

Lattices and the Knaster-Tarski Theorem

Deepak D'Souza

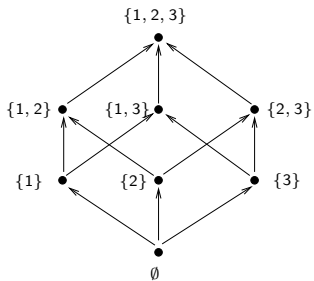
Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

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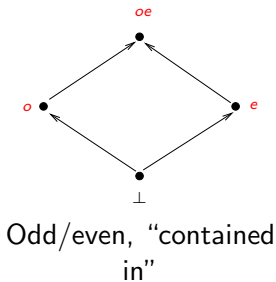
Outline

- 1 Why study lattices
- 2 Partial Orders
- 3 Lattices
- 4 Knaster-Tarski Theorem
- 5 Computing LFP

What a lattice looks like



Subsets of $\{1, 2, 3\}$,
"subset"



Odd/even, "contained
in"

Why study lattices in program analysis?

Why lattices?

- Natural way to obtain the “collecting state” at a point is to take union of states reached along each path leading to the point.
- With abstract states also we want a “union” or “join” over all paths (JOP).

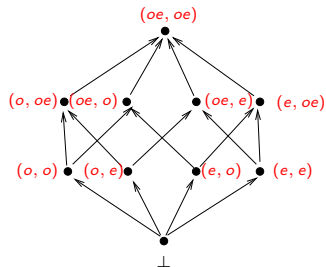
Why fixpoints?

- Guaranteed to safely approximate JOP (* Conditions apply).
- Easier to compute than JOP.
- Knaster-Tarski theorem tells us when a fixpoint exists and what it looks like.

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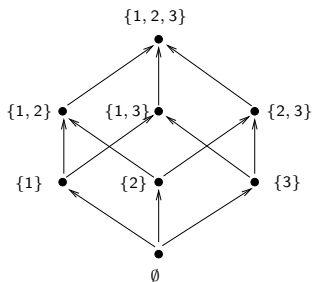
1: p := 5;
2: q := 2;
3: while (p > q) {
4:   p := p+1;
5:   q := q+2;
6: }
6: print p;

```

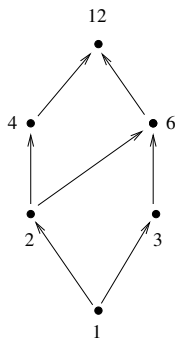


Partial Orders

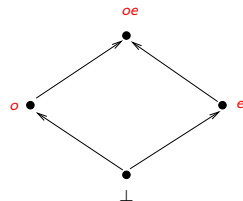
- Usual order (or **total** order) on numbers: $1 \leq 2 \leq 3$.
- Some domains are naturally “partially” ordered:



Subsets of $\{1, 2, 3\}$,
“subset”



Divisors of 12, “divides”



Odd/even, “contained
in”

Partial orders: definition

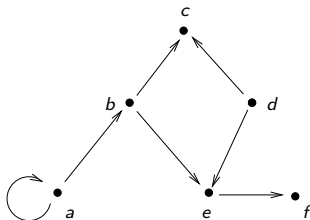
- A **partially ordered set** is a non-empty set D along with a partial order \leq on D . Thus \leq is a binary relation on D satisfying:
 - \leq is reflexive ($d \leq d$ for each $d \in D$)
 - \leq is transitive ($d \leq d'$ and $d' \leq d''$ implies $d \leq d''$)
 - \leq is anti-symmetric ($d \leq d'$ and $d' \leq d$ implies $d = d'$).

Binary relations as Graphs

We can view a binary relation on a set as a **directed graph**.
For example, the binary relation

$$\{(a, a), (a, b), (b, c), (b, e), (d, e), (d, c), (e, f)\}$$

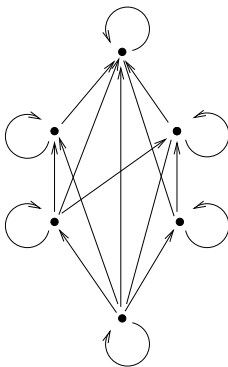
can be represented as the graph:



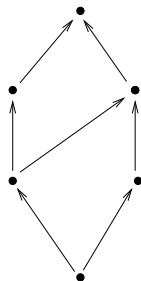
Partial Order as a graph

A **partial order** is then a special kind of directed graph:

- Reflexive = self-loop on each node
- Antisymmetric = no 2-length cycles
- Transitive = “transitivity” of edges.



Graph
representation

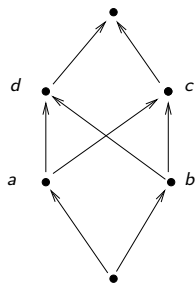


Hasse-diagram
representation

Upper bounds etc.

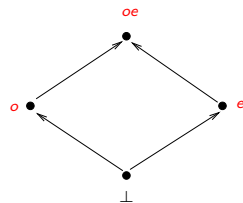
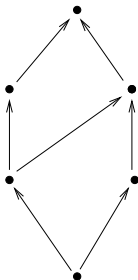
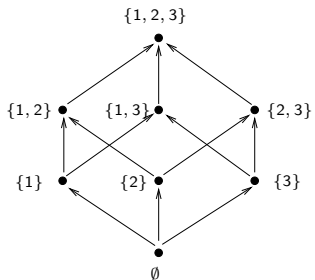
In a partially ordered set (D, \leq) :

- An element $u \in D$ is an **upper bound** of a set of elements $X \subseteq D$, if $x \leq u$ for all $x \in X$.
- u is the **least upper bound** (or **lub** or **join**) of X if u is an upper bound for X , and for every upper bound y of X , we have $u \leq y$. We write $u = \bigsqcup X$.
- Similarly, $v = \bigsqcap X$ (v is the **greatest lower bound** or **glb** or **meet** of X).



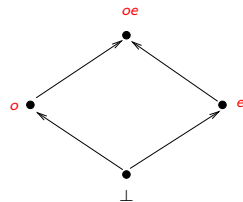
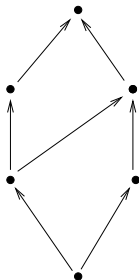
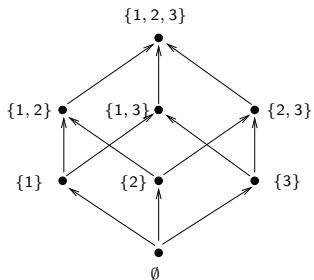
Lattices

- A **lattice** is a partially order set in which every pair of elements has an lub and a glb.
- A **complete** lattice is a lattice in which every **subset** of elements has a lub and a glb.

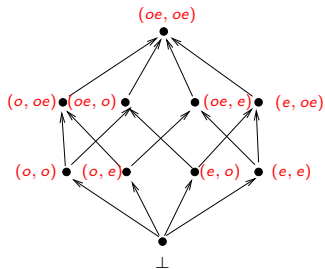


Lattices

- A **lattice** is a partially order set in which every pair of elements has an lub and a glb.
- A **complete** lattice is a lattice in which every **subset** of elements has a lub and a glb.
- Examples below are all complete lattices.



More lattices

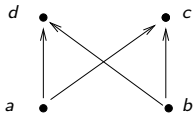


Exercise

- 1 Example of a partial order that is not a lattice?

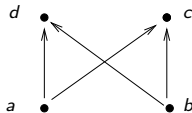
Exercise

- ① Example of a partial order that is not a lattice?



Exercise

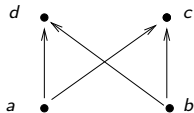
- ① Example of a partial order that is not a lattice?



- ② “Simplest” example of a partial order that is not a lattice?

Exercise

- 1 Example of a partial order that is not a lattice?

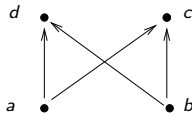


- 2 “Simplest” example of a partial order that is not a lattice?



Exercise

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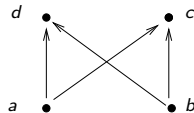
- 2 “Simplest” example of a partial order that is not a lattice?



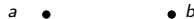
- 3 Example of a lattice which is **not** complete?

Exercise

- 1 Example of a partial order that is not a lattice?



- 2 “Simplest” example of a partial order that is not a lattice?



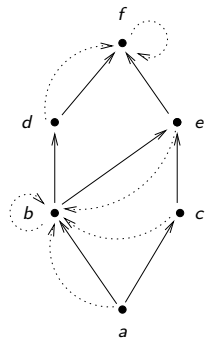
- 3 Example of a lattice which is **not** complete?



Monotonic functions

Let (D, \leq) be a partially ordered set.

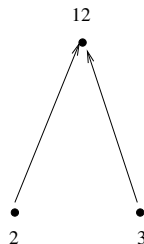
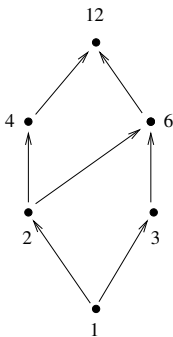
- A function $f : D \rightarrow D$ is **monotonic** or **order-preserving** if whenever $x \leq y$ we have $f(x) \leq f(y)$.



Partial order induced by a subset of elements

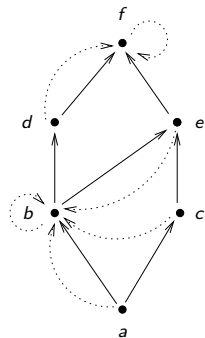
Let (D, \leq) be a partially ordered set, and X be a non-empty subset of D . Then X induces a partial order, which we call the partial order *induced by X* in (D, \leq) , and defined to be $(X, \leq \cap (X \times X))$.

Example: the partial order induced by the set of elements $X = \{2, 3, 12\}$.



Fixpoints

- A **fixpoint** of a function $f : D \rightarrow D$ is an element $x \in D$ such that $f(x) = x$.
- A **pre-fixpoint** of f is an element x such that $x \leq f(x)$.
- A **post-fixpoint** of f is an element x such that $f(x) \leq x$.



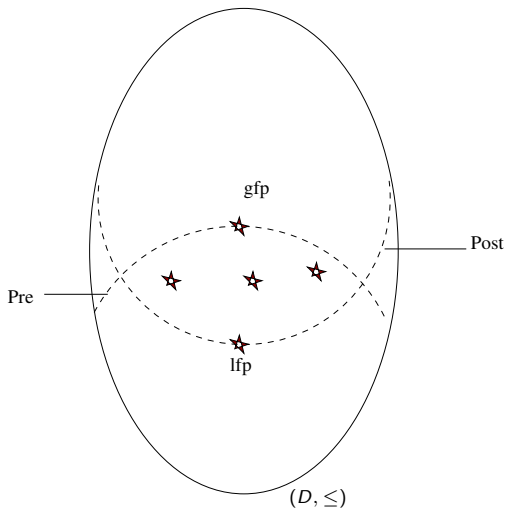
Knaster-Tarski Fixpoint Theorem

Theorem (Knaster-Tarski)

Let (D, \leq) be a complete lattice, and $f : D \rightarrow D$ a monotonic function on (D, \leq) . Then:

- (a) f has at least one fixpoint.
- (b) f has a **least fixpoint** which coincides with the glb of the set of postfixpoints of f , and a **greatest fixpoint** which coincides with the lub of the prefixpoints of f .
- (c) The set of fixpoints P of f itself forms a complete lattice under \leq .

Fixpoints of f

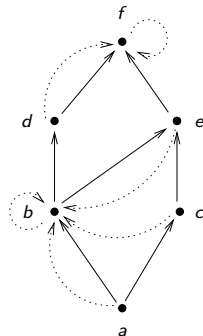


Stars denote fixpoints.

Exercise

Consider the complete lattice and monotone function f below.

- 1 Mark the pre-fixpoints with up-triangles (\triangle).
- 2 What is the lub of the pre-fixpoints?
- 3 Mark post-fixpoints with down-triangles (∇).
- 4 Fixpoints are the stars (\boxtimes).

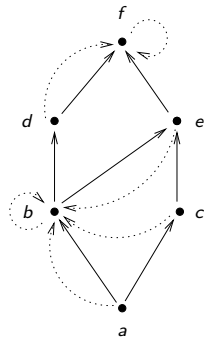


Check that claims of K-T theorem hold here.

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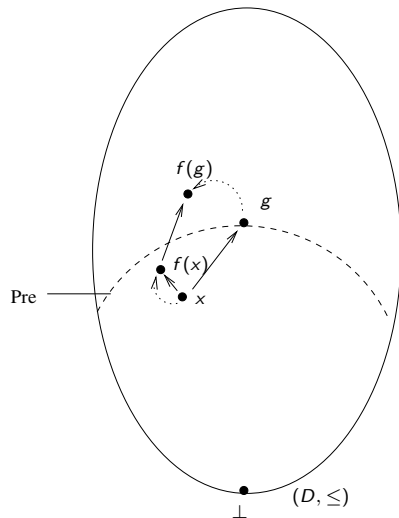
Check that claims of K-T theorem hold here.

If you drop completeness of lattice or monotonicity of f , does K-T still hold?

Proof of Knaster-Tarski theorem

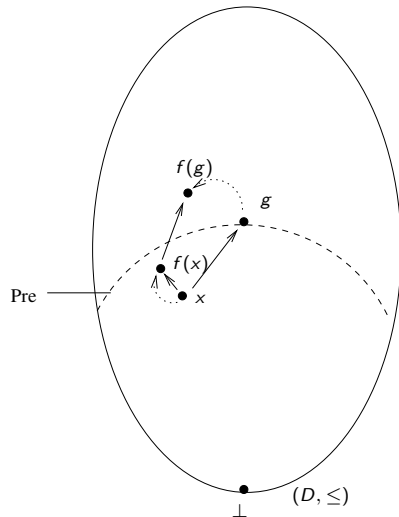
- (a) $g = \bigsqcup Pre$ is a fixpoint of f .
- (b) g is the greatest fixpoint of f .
- (c) Similarly $l = \bigsqcap Post$ is the least fixpoint of f .
- (d) Let P be the set of fixpoints of f . Then (P, \leq) is a *complete* lattice.

Proof of K-T theorem: (a)



Proof of K-T theorem: (a)

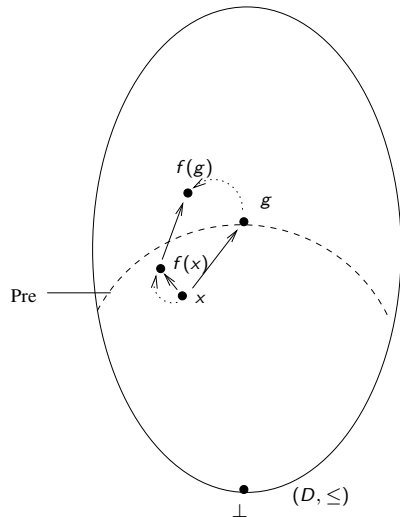
To show $g = f(g)$:



Proof of K-T theorem: (a)

To show $g = f(g)$:

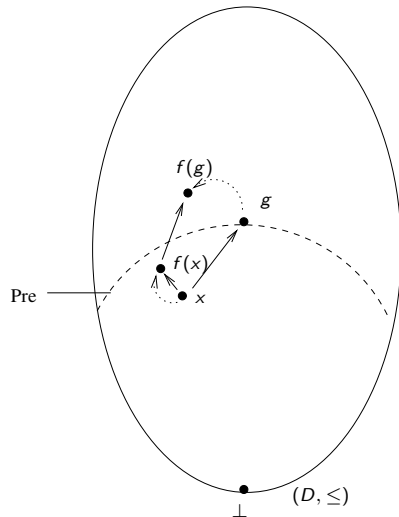
- $g \leq f(g)$



Proof of K-T theorem: (a)

To show $g = f(g)$:

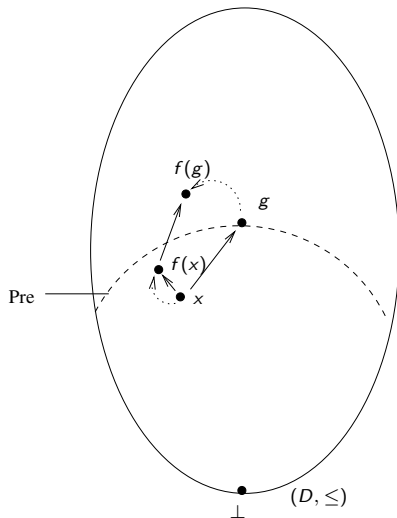
- $g \leq f(g)$
 - Since $f(g)$ can be seen to be u.b. of Pre .



Proof of K-T theorem: (a)

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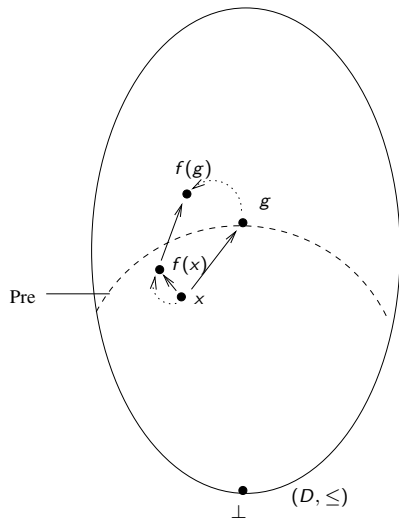
- $g \leq f(g)$
 - Since $f(g)$ can be seen to be u.b. of Pre .
- $f(g) \leq g$



Proof of K-T theorem: (a)

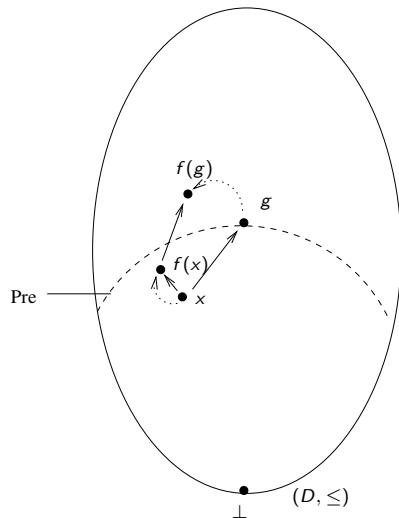
To show $g = f(g)$:

- $g \leq f(g)$
 - Since $f(g)$ can be seen to be u.b. of Pre .
- $f(g) \leq g$
 - Since $f(g)$ can be seen to be prefixpoint of f .



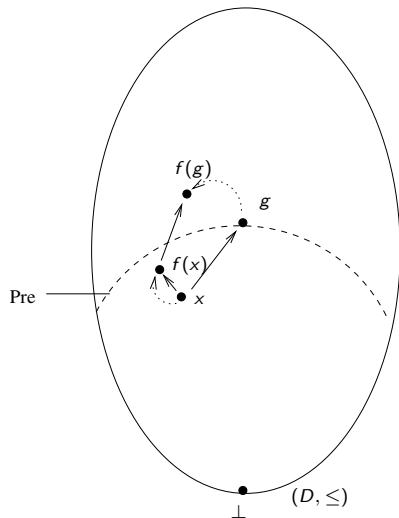
Proof of K-T theorem: (b)

g is the greatest fixpoint of f .



Proof of K-T theorem: (b)

g is the greatest fixpoint of f .
Any other fixpoint is also a
pre-fixpoint of f , and hence g
must dominate it.



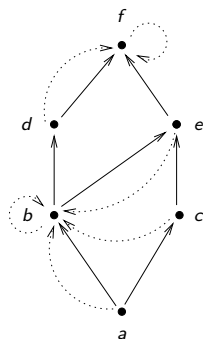
Exercise: intervals and closure

Let (D, \leq) be a partial order, and let $f : D \rightarrow D$.

- Let $a, b \in D$. The **interval** from a to b , written $[a, b]$, is the set $\{d \mid a \leq d \leq b\}$.
- A subset $X \subseteq D$ is said to be **closed** wrt to f , if $f(x) \in X$ for each $x \in X$.

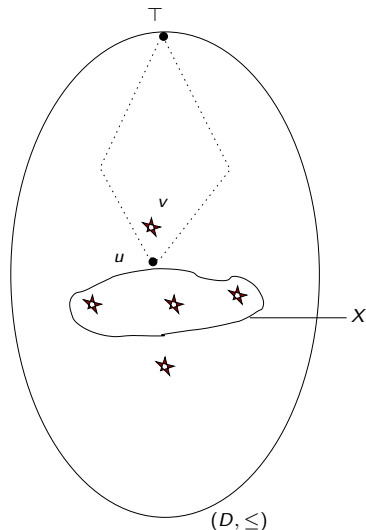
Exercise: Let (D, \leq) be a partial order with a \top element, and let $f : D \rightarrow D$ be a monotone function on D .

- Show that an interval in D need *not* be closed wrt f .
- Let $u \in D$ be the lub of a set X of fixpoints of f . Prove that the interval $[u, \top]$ is closed wrt f .



Proof of K-T theorem: (d)

- (P, \leq) is also a partial order.
- (P, \leq) is a complete lattice
 - Let $X \subseteq P$. We show there is an lub of X in (P, \leq) .
 - Let u be lub of X in (D, \leq) .
 - Consider “interval” $I = [u, \top] = \{x \in D \mid u \leq x\}$. (I, \leq) is also a complete lattice.
 - $f : I \rightarrow I$ as well, and monotonic on (I, \leq) .
 - Hence by part (a) f has a least fixpoint in I , say v .
 - Argue that v is the lub of X in (P, \leq) .



Chains in partial orders

- A **chain** in a partial order (D, \leq) is a totally ordered subset of D .
- An **ascending chain** is an infinite sequence of elements of D of the form:

$$d_0 \leq d_1 \leq d_2 \leq \dots .$$

- An ascending chain $\langle d_i \rangle$ is **eventually stable** if there exists n_0 such that $d_i = d_{n_0}$ for each $i \geq n_0$.
- (D, \leq) has **finite** height if each chain in it is finite.
- (D, \leq) has **bounded** height if there exists k such that each chain in D has height at most k (i.e. number of elements in each chain is at most $k + 1$.)

Monotonicity, distributivity, and continuity

- f is monotone:

$$x \leq y \implies f(x) \leq f(y).$$

- f is distributive:

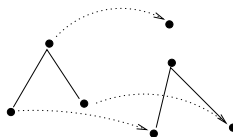
$$f(x \sqcup y) = f(x) \sqcup f(y).$$

- f is continuous: For any asc chain X :

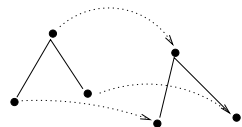
$$f(\bigsqcup X) = \bigsqcup (f(X)).$$

- f is inf distributive: For any $X \subseteq D$:

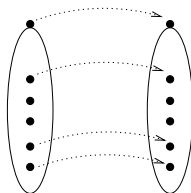
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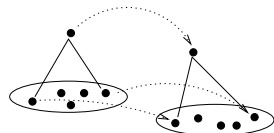
Monotonic



Distributive



Continuous



Inf-Distributive

Characterising lfp's and gfp's of a function f in a complete lattice (D, \leq)

- f is **continuous** if for any ascending chain X in D ,

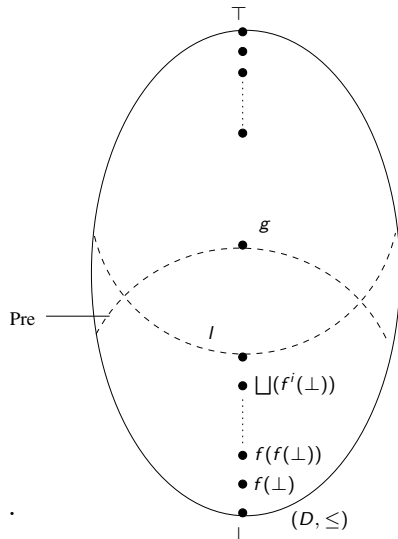
$$f(\bigsqcup X) = \bigsqcup (f(X)).$$

- If f is **continuous** then

$$\text{lfp}(f) = \bigsqcup_{i \geq 0} (f^i(\perp)).$$

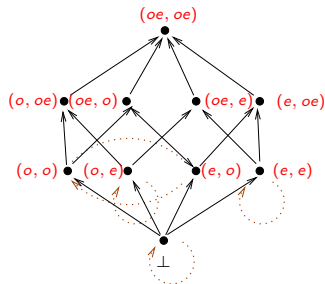
- If f is **monotonic** and (D, \leq) has **finite height** then we can compute $\text{lfp}(f)$ by finding the stable value of the asc. chain

$$\perp \leq f(\perp) \leq f^2(\perp) \leq f^3(\perp) \leq \dots$$



Exercise

Consider the statement “ $p := p + q$ ”. Show the transfer function of this statement in the parity lattice below.



Is it monotonic/distributive/continuous?