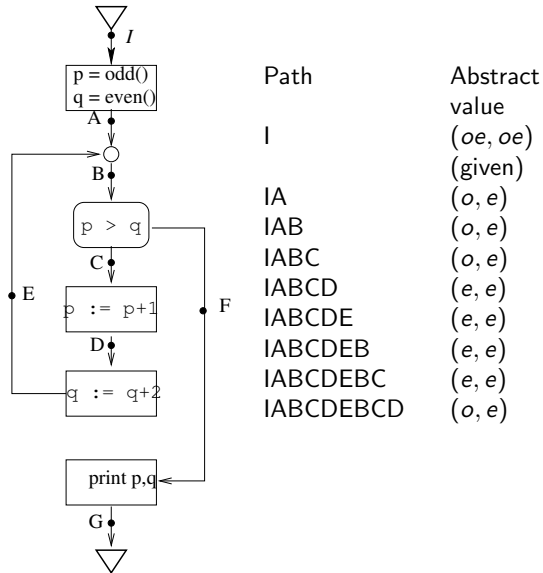


Path	Abstract value
I	(oe, oe)
(given)	
IA	(o, e)
IAB	(o, e)
IABC	(o, e)
IABCD	(e, e)
IABCDE	(e, e)
IABCDEB	(e, e)

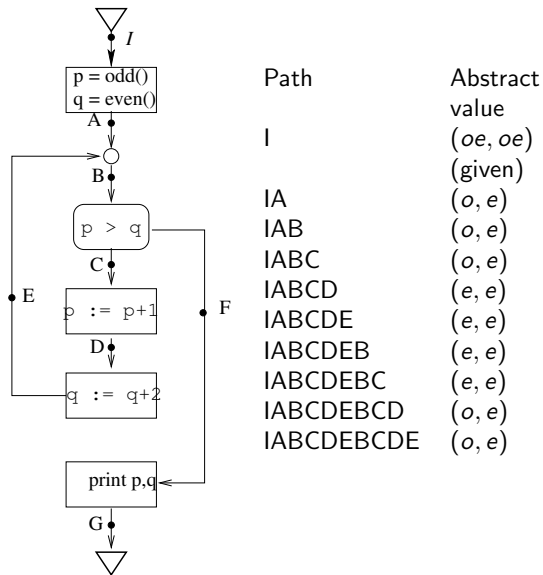
Collecting abstract values – example



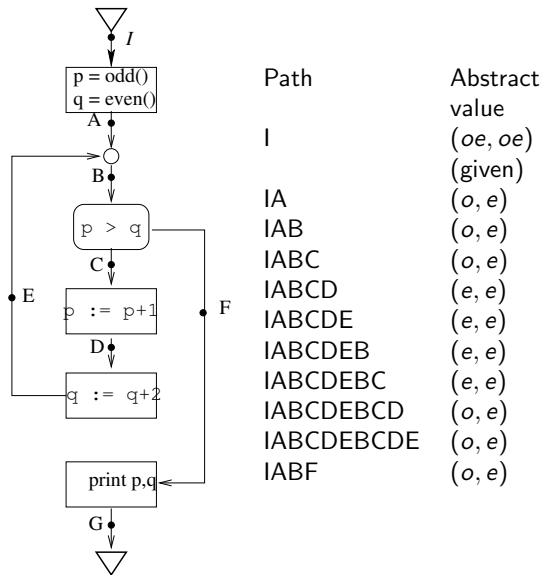
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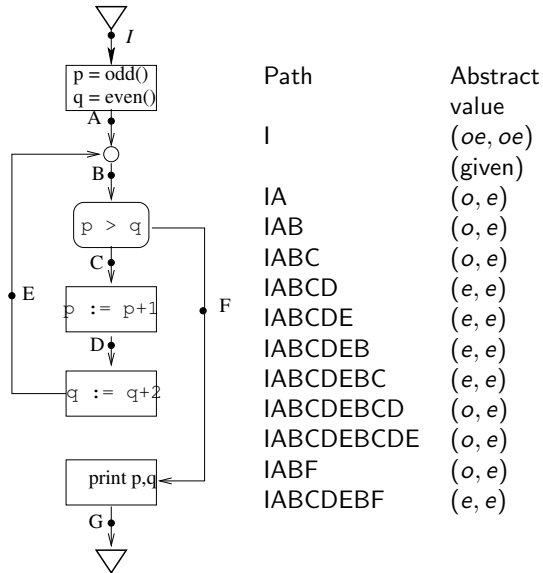
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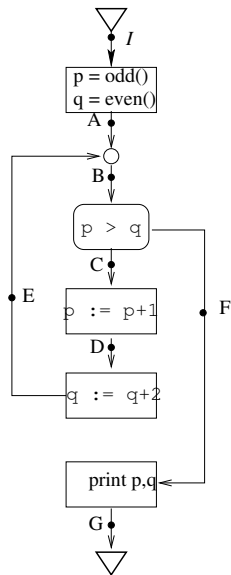
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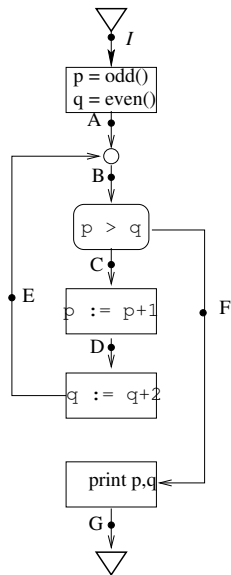


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IABCD	(e, e)
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IABCDEB	(e, e)
IABCDEBC	(e, e)
IABCDEBCD	(o, e)
IABCDEBCDE	(o, e)
IABF	(o, e)
IABCDEBF	(e, e)

Therefore, joining all abstract values at each point:

A	(o, e)
B	$(o, e) \sqcup (e, e) = (oe, e)$
C	$(o, e) \sqcup (e, e) = (oe, e)$
D	$(e, e) \sqcup (o, e) = (oe, e)$
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Collecting abstract values – example



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This is *abstract join-over-all-paths* (JOP) solution.

Comparison of abstract JOP states and collecting states

Abstract JOP:

A (o, e)

B (oe, e)

C (oe, e)

D (oe, e)

E (oe, e)

F (oe, e)

Collecting states:

A $\{(i, j) \mid i \text{ odd}, j \text{ even}\}$

B $\{(i, j) \mid i \text{ odd}, j \text{ even}\} \cup \{(i, j) \mid i \text{ even}, j \text{ even}, i \geq j\}$

C $\{(i, j) \mid j \text{ even}, i > j\}$

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Comparison of abstract JOP states and collecting states

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B (oe, e)

C (oe, e)

D (oe, e)

E (oe, e)

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Note that at each point γ image of abstract solution is over-approximation of collecting states.

A given abstract interpretation is said to be *correct* if, for all abstract states $d_0 \in D$, for all programs P and for all program points p in P ,

γ image of join of all abstract states arising at p (i.e., abstract JOP solution at p), with d_0 as the initial abstract value at P 's

entry

\supseteq

collecting semantics at p , with $\gamma(d_0)$ as the initial set of concrete states at P 's entry

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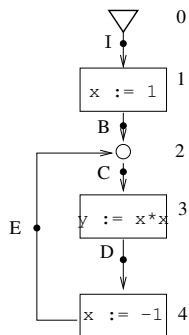
entry

\supseteq

collecting semantics at p , with $\gamma(d_0)$ as the initial set of concrete states at P 's entry

We will study later certain sufficient conditions for a given abstract interpretation to be correct.

Another example program



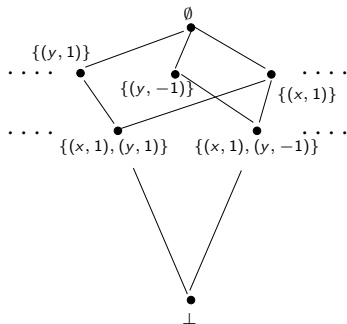
Path	Characterization of concrete states
I	<i>true</i> (given)
IB	$x = 1$
IBC	$x = 1$
IBCD	$x = 1 \wedge y = 1$
IBCDE	$x = -1 \wedge y = 1$
IBCDEC	$x = -1 \wedge y = 1$
IBCDECD	$x = -1 \wedge y = 1$
...	$x = -1 \wedge y = 1$

Therefore, collecting semantics:

Point	Characterization of concrete states
I	<i>true</i>
B	$x = 1$
C	$(x = 1) \vee (x = -1 \wedge y = 1)$
D	$(y = 1) \wedge (x = -1 \vee x = 1)$
E	$x = -1 \wedge y = 1$

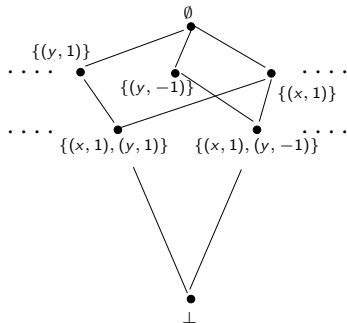
Abstract interpretation for constant propagation

- Abstract lattice D



- Concretization function: What is $\gamma(d)$?

- Abstract lattice D



- Concretization function: What is $\gamma(d)$?

\perp	\mapsto	$\{\}$
\emptyset	\mapsto	<i>State</i>
$\{(v, c)\}$	\mapsto	$\{s \mid s \in \text{State} \wedge s[v] = c\}$
$\{(x, c), (y, d)\}$	\mapsto	$\{(c, d)\}$

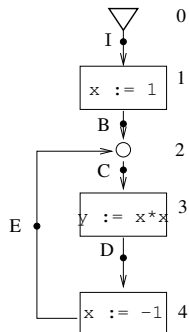
Transfer function for assignment node n of the form $x := exp$.

$$\begin{aligned} f_n(P) &= \perp, \text{ if } P \text{ is } \perp \\ &= \{(y, c) \in P \mid y \neq x\} \cup \{(x, d)\}, \\ &\quad \text{if all variables in } exp \text{ have constant values in } P, \text{ and if} \\ &\quad \text{exp evaluates to } d \text{ with these constant values} \\ &= \{(y, c) \in P \mid y \neq x\}, \text{ otherwise} \end{aligned}$$

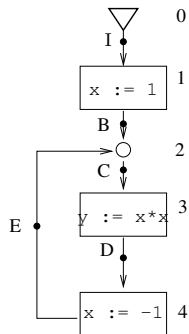
JOP using abstract lattice

Path
1

Abstract value at end of path
 \emptyset



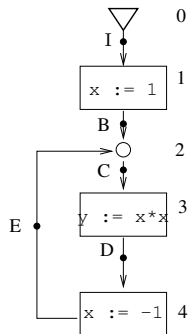
JOP using abstract lattice



Path
I
IB

Abstract value at end of path
 \emptyset
 $\{(x, 1)\}$

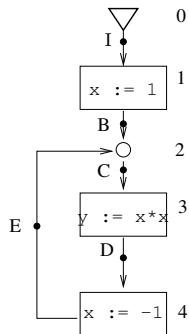
JOP using abstract lattice



Path
I
IB
IBC

Abstract value at end of path
 \emptyset
 $\{(x, 1)\}$
 $\{(x, 1)\}$

JOP using abstract lattice



Path

I

IB

IBC

IBCD

Abstract value at end of path

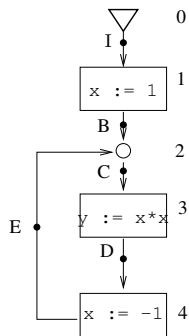
\emptyset

$\{(x, 1)\}$

$\{(x, 1)\}$

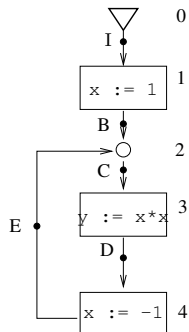
$\{(x, 1), (y, 1)\}$

JOP using abstract lattice



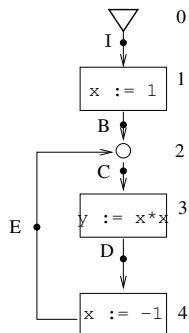
Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$

JOP using abstract lattice



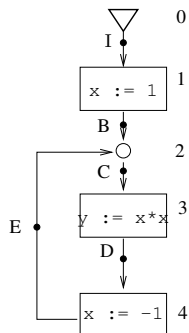
Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$

JOP using abstract lattice



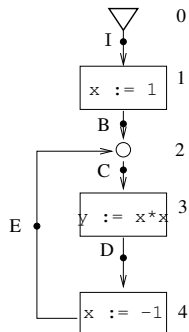
Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$

JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

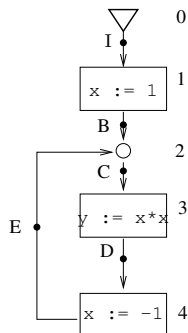
JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	\emptyset

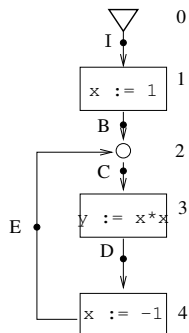
JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	\emptyset
B	$\{(x, 1)\}$

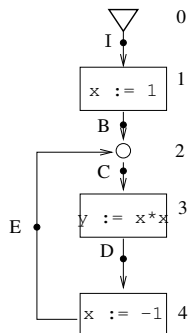
JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	\emptyset
B	$\{(x, 1)\}$
C	\emptyset

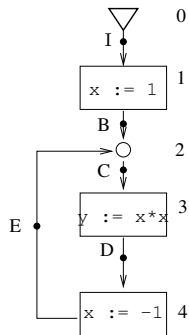
JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	\emptyset
B	$\{(x, 1)\}$
C	\emptyset
D	$\{(y, 1)\}$

JOP using abstract lattice



Path	Abstract value at end of path
I	\emptyset
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	\emptyset
B	$\{(x, 1)\}$
C	\emptyset
D	$\{(y, 1)\}$
E	$\{(x, -1), (y, 1)\}$

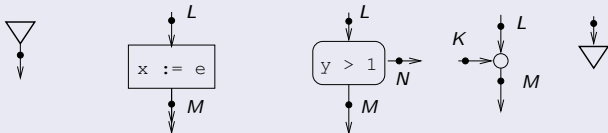
Verify that

- at points I, B and E
 $\gamma(\text{abstract JOP value}) = \text{collecting semantics.}$
- at points C and D
 $\gamma(\text{abstract JOP value}) \supseteq \text{collecting semantics.}$
- the abstract transfer functions given are the *best* possible for the given lattice L . That is, imprecision is due to the lattice, not the transfer functions.

Formal definition of control-flow graphs

Programs are finite directed graphs with following nodes (statements):

Nodes or statements in a program



- Expressions:

$$e ::= c \mid x \mid e + e \mid e - e \mid e * e.$$

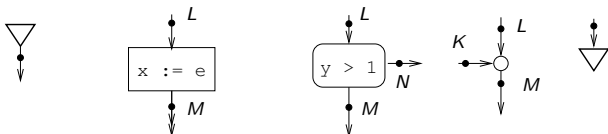
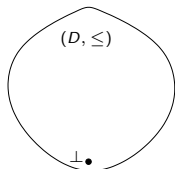
- Boolean expressions:

$$be ::= tt \mid ff \mid e \leq e \mid e = e \mid \neg be \mid be \vee be \mid be \wedge be.$$

- Assume unique initial program point I .

Formal definition of an abstract interpretation

- Complete join semi-lattice (D, \leq) , with a least element \perp .
- Concretization function $\gamma : D \rightarrow 2^{State}$
- $\perp \in D$ represents unreachability of the program point (i.e., $\gamma(\perp) = \emptyset$)
- Transfer function $f_{LM} : D \rightarrow D$ for each node n and incoming edge L into n and outgoing edge M from n .



- Junction nodes have identity transfer function.

What we want to compute for a given program

- Path in a program: Sequence of connected edges or program points.
- Transfer functions extend to paths in program:

$$f_{ABCD} = f_{CD} \circ f_{BC} \circ f_{AB}.$$

where $(f_a \circ f_b)(x)$ is defined as $f_a(f_b(x))$.

- f_p is $\lambda d. \perp \Rightarrow$ path p is *infeasible*.
- Join over all paths (JOP) definition: For each program point N

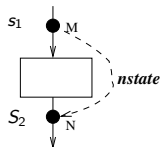
$$d_N = \bigsqcup_{\text{paths } p \text{ from } I \text{ to } N} f_p(d_0).$$

where d_0 is a given initial abstract value at entry node.

Formalization of collecting semantics

- Let Val be the set of all *concrete* values; e.g., $Integer \cup Boolean$.
- $State$ is normally the domain $Var \rightarrow Val$. However, in general, it can be any semantic domain.

- Program semantics is given by the functions $nstate_{MN} : State \rightarrow 2^{State}$



- These induce the functions $nstate' : 2^{State} \rightarrow 2^{State}$

$$nstate'_{MN}(S_1 \in 2^{State}) = \bigcup_{s_1 \in S_1} nstate_{MN}(s_1)$$

- Collecting semantics SS is a map $ProgramPoints \rightarrow 2^{State}$
- At each program point N ,

$$SS(N) = \bigcup_{p \text{ path from } I \text{ to } N} nstate'_p(S_0).$$

where I is entry point of CFG, S_0 is the given initial set of states, and $nstate'_p$ is composition of $nstate'$ functions of edges that constitute p .