Correctness of Abstract Interpretation

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IISc

Recollection of Abstract Interpretation

It is a tuple (D, F_D, γ), such that

- $(D, \leq)$ is a complete join semi-lattice (aka the abstract lattice), with a least element $\bot$.
- Concretization function $\gamma : D \to 2^{State}$
- Monotone transfer function $(f_{LM} : D \to D) \in F_D$ for each node n and incoming edge L into n and outgoing edge M from n.
  - Junction nodes have identity transfer function.

An aside: Collecting semantics stated as an abstract interpretation

- Concrete lattice $C : (2^{State}, \subseteq)$, $\bot = \emptyset$, $\top = State$, $\sqcup = \cup$.
- Transfer function $f_{LM} = nstate'_{LM}$ for each node n and incoming edge L into n and outgoing edge M from n.
- $\gamma : C \rightarrow C$ is identity

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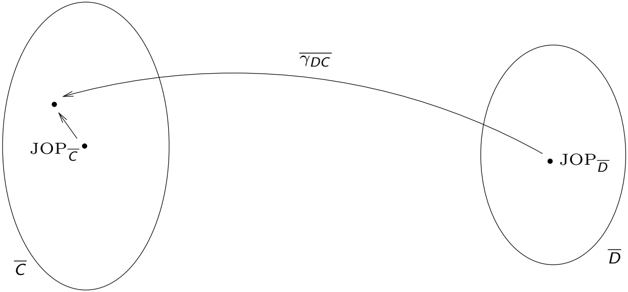
- $\gamma : C \to C$ is identity
- Therefore, collecting states at any point $N =$ JOP at this point using this interpretation
- This particular abstract interpretation is also known as the concrete interpretation.

Definition: consistent abstractions

An A.I. $(D, F_D, \gamma_D : D \rightarrow 2^{State})$ is said to be a consistent abstraction of (or, be correct wrt) another A.I. $(C, F_C, \gamma_C : C \rightarrow 2^{State})$ *under* a pair of monotone functions $\gamma_{DC} : D \rightarrow C$ and $\alpha_{CD} : C \rightarrow D$ iff:
(a) $(\alpha_{CD}, \gamma_{DC})$ form a Galois connection, and

(b) for all programs, and for all $d_0 \in D$ and $c_0 \in C$ such that $\gamma_{DC}(d_0) \geq c_0$:

$$\mathrm{JOP}_{\overline{C}} \leq \overline{\gamma_{DC}}(\mathrm{JOP}_{\overline{D}})$$



Definition – contd.

where

- $\mathrm{JOP}_{\overline{C}}$ is obtained by using (C, f_C), with c_0 as the initial state,
- $\mathrm{JOP}_{\overline{D}}$ is by obtained using (D, f_D), with d_0 as the initial state, and
- $\overline{x}$ is the "vectorized" form of x, i.e., x for all points in a program.

Note: Throughout remaining slides we use γ to mean γ_{DC} and α to mean α_{CD}.

Definition: (α, γ) form Galois Connection

- α and γ are monotonic
- $\gamma(\alpha(e)) \geq e$, for all $e \in C$
- $\alpha(\gamma(d)) = d$, for all $d \in D$

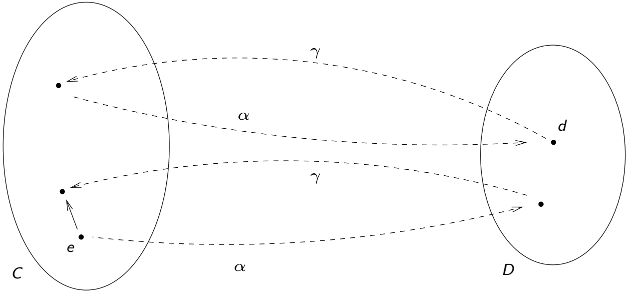


Illustration of consistent abstraction

- Consider the lattices L_1 and L_2 from the introduction slides.
- L_1 is a consistent abstraction of L_2 under the following (α, γ):

$$\begin{array}{rcl} \alpha(S \in L_2) & = & \bot, \text{ if } S = \emptyset \\ & = & (coll(\{x \mid (x,y) \in S\}), coll(\{y \mid (x,y) \in S\})), \\ & & \text{ otherwise} \\ \gamma((c,d) \in L_1) & = & \{(x,y) \mid \text{if } c \text{ is } oe \text{ then } x = o \vee x = e \text{ else } x = c, \\ & & \quad \text{if } d \text{ is } oe \text{ then } y = o \vee y = e \text{ else } y = d\} \end{array}$$

where

$$\begin{array}{rcl} coll(W) & = & o, \text{ if } W = \{o\} \\ & = & e, \text{ if } W = \{e\} \\ & = & oe, \text{ if } W = \{o, e\} \end{array}$$

Another illustration of consistent abstraction

Constant propagation (CP) is a consistent intepretation of the concrete interpretation, under the following (α, γ):

$$
\begin{array}{rcl}
\alpha(S \in 2^{State}) & = & \bot, \\
 & & \text{if } S \text{ is empty} \\
 & = & \{(x, c) \mid \forall e \in S: \ e(x) = c\}, \\
 & & \text{otherwise} \\
\gamma(p) & = & \emptyset, \\
 & & \text{if } p = \bot \\
 & = & \{e \in State \mid \text{for each } (x, c) \in p : e(x) = c\}, \\
 & & \text{if } p \text{ is any other element of the lattice}
\end{array}
$$

Properties of consistent abstractions

- Note: If an abstract interpretation $(D, F_D, \gamma : D \to 2^{State})$ is a consistent abstraction of $(2^{State}, nstate', identity)$, then we say that $(D, F_D, \gamma : D \to 2^{State})$ is correct.
- Consistent-abstraction-of is a transitive property. That is, if $(D, F_D, \gamma_D : D \to 2^{State})$ is a consistent abstraction of $(C, F_C, \gamma_C : C \to 2^{State})$ under $\gamma_{DC} : D \to C$, and $(C, F_C, \gamma_C : C \to 2^{State})$ is a consistent abstraction of $(B, F_B, \gamma_B : B \to 2^{State})$ under $\gamma_{CB} : C \to B$, then $(D, F_D, \gamma_D : D \to 2^{State})$ is a consistent abstraction of $(B, F_B, \gamma_B : B \to 2^{State})$ under $\gamma_{CB} \circ \gamma_{DC}$.

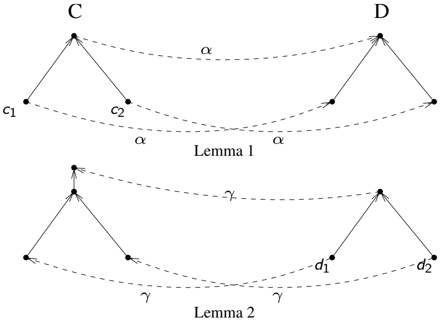
A sufficient condition for correctness

Theorem: An abstract interpretation (D, F_D, γ_D) is a consistent abstraction of another abstract interpretation (C, F_C, γ_C) under a pair (α, γ) if

- (α, γ) form a Galois connection, and
- Each transfer function $f_{LM,D} \in F_D$ is an abstraction of the corresponding function $f_{LM,C} \in F_C$.

Lemmas

If (α, γ) form a Galois connection then the concrete and abstract join operators satisfy the following properties.



Proof of lemmas

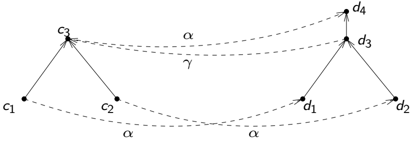
Proof of Lemma 2:

- $d_1 \sqcup d_2$ is $\geq$ both d_1 and d_2 (property of join)
- Therefore, due to monotonicity of γ, $\gamma(d_1 \sqcup d_2)$ is $\geq$ both $\gamma(d_1)$ and $\gamma(d_2)$.
- Therefore, by property of join, $\gamma(d_1 \sqcup d_2) \ \geq \ \gamma(d_1) \sqcup \gamma(d_2)$. $\Box$.

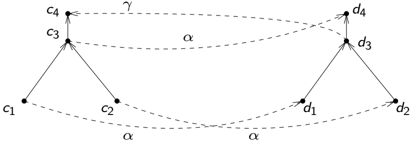
Proof of Lemma 1:

- Using an argument similar to above it can be shown that $\alpha(c_1 \sqcup c_2) \ \geq \ \alpha(c_1) \sqcup \alpha(c_2)$.
- Let $c_3 \equiv c_1 \sqcup c_2$, $d_1 \equiv \alpha(c_1)$, $d_2 \equiv \alpha(c_2)$, $d_3 \equiv d_1 \sqcup d_2$, and $d_4 \equiv \alpha(c_3)$.
- We now prove that $\alpha(c_1 \sqcup c_2) \ \sqsupset \ \alpha(c_1) \sqcup \alpha(c_2)$ is *not* possible. Assume, for contradiction, that $d_4 \sqsupset d_3$.
- Due to Galois connection property, $\gamma(d_1) \sqsupseteq c_1$ and $\gamma(d_2) \sqsupseteq c_2$. Now, since d_3 dominates d_1 and d_2, due to monotonicity of γ, it follows that $\gamma(d_3)$ dominates c_1 and c_2. Therefore, $\gamma(d_3)$ dominates c_3.

Proof of Lemma 1 – continued



Case 1



Case 2

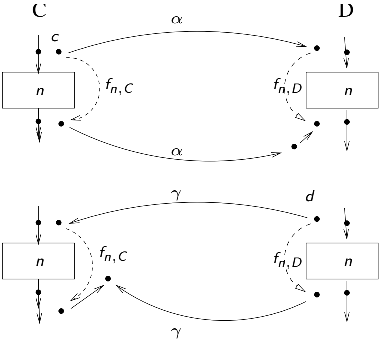
Now, one of the following two cases has to hold.

- Case 1: $\gamma(d_3) = c_3$. Following the property of Galois connections, we get $\alpha(c_3) = d_3$. But this contradicts $\alpha(c_3) = d_4$.
- Case 2: $(\gamma(d_3) = c_4) \sqsupset c_3$. Therefore, due to the Galois connection property, $\alpha(c_4) = d_3$. This, in conjunction with $\alpha(c_3) = d_4$, violates monotonicity of α.

Therefore, $\alpha(c_1 \sqcup c_2) \ = \ \alpha(c_1) \sqcup \alpha(c_2)$.

Definition: $f_{n,D}$ is an abstraction of $f_{n,C}$

$f_{n,C}$ and $f_{n,D}$ satisfy *one* of the following (each of them implies the other):



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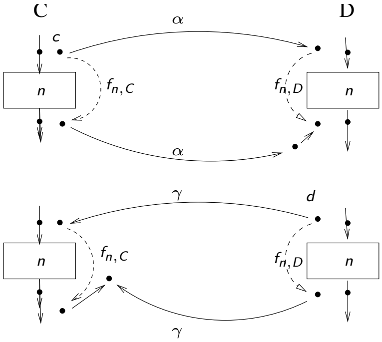
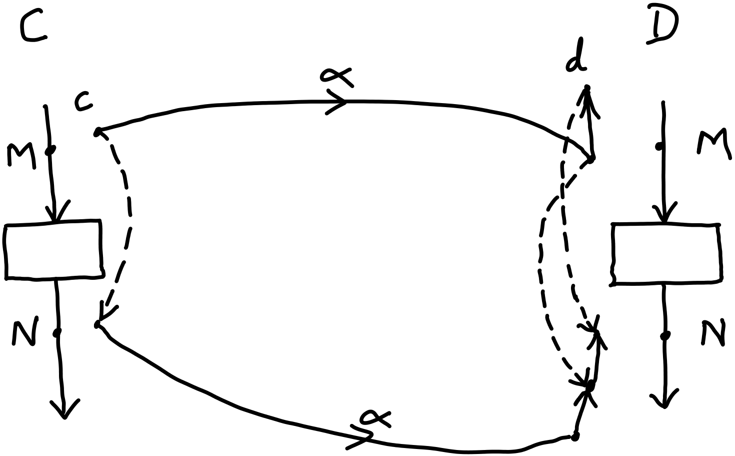


Exercise: Given a statement $n =$ "`x = x + y`", and treating the constant propagation lattice as D, identify an element c of the 2^{State} domain such that $\alpha(f_{n,C}(c)) \sqsubset f_{n,D}(\alpha(c))$.

Lemma 3

Statement: Consider any edge $M \to N$. If d is any element of D and c is any element of C such that $\alpha(c) \leq d$, then $\alpha(f_{MN,C}(c)) \leq f_{MN,D}(d)$.
Proof: The first condition on transfer functions tells us that $\alpha(f_{MN,C}(c)) \leq f_{MN,D}(\alpha(c))$. Using the lemma's prerequisite $\alpha(c) \leq d$, and by monotonicity of $f_{MN,D}$, we get $f_{MN,D}(\alpha(c)) \leq f_{MN,D}(d)$. Therefore $\alpha(f_{MN,C}(c)) \leq f_{MN,D}(d)$ □

Lemma 3 proof illustration

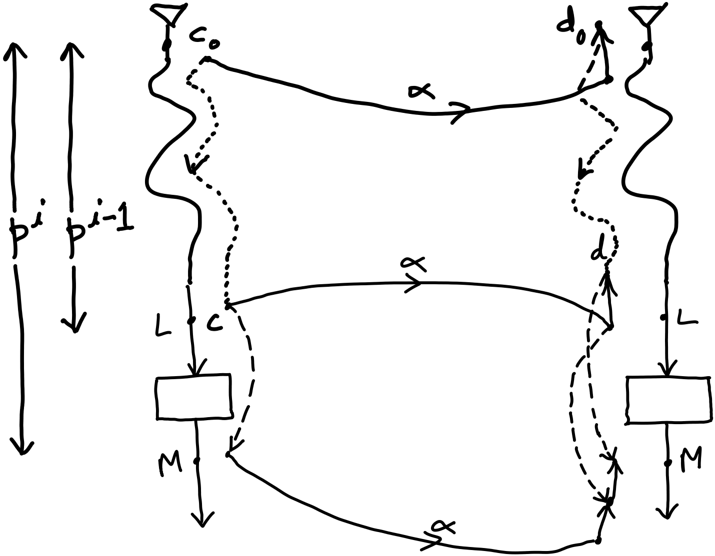


Lemma 4

Lemma 4: If $\alpha(c_0) \leq d_0$, then $\overline{\alpha}(\mathrm{JOP}_{\overline{C}}) \leq \mathrm{JOP}_{\overline{D}}$.
Proof:
Consider any path p in the CFG starting from the entry point E. We will first prove using induction that for any $i >= 0$, where p^i is the prefix of p containing i edges, $\alpha(f_{p^i,C}(c_0)) \leq f_{p^i,D}(d_0)$, where $f_{p^i,C}$ $(f_{p^i,D})$ is the composition of the concrete (abstract) transfer functions of the edges in p^i.

- Base case ($i = 0$): The property to prove reduces to $\alpha(c_0) \leq d_0$. Recall that this is a pre-requisite of this lemma.
- Inductive case: The inductive hypothesis is that $\alpha(f_{p^{i-1},C}(c_0)) \leq f_{p^{i-1},D}(d_0)$. Let the i^{th} edge of p be $L \rightarrow M$. Applying Lemma 3 on this edge we get $\alpha(f_{LM,C}(f_{p^{i-1},C}(c_0))) \leq f_{LM,D}(f_{p^{i-1},D}(d_0))$. This reduces to $\alpha(f_{p^i,C}(c_0)) \leq f_{p^i,D}(d_0)$. The inductive case is done.

Illustration of inductive case of Lemma 4



Lemma 4 – continued

- From the result proved above we derive
$$\alpha(c_p) \leq d_p \tag{1}$$
where p is any path, $c_p = f_{p,C}(c_0)$ and $d_p = f_{p,D}(d_0)$.
- Let N be any program point, and let
$P_N = \{p \mid p \text{ is a path from } I \text{ to } N\}$.

Lemma 4 – continued

- Property (1), plus the property of joins, gives us

$$\bigsqcup_{p \in P_N} (\alpha(c_p)) \leq \bigsqcup_{p \in P_N} (d_p) \tag{2}$$

$$= \mathrm{JOP}_{\overline{D}}[N] \tag{3}$$

- By Lemma 1 we have

$$\bigsqcup_{p \in P_N} (\alpha(c_p)) = \alpha(\bigsqcup_{p \in P_N} (c_p)) \tag{4}$$

$$= \alpha(\mathrm{JOP}_{\overline{C}}[N]) \tag{5}$$

- Using Properties 3 and 5, and extending over all program points N we get

$$\overline{\alpha}(\mathrm{JOP}_{\overline{C}}) \leq \mathrm{JOP}_{\overline{D}}$$

  We are done.

Proof of main theorem

Pick any $c_0 \in C$ and $d_0 \in D$ such that $\gamma(d_0) \geq c_0$.

$$\begin{array}{lll}
\alpha(\gamma(d_0)) & \geq \alpha(c_0) & \text{(monotonicity of } \alpha\text{)} \\
d_0 & \geq \alpha(c_0) & \text{(Galois connection property)} \\
\overline{\alpha}(\mathrm{JOP}_{\overline{C}}) & \leq \mathrm{JOP}_{\overline{D}} & \text{(Lemma 4)} \\
\overline{\gamma}(\overline{\alpha}(\mathrm{JOP}_{\overline{C}})) & \leq \overline{\gamma}(\mathrm{JOP}_{\overline{D}}) & \text{(monotonicity of } \gamma\text{)} \\
\mathrm{JOP}_{\overline{C}} & \leq \overline{\gamma}(\mathrm{JOP}_{\overline{D}}) & \text{(property of Galois connection)}
\end{array}$$