

Floyd-Hoare Style Program Verification

Deepak D'Souza

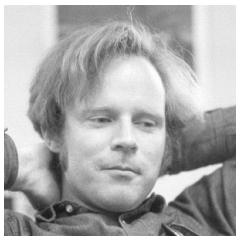
Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

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Outline of these lectures

- 1 Overview
- 2 Hoare Triples
- 3 Proving assertions
- 4 Inductive Annotation
- 5 Hoare Logic
- 6 Weakest Preconditions
- 7 Completeness

Floyd-Hoare Style of Program Verification



Robert W. Floyd: “Assigning meanings to programs” *Proceedings of the American Mathematical Society Symposia on Applied Mathematics* (1967)



C A R Hoare: “An axiomatic basis for computer programming”, *Communications of the ACM* (1969).

Floyd-Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that “Whenever program P is started in a state satisfying condition A , if it terminates, it will terminate in a state satisfying condition B .”
- Example assertion: $\{n \geq 0\} P \{a = n + m\}$, where P is the program:

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

- Inductive Annotation (“consistent interpretation”) (due to Floyd)
- A proof system (due to Hoare) for proving such assertions.
- A way of reasoning about such assertions using the notion of “Weakest Preconditions” (due to Dijkstra).

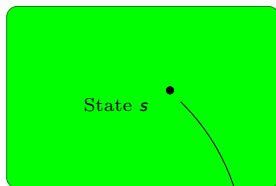
A simple programming language

- skip
- $x := e$ (assignment)
- if b then S else T (if-then-else)
- while b do S (while)
- $S ; T$ (sequencing)

Programs as State Transformers

View program P as a **partial** map $[P] : \text{States} \rightarrow \text{States}$. (Assume that $\text{States} = \text{Var} \rightarrow \mathbb{Z}$)

All States



$\langle x \mapsto 2, y \mapsto 10, z \mapsto 3 \rangle$

P



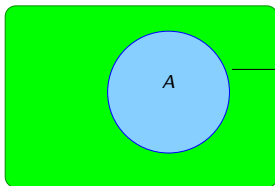
$y := y + 1;$

$z := x + y$

$\langle x \mapsto 2, y \mapsto 11, z \mapsto 13 \rangle$

Predicates on States

All States



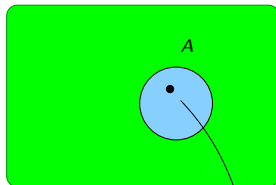
States satisfying
Predicate A

Eg. $0 \leq x \wedge x < y$

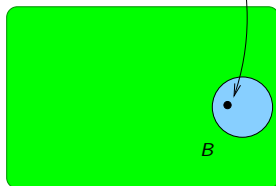
Assertion of “Partial Correctness” $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that “Whenever program P is started in a state satisfying condition A , either it will not terminate, or it will terminate in a state satisfying condition B .”

All States



$$\{10 \leq y\}$$

 P 

$$y := y + 1;$$

$$z := x + y$$

$$\{x < z\}$$

Mathematical meaning of a Hoare triple

- View program P as a **relation** on States (allows non-termination as well as non-determinism)

$$[P] \subseteq \text{States} \times \text{States}.$$

Here $(s, t) \in [P]$ iff it is possible to start P in the state s and terminate in state t .

- As usual, elements of States are maps from variables to integers.
- $[P]$ is possibly non-deterministic, in case we also want to model non-deterministic assignment etc.
- Then the Hoare triple $\{A\} P \{B\}$ is true iff for all states s and t : whenever $s \models A$ and $(s, t) \in [P]$, then $t \models B$.
- In other words $Post_{[P]}([A]) \subseteq [B]$.

Example programs and pre/post conditions

```
// Pre: true
```

```
if (a <= b)
```

```
    min := a;
```

```
else
```

```
    min := b;
```

```
// Post: min <= a && min <= b
```

```
// Pre: 0 <= n
```

```
int a := m;
```

```
int x := 0;
```

```
while (x < n) {
```

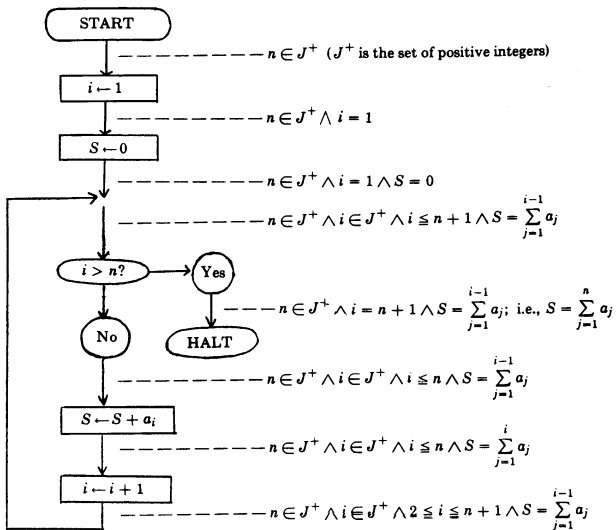
```
    a := a + 1;
```

```
    x := x + 1;
```

```
}
```

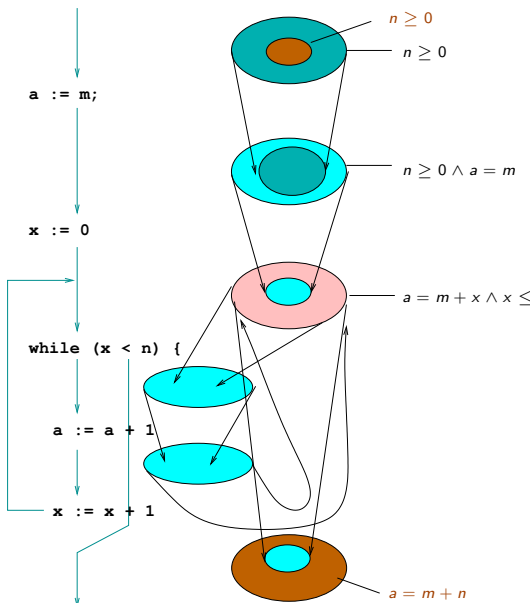
```
// Post: a = m + n
```

Floyd's style of proof: Inductive Annotation



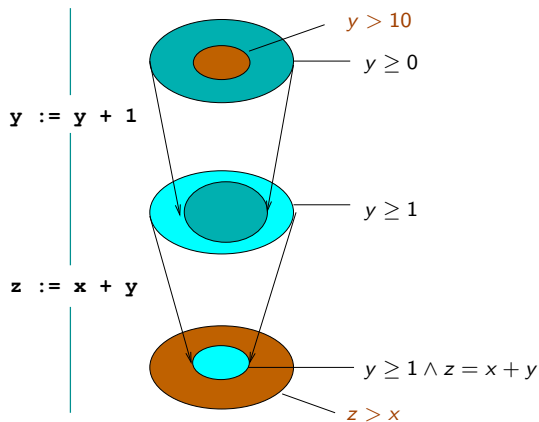
Inductive annotation based proof of a pre/post specification

- Annotate each program point i with a predicate A_i
- Successive annotations must be **inductive**:
 $A_i \wedge [S_i] \implies A'_{i+1}$.
- Annotation is **adequate**:
 $Pre \implies A_1$ and
 $A_n \implies Post$.
- Adequate annotation constitutes a proof of $\{Pre\} Prog \{Post\}$.



Example of inductive annotation

To prove: $\{y > 10\} y := y+1; z := x+y \{z > x\}$



Exercise

Prove using Floyd's inductive annotation:

$$\{n \geq 1\} P \{a = n!\},$$

where P is the program:

```
x := n;  
a := 1;  
while (x ≥ 1) {  
    a := a * x;  
    x := x - 1  
}
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n - 1) \times \cdots \times 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

Exercise

Prove using Floyd's inductive annotation:

$$\{n \geq 1\} P \{a = n!\},$$

where P is the program:

```
S1: x := n;
S2: a := 1;
S3: while (x ≥ 1) {
S4:     a := a * x;
S5:     x := x - 1
    }
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n - 1) \times \cdots \times 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

Solution

Need a loop invariant P satisfying:

- 1 $\{n \geq 1\} S1; S2 \{P\}$
- 2 $\{P \wedge (x \geq 1)\} S4; S5 \{P\}$
- 3 $(P \wedge \neg(x \geq 1)) \implies (a = n!)$

A potential P : $(x \geq 0) \wedge (a \times x! = n!)$.

Hoare's view: Program as a composition of statements

```
int a := m;  
int x := 0;  
while (x < n) {  
    a := a + 1;  
    x := x + 1;  
}
```

Hoare's view: Program as a composition of statements

```
int a := m;  
int x := 0;  
while (x < n) {  
    a := a + 1;  
    x := x + 1;  
}
```

```
S1: int a := m;  
S2: int x := 0;  
S3: while (x < n) {  
    a := a + 1;  
    x := x + 1;  
}
```

Program is S1;S2;S3

Proof rules of Hoare Logic

Axiom of Valid formulas:

$$\frac{}{A}$$

provided " $\models A$ " (i.e. A is a valid logical formula, eg. $x > 10 \implies x > 0$).

Skip:

$$\frac{}{\{A\} \text{ skip } \{A\}}$$

Assignment

$$\frac{}{\{A[e/x]\} x := e \{A\}}$$

Proof rules of Hoare Logic

If-then-else:

$$\frac{\{P \wedge b\} S \{Q\}, \{P \wedge \neg b\} T \{Q\}}{\{P\} \text{if } b \text{ then } S \text{ else } T \{Q\}}$$

While (here P is called a *loop invariant*)

$$\frac{\{P \wedge b\} S \{P\}}{\{P\} \text{while } b \text{ do } S \{P \wedge \neg b\}}$$

Sequencing:

$$\frac{\{P\} S \{Q\}, \{Q\} T \{R\}}{\{P\} S; T \{R\}}$$

Weakening:

$$\frac{P \implies Q, \{Q\} S \{R\}, R \implies T}{\{P\} S \{T\}}$$

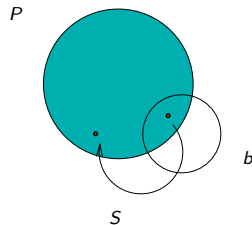
Loop invariants

A predicate P is a **loop invariant** for the while loop:

```
while (b) {  
  S  
}
```

if $\{P \wedge b\} S \{P\}$ holds.

If P is a loop invariant then we can infer that:

$$\{P\} \text{ while } b \text{ do } S \{P \wedge \neg b\}$$


Some examples to work on

Use the rules of Hoare logic to prove the following assertions:

- 1 $\{x \geq 3\} \ x := x + 2 \ \{x \geq 5\}$
- 2 $\{(y \leq 0) \wedge (-1 < x)\} \ \text{if } (y < 0) \ \text{then } x := x + 1 \ \text{else } x := y$
 $\{0 \leq x\}$
- 3 $\{x \leq 0\} \ \text{while } (x \leq 5) \ \text{do } x := x + 1 \ \{x = 6\}$

Example proof using Hoare Logic

- 1 $\{n \geq 0\}$ S1 $\{n \geq 0 \wedge a = m\}$
- 2 $\{n \geq 0 \wedge a = m\}$ S2 $\{n \geq 0 \wedge a = m \wedge x = 0\}$
- 3 $\{a = m + x \wedge 0 \leq x \leq n \wedge x < n\}$ S4;S5
 $\{a = m + x \wedge 0 \leq x \leq n\}$
- 4 $\{a = m + x \wedge 0 \leq x \leq n\}$ S3
 $\{a = m + x \wedge 0 \leq x \leq n \wedge x \geq n\}$
- 5 $\{n \geq 0\}$ S1;S2 $\{n \geq 0 \wedge a = m \wedge x = 0\}$ (From Seq rule, 1 and 2)
- 6 $(n \geq 0 \wedge a = m \wedge x = 0) \implies (a = m + x \wedge 0 \leq x \leq n)$ (From logical axiom)
- 7 $\{n \geq 0\}$ S1;S2 $\{a = m + x \wedge 0 \leq x \leq n\}$ (From Weakening rule, 5 and 6)
- 8 $\{n \geq 0\}$ (S1;S2);S3
 $\{a = m + x \wedge 0 \leq x \leq n \wedge x \geq n\}$ (From Seq rule, 7, 4)
- 9 $(a = m + x \wedge 0 \leq x \leq n \wedge x \geq n) \implies (a = m + n)$
- 10 $\{n \geq 0\}$ (S1;S2);S3 $\{a = m + n\}$ (From Weakening rule, 8, 9).

```
// pre: n >= 0
S1: int a := m;
S2: int x := 0;
S3: while (x < n) {
S4:   a := a + 1;
S5:   x := x + 1;
}
// post: a = m + n
```

Program is S1;S2;S3

Exercise

Prove using Hoare logic:

$$\{n \geq 1\} P \{a = n!\},$$

where P is the program:

```
x := n;  
a := 1;  
while (x ≥ 1) {  
  a := a * x;  
  x := x - 1  
}
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n - 1) \times \dots \times 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

Exercise

Prove using Hoare logic:

$$\{n \geq 1\} P \{a = n!\},$$

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S1: x := n;
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S4:     a := a * x;
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      }
```

Assume that factorial is defined as follows:

$$n! = \begin{cases} n \times (n - 1) \times \dots \times 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

Solution

Need a loop invariant P satisfying:

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A potential P : $(x \geq 0) \wedge (a \times x! = n!)$.

Soundness and Completeness

Soundness: If our proof system proves $\{A\} P \{B\}$ then $\{A\} P \{B\}$ indeed holds.

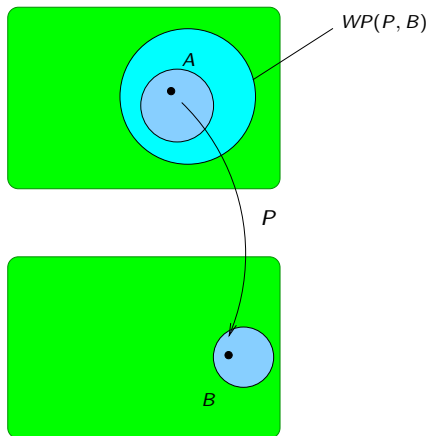
Completeness: If $\{A\} P \{B\}$ is true then our proof system can prove $\{A\} P \{B\}$.

- Floyd proof style is sound since any execution must stay within the annotations. Complete because the “collecting” set is an adequate inductive annotation for any program and any true pre/post condition.
- Hoare logic is sound, essentially because the individual rules can be seen to be sound.
- For completeness of Hoare logic, we need weakest preconditions.

Weakest Precondition $WP(P, B)$

$WP(P, B)$ is “a predicate that describes the exact set of states s such that when program P is started in s , if it terminates it will terminate in a state satisfying condition B .”

All States



$$\{10 < y\}$$

$y := y + 1;$

$z := x + y;$

$$\{x < z\}$$

Exercise: Give “weakest” preconditions

$$\textcircled{1} \{? \} x := x + 2 \{x \geq 5\}$$

Exercise: Give “weakest” preconditions

1 $\{x \geq 3\} x := x + 2 \{x \geq 5\}$

2 $\{?\} \text{if } (y < 0) \text{ then } x := x+1 \text{ else } x := y \{x > 0\}$

Exercise: Give “weakest” preconditions

1 $\{ x \geq 3 \} x := x + 2 \{ x \geq 5 \}$

2 $\{ (y < 0 \wedge x > -1) \vee (y > 0) \}$
if $(y < 0)$ then $x := x+1$ else $x := y$
 $\{ x > 0 \}$

3 $\{ ? \} \text{ while } (x \leq 5) \text{ do } x := x+1 \{ x = 6 \}$

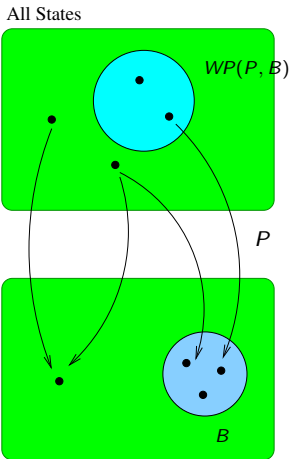
Exercise: Give “weakest” preconditions

1 $\{x \geq 3\} x := x + 2 \{x \geq 5\}$

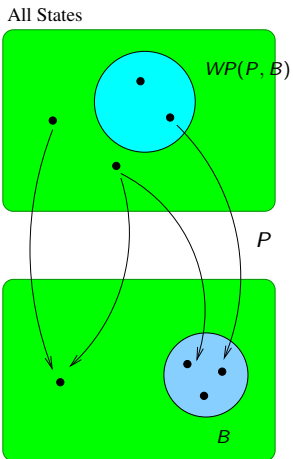
2 $\{ (y < 0 \wedge x > -1) \vee (y > 0) \}$
if $(y < 0)$ then $x := x+1$ else $x := y$
 $\{x > 0\}$

3 $\{x \leq 6\}$ while $(x \leq 5)$ do $x := x+1$ $\{x = 6\}$

Exercise: How will you define $WP(P, B)$?



Exercise: How will you define $WP(P, B)$?



$$WP(P, B) = \{s \mid \forall t : (s, t) \in [P] \text{ we have } t \models B\}$$

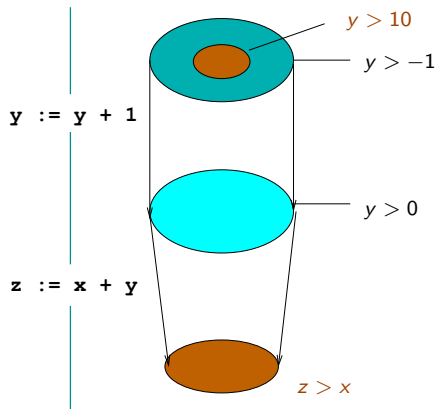
Using weakest preconditions to partially automate inductive proofs

Weakest preconditions give us a way to:

- Check inductiveness of annotations

$$\{A_i\} S_i \{A_{i+1}\} \text{ iff } A_i \implies WP(S_i, A_{i+1})$$

- Reduce the amount of user-annotation needed
 - Programs **without loops** don't need any user-annotation
 - For programs with loops, user only needs to provide **loop invariants**

Checking $\{A\} P \{B\}$ using WP

Check that

$$(y > 10) \implies WP(P, z > x)$$

WP rules

- Hoare's rules for **skip**, **assignment**, and **if-then-else** are already WP rules.
- For **Sequencing**:

$$WP(S; T, B) = WP(S, WP(T, B)).$$

Weakest Precondition for `while` statements

- We can “approximate” $WP(\text{while } b \text{ do } c)$.
- $WP_i(w, A)$ = the set of states from which the body c of the loop is either entered more than i times or we exit the loop in a state satisfying A .
- WP_i defined inductively as follows:

$$\begin{aligned} WP_0 &= b \vee A \\ WP_{i+1} &= (\neg b \wedge A) \vee (b \wedge WP(c, WP_i)) \end{aligned}$$

- Then $WP(w, A)$ can be shown to be the “limit” or least upper bound of the chain $WP_0(w, A), WP_1(w, A), \dots$ in a suitably defined lattice (here the join operation is “And” or intersection).

Illustration of WP_i through example

Consider the program w below:

```
while ( $x \geq 10$ ) do  
   $x := x - 1$ 
```

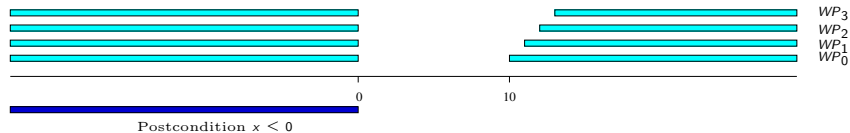
- What is the weakest precondition of w with respect to the postcondition $(x \leq 0)$?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$, \dots

Illustration of WP_i through example

Consider the program w below:

```
while ( $x \geq 10$ ) do
   $x := x - 1$ 
```

- What is the weakest precondition of w with respect to the postcondition ($x \leq 0$)?
- Compute $WP_0(w, (x \leq 0))$, $WP_1(w, (x \leq 0))$,



Reducing verification to satisfiability: Generating Verification Conditions

To check:

$\{y > 10\}$

$y := y + 1;$

$z := x + y;$

$\{x < z\}$

Use the weakest precondition rules to generate the **verification condition**:

$$(y > 10) \implies (y > -1).$$

Check the verification condition by asking a theorem prover / SMT solver if the formula

$$(y > 10) \wedge \neg(y > -1).$$

is satisfiable.

What about while loops?

Pre: $0 \leq n$

```
int a := m;
int x := 0;
while (x < n) {
  a := a + 1;
  x := x + 1;
}
```

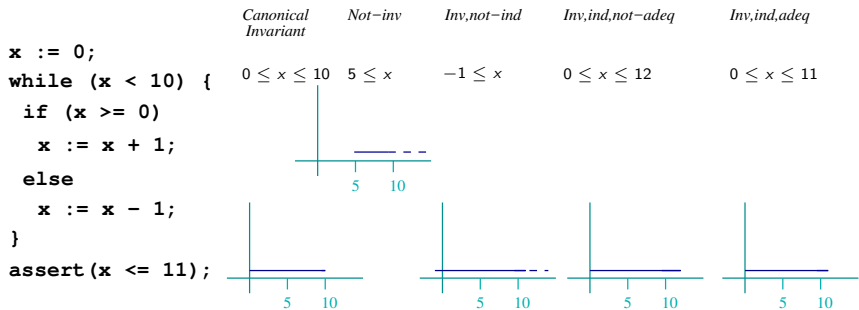
Post: $a = m + n$

Adequate loop invariant

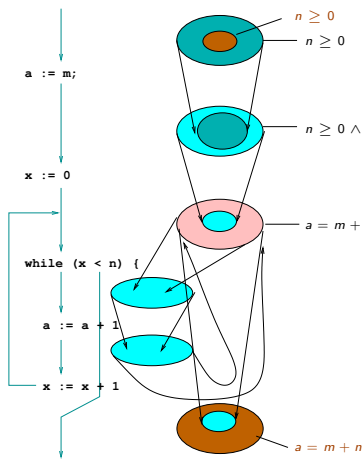
What is a “good” loop invariant for this program?

```
x := 0;
while (x < 10) {
  if (x >= 0)
    x := x + 1;
  else
    x := x - 1;
}
assert(x <= 11);
```

Adequate loop invariant



Adequate loop invariant



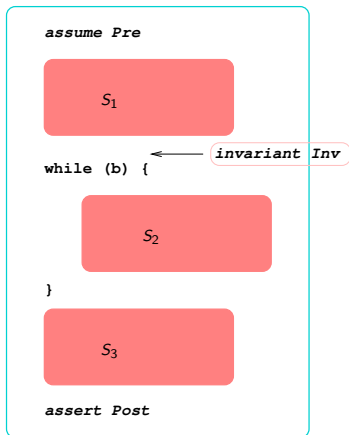
An **adequate** loop invariant needs to satisfy:

- $\{n \geq 0\} a := m; x := 0$
 $\{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x < n\} a := a + 1;$
 $x := x + 1 \{a = m + x \wedge x \leq n\}$.
- $\{a = m + x \wedge x \leq n \wedge x \geq n\}$ skip
 $\{a = m + n\}$.

Verification conditions are generated accordingly.

Note that $a = m + x$ is **not** an adequate loop invariant.

Generating Verification Conditions for a program



The following VCs are generated:

- $Pre \wedge [S_1] \implies Inv'$
Or: $Pre \implies WP(S_1, Inv)$
- $Inv \wedge b \wedge [S_2] \implies Inv'$
Or: $(Inv \wedge b) \implies WP(S_2, Inv)$
- $Inv \wedge \neg b \wedge [S_3] \implies Post'$
Or: $Inv \wedge \neg b \implies WP(S_3, Post)$

Relative completeness of Hoare logic

Theorem (Cook 1974)

Hoare logic is complete provided the assertion language L can express the WP for any program P and post-condition B .

Proof uses WP predicates and proceeds by induction on the structure of the program P .

- Suppose $\{A\} \text{ skip } \{B\}$ holds. Then it must be the case that $A \implies B$ is true. By Skip rule we know that $\{B\} \text{ skip } \{B\}$. Hence by Weakening rule, we get that $\{A\} \text{ skip } \{B\}$ holds.
- Suppose $\{A\} x := e \{B\}$ holds. Then it must be the case that $A \implies B[e/x]$. By Assignment rule we know that $\{B[e/x]\} x := e \{B\}$ is true. Hence by Weakening rule, we get that $\{A\} x := e \{B\}$ holds.
- Similarly for sequencing $S;T$.
- Similarly for if-then-else.

Relative completeness of Hoare logic

- Suppose $\{A\} \text{ while } b \text{ do } S \{B\}$ holds. Let $P = WP(\text{while } b \text{ do } S, B)$. Then it is not difficult to check that P is a loop invariant for the while statement. I.e. $\{P \wedge b\} S \{P\}$ is true. By induction hypothesis, this triple must be provable in Hoare logic. Hence we can conclude using the While rule, that $\{P\} \text{ while } b \text{ do } S \{P \wedge \neg b\}$. But since P was a valid precondition, it follows that $(P \wedge \neg b) \implies B$. For the same reason, we should have $A \implies P$. By the weakening rule, we have a proof of $\{A\} \text{ while } b \text{ do } S \{B\}$.

Conclusion

- Features of this Floyd-Hoare style of verification:
 - Tries to find a proof in the form of an **inductive annotation**.
 - A Floyd-style proof can be used to obtain a Hoare-style proof; and vice-versa.
 - Reduces verification (given key annotations) to checking satisfiability of a logical formula (VCs).
 - Is flexible about predicates, logic used (for example can add quantifiers to reason about arrays).
- Main challenge is the need for user annotation (adequate loop invariants).
- Can be increasingly automated (using learning techniques).