Kildall's algorithm for over-approximate JOP

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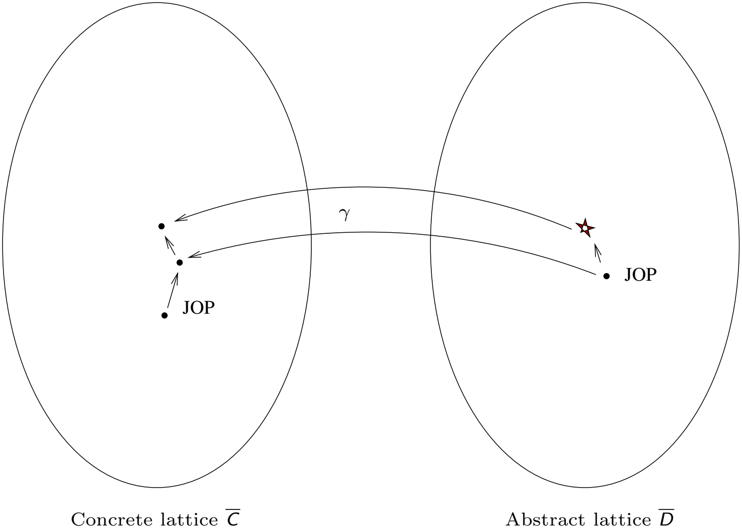
Why over-approximation of JOP in abstract lattice is useful



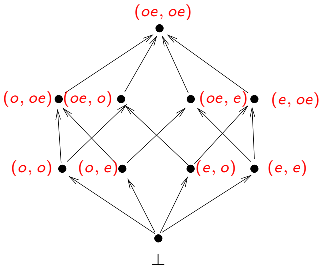
Kildall's algorithm to compute over-approximation of JOP

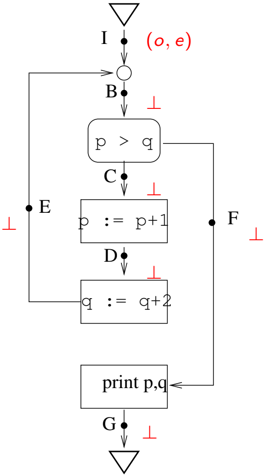
Input: An instance (P, d_0) of a monotone data-flow framework $((D, \leq), F)$.
Output: For each program point N in P, a data-value d_N such that $\mathrm{JOP}_N^{d_0} \leq d_N$.

- Initialize data value at each program point to $\perp$, entry point to d_0.
- Mark all points.
- Repeat while there is a marked point:
  - Choose a marked point M with value d_M, unmark it, and "propagate" it to successor points (i.e. for each successor N, replace value at N by $f_{MN}(d_M) \sqcup d_N$).
  - Mark successor point if old value was marked, or new value strictly dominates than old value.
- Return data values at each point as over-approx of JOP.

Kildall's algo on parity interpretation example

Underlying lattice





Kildall's algo on parity interpretation example

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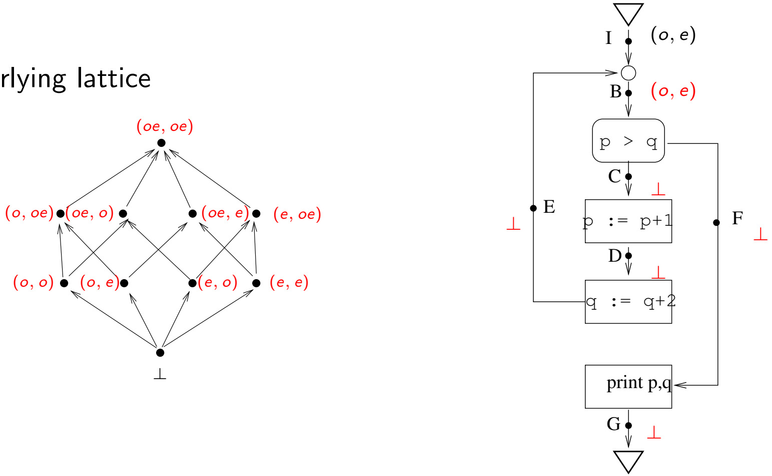

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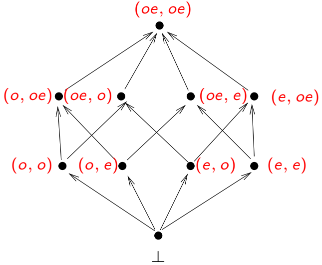

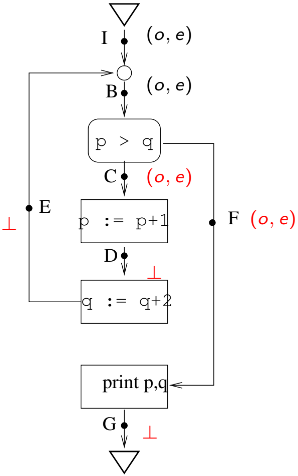

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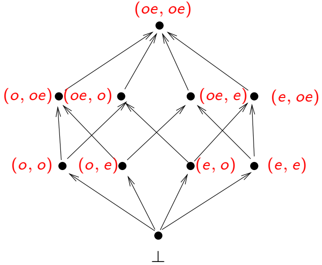

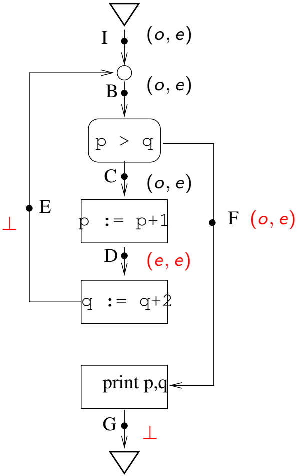

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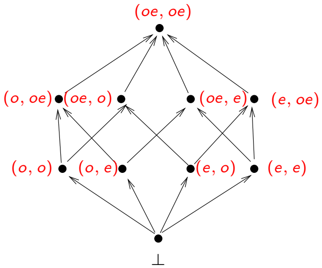
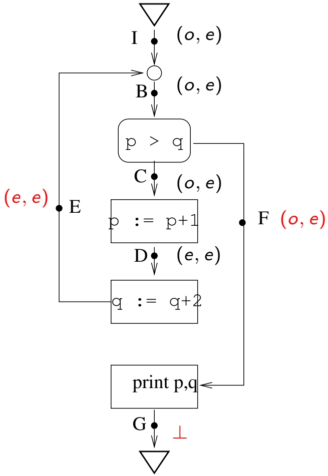
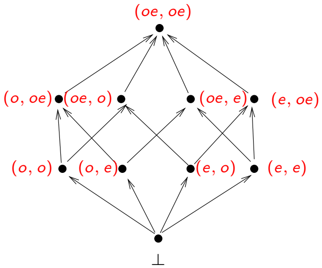
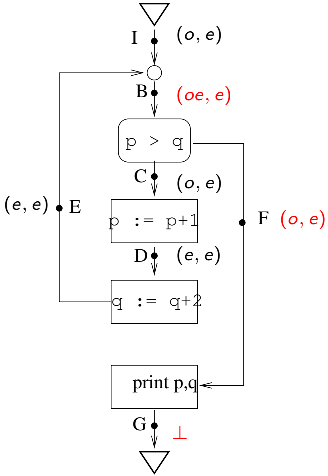




Kildall's algo on parity interpretation example

Underlying lattice





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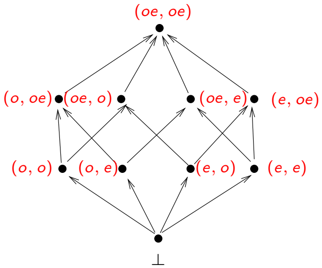

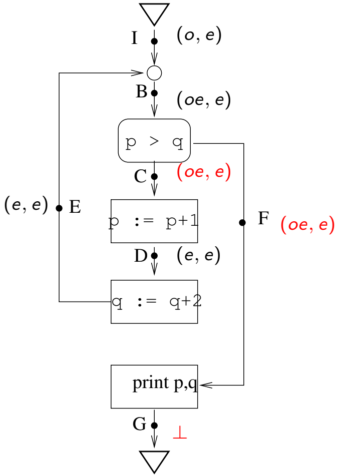

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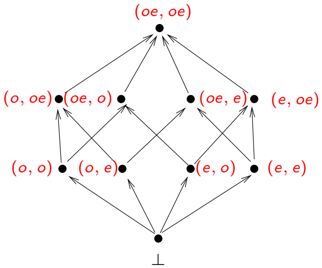

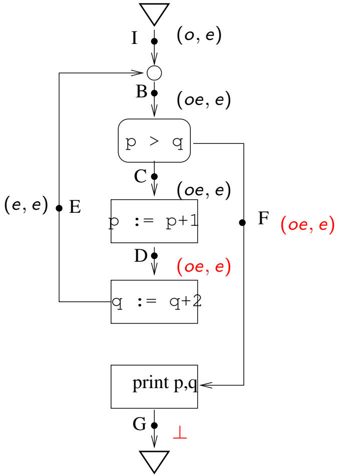

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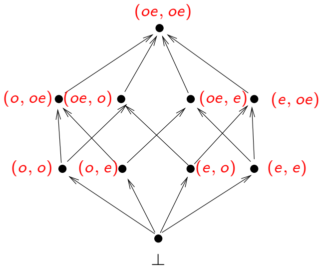

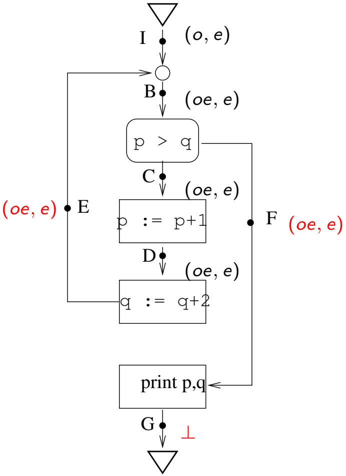

Kildall's algo on parity interpretation example

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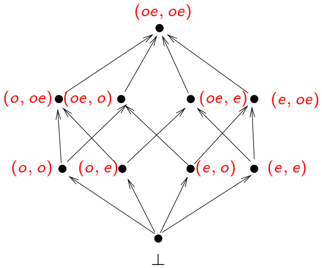
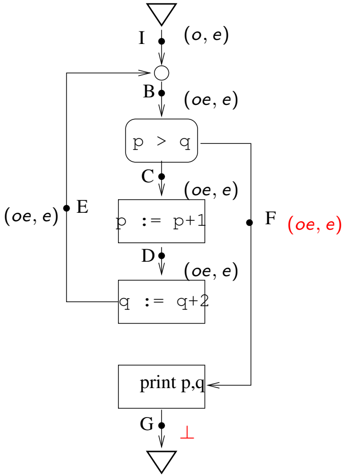
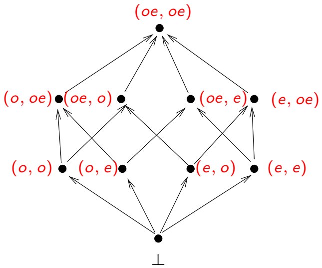
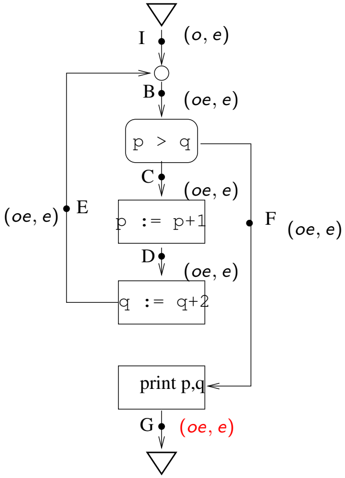




Kildall's algo on parity interpretation example

Underlying lattice





Kildall's algo on parity interpretation example

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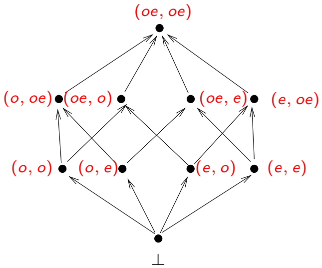
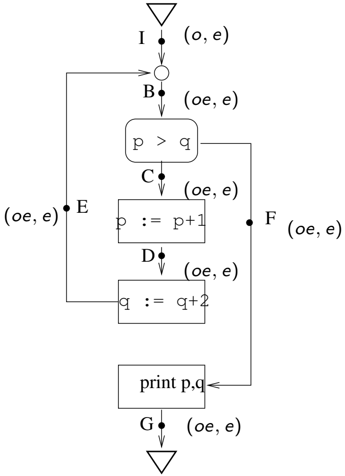
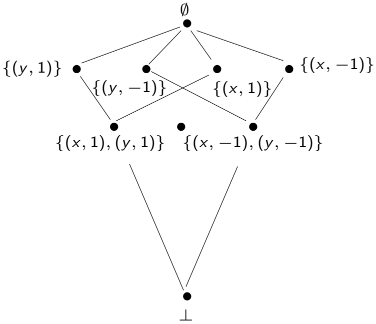
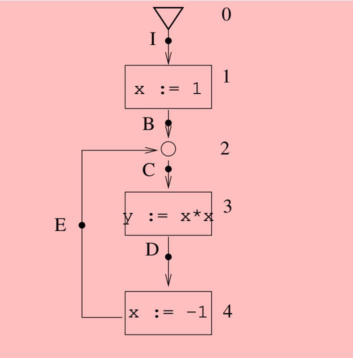




Values computed coincide with JOP values.

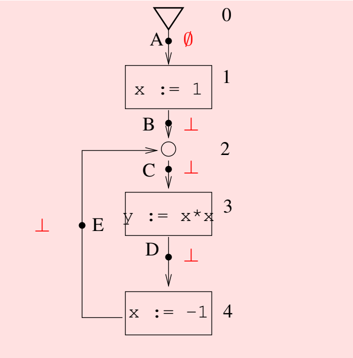
Constant propagation example



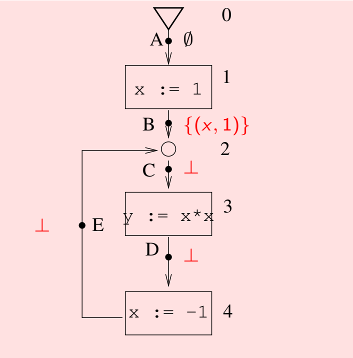


ProgPt	JOP values
A	$\emptyset$
B	$\{(x, 1)\}$
C	$\emptyset$
D	$\{(y, 1)\}$
E	$\{(x, -1), (y, 1)\}$

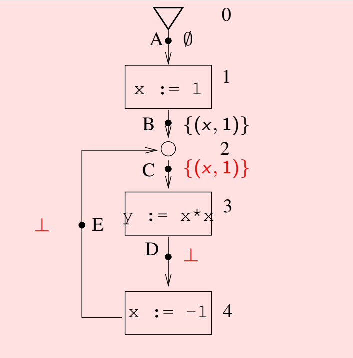
Kildall's algo on CP example: 1



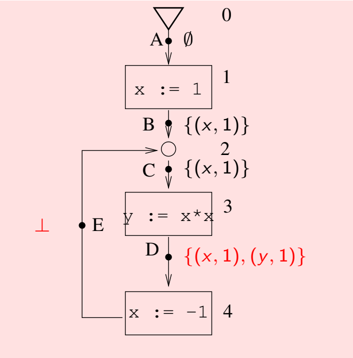
Kildall's algo on CP example: 2



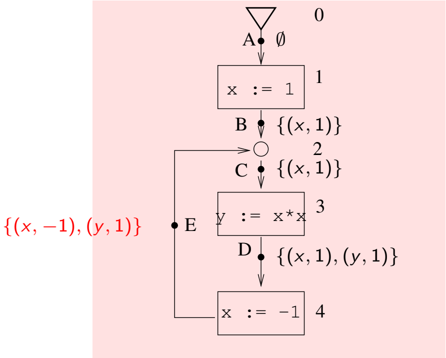
Kildall's algo on CP example: 3



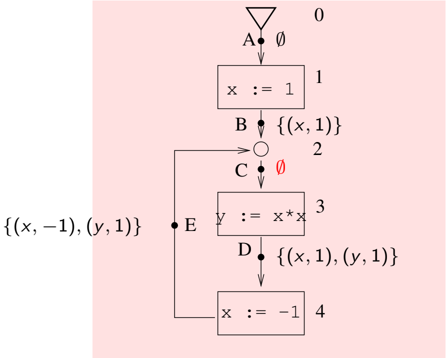
Kildall's algo on CP example: 4



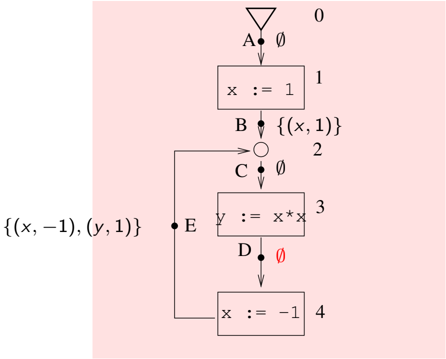
Kildall's algo on CP example: 5



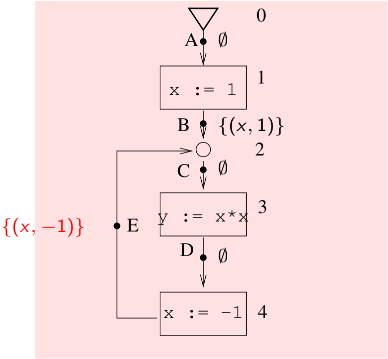
Kildall's algo on CP example: 6



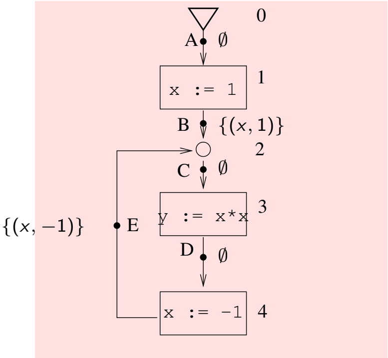
Kildall's algo on CP example: 7



Kildall's algo on CP example: 8

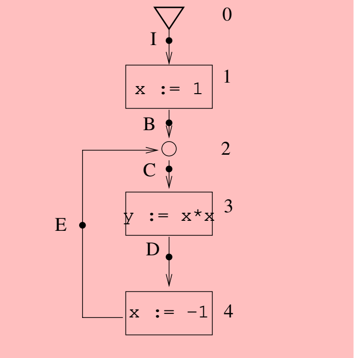


Kildall's algo on CP example: 9



Kildall's algo vs Actual Constant data

ProgPt	Actual JOP values	Kildall's data
A	$\emptyset$	$\emptyset$
B	$\{(x, 1)\}$	$\{(x, 1)\}$
C	$\emptyset$	$\emptyset$
D	$\{(y, 1)\}$	$\emptyset$
E	$\{(x, -1), (y, 1)\}$	$\{(x, -1)\}$



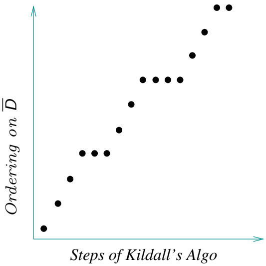
Note that Kildall's values are $\geq$ the actual JOP values at all points.

What Kildall's algo computes

- Always terminates if lattice has no infinite ascending chains.
- In general, computes the least solution to a system of equations induced by the given instance of the analysis.
- This value is always an over-approximation of the JOP for the given instance.

Termination of Kildall's algo

- Let $\overline{d}_i$ be the vector of values after the i-th step of algo.
- At step $i+1$ either $\overline{d}_{i+1}$ strictly dominates $\overline{d}_i$, or $\overline{d}_{i+1} = \overline{d}_i$. In the latter case number of marks *decreases*.
- The maximum length of any contiguous non-"climbing" sequence is equal to the number of program points.
- Moreover, the maximum number of "climbing" steps in algorithm is at most the length of any chain in the lattice $\overline{D}$.
- Therefore, the algorithm is guaranteed to terminate on finite-height lattices.



Induced Equations

The program induces a set of data-flow equations:

$$
\begin{array}{lcll}
x_I & = & d_0 & \text{for entry point } I \\
x_N & = & f_{MN}(x_M) & \text{for an assignment or conditional node } n \text{ with} \\
 & & & \text{with incoming point } M \text{ and outgoing point } N \\
x_N & = & x_L \sqcup x_M & \text{for a junction node with incoming points L,M} \\
 & & & \text{ and outgoing N.} \\
 & \cdots & & \text{etc.}
\end{array}
$$

Induced Equations

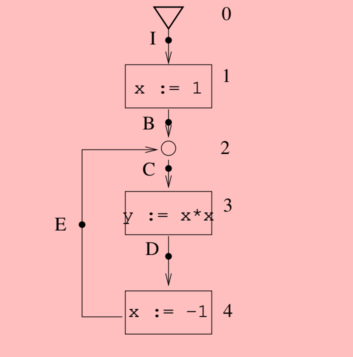
The program induces a set of data-flow equations:

x_I	$=$	d_0	for entry point I
x_N	$=$	$f_{MN}(x_M)$	for an assignment or conditional node n with with incoming point M and outgoing point N
x_N	$=$	$x_L \sqcup x_M$	for a junction node with incoming points L,M and outgoing N.
	$\cdots$		etc.

Note: The collecting semantics is a solution to the above equations.

Example equations

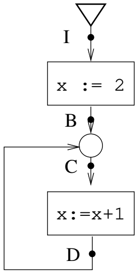
$$
\begin{aligned}
x_I &= d_0\\
x_B &= f_1(x_I)\\
x_C &= x_B \sqcup x_E\\
x_D &= f_3(x_C)\\
x_E &= f_4(x_D).
\end{aligned}
$$



Equations can have multiple solutions

Exercise: Give two solutions to equations induced for this program

- Use lattice of subsets of concrete stores, with integer values for `x`.
- Write down induced equations.
- Give two different solutions to the equations. Let $d_0 = \mathit{State}$.



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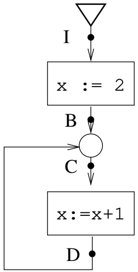
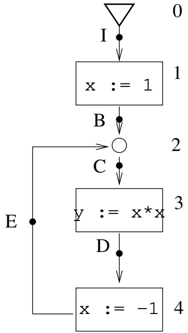


Note: collecting semantics of any program is the least solution to its data-flow equations using the concrete lattice (to be shown).

Function $\overline{f}$ induced by equations

Equations:

$$\begin{array}{rcl} x_I & = & d_0 \\ x_B & = & f_1(x_I) \\ x_C & = & x_B \sqcup x_E \\ x_D & = & f_3(x_C) \\ x_E & = & f_4(x_D). \end{array}$$



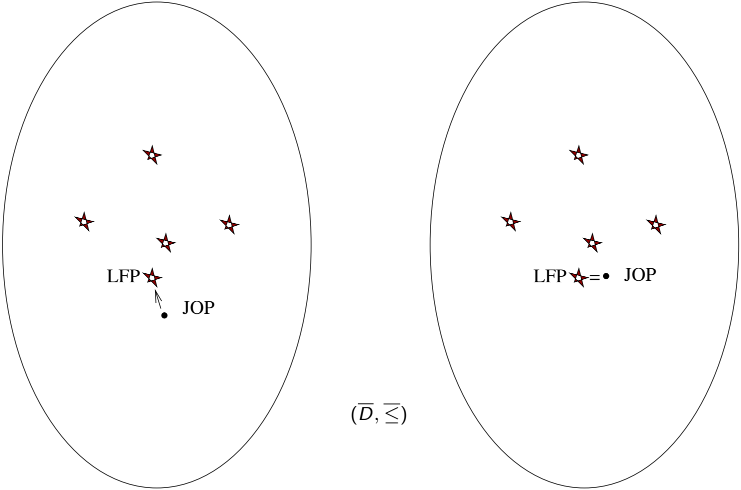
Corresponding $\overline{f}$ function:

$$\overline{f}(d_I, d_B, d_C, d_D, d_E) = (d_0, f_1(d_I), d_B \sqcup d_E, f_3(d_C), f_4(d_D)).$$

Natural ordering on solutions to Eq

- Consider "vectorised" lattice $\overline{D} = (D^k, \overline{\leq})$, where D is the underlying lattice.
- Each solution to the equations is a point in this vectorised lattice.
- The solutions are precisely the fix-points of the function $\overline{f}: \overline{D} \rightarrow \overline{D}$.
- If D is a complete lattice and f_i's are monotone, then $\overline{D}$ is complete and $\overline{f}$ is monotone.
  - Note: Concrete analysis satisfies these properties. So do many abstract interpretations.
- Therefore, Knaster-Tarski theorem applies. Therefore, there exists a least solution to $\overline{f}$.
- Kildall's algorithm computes this lfp (if it terminates).
  - So does the Kleene iteration $\perp_{\overline{D}}, \overline{f}(\perp_{\overline{D}}), \overline{f}^2(\perp_{\overline{D}}), \ldots$ if it reaches a stable value.

Correctness

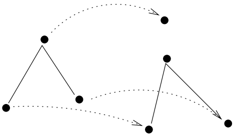


Monotonic Framework

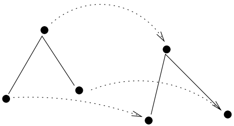
Infinitely–Distributive Framework

Kildall's algo always computes LFP of $\overline{f}$.

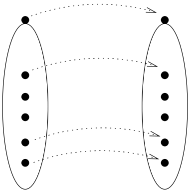
Monotonicity, distributivity, and continuity



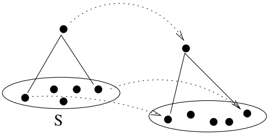
Monotonic



Distributive



Continuous



Inf−Distributive
(S is any subset of the lattice,
(including empty subset, or an infinite subset)

1. JOP $\leq$ LFP for monotone framework

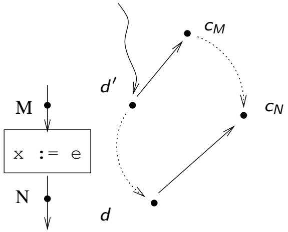
- Let $\overline{c}$ be any FP of $\overline{f}$. Consider any program point N. Let $c_N \equiv \overline{c}[N]$.
- **Claim**: For any path p, if N is the point at the end of p, c_N dominates $d \equiv f_p(d_0)$ reaching N.
  The argument is by induction on length of path p.
  - Base case $|p| = 0$: Then $N = I$, and $d = c_N = d_0$.
  - Let path p be of length $i + 1$. Let M be the program that p passes through just before reaching N. Let d' be $f_p^M(d_0)$, where f_p^M is the path transfer function of the prefix of path p that ends at point M. The inductive hypothesis is that $d' \sqsubseteq c_M$.
    The rest of the proof is in two cases.

1. JOP $\leq$ LFP for monotone framework

Case (node between M and N is not a join node):

Since $\overline{c}$ is a solution to the equations, and since the equation for x_N is $x_N = f_{MN}(x_M)$, we have $c_N = f_{MN}(c_M)$.

Now, since $d = f_{MN}(d')$, by monotinicity of f_{MN}, and from the hypothesis $d' \sqsubseteq c_M$, it follows that $d \sqsubseteq c_N$.

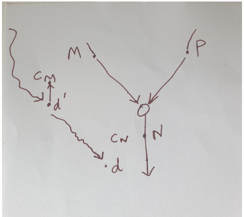


1. JOP $\leq$ LFP for monotone framework

Case (node between M and N is a join node):
Let P be the other predecessor of the join node.

1. $d = d'$ (because join nodes have identity transfer function)
2. The dataflow equation for x_N is $x_N = x_M \sqcup x_P$. Since $\overline{c}$ is a solution to the equations, $c_N = c_M \sqcup c_P$. That is, $C_M \sqsubseteq C_N$.
3. By inductive hypothesis, $d' \sqsubseteq c_M$.

The observations above imply that $d \sqsubseteq c_N$.



1. JOP $\leq$ LFP for monotone framework

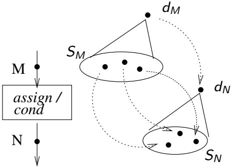
- That is, for every path p that reaches a point N, $f_p(d_0) \sqsubseteq c_N$.
- Therefore, JOP d_N at N is $\sqsubseteq c_N$

2. JOP = LFP for infinitely-distributive framework

Proof: Enough to show that the JOP $\overline{d}$ is a fixpoint of $\overline{f}$. We denote $\overline{d}[M]$ as d_M, $\overline{d}[N]$ as d_N, etc.

2. JOP = LFP for infinitely-distributive framework

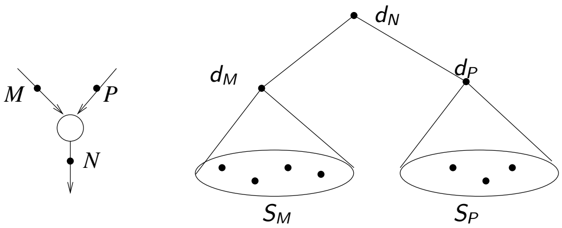
Proof: Enough to show that the JOP $\overline{d}$ is a fixpoint of $\overline{f}$. We denote $\overline{d}[M]$ as d_M, $\overline{d}[N]$ as d_N, etc.
Let N be any program point.
Case (the node before N is not a join node):



- Let S_M (resp. S_N) be the set of all facts that reach M (resp. N) along all paths.
- It is clear that $S_N = \{f_{MN}(s) | s \in S_M\}$.
- It is clear that the JOP d_M at M is equal to $\sqcup S_M$, and the JOP d_N at N is equal to $\sqcup S_N$.
- Therefore, by the previous two observations, and due to infinite distributivity, it follows that $d_N = f_{MN}(d_M)$.
- Therefore, $\overline{d}$ satisfies N's equation, which is $x_N = f_{MN}(x_M)$.

2. JOP = LFP for infinitely-distributive framework

Case (the node before N is a join node):



- Say S_M (resp. S_P resp. S_N) is the set of lattice values reaching M along all paths (resp. reaching P resp. reaching N).
- Clearly, d_M (resp. d_P resp. d_N) is equal to $\sqcup S_M$ (resp. $\sqcup S_P$ resp. $\sqcup S_N$).
- It is clear that $S_N = S_M \cup S_P$. Therefore, $d_N = d_M \sqcup d_P$.
- Therefore, $\overline{d}$ satisfies N's equation, which is $x_N = x_M \sqcup x_P$.

2. JOP = LFP for infinitely-distributive framework

- Since the argument in the previous two slides applies at all points N, we have shown that the vector $\overline{d}$ satisfies all the equations, and is hence a fix-point of $\overline{f}$.
- Note: Lattice is finite, and functions are pairwise distributive, and $f_i(\perp) = \perp$ implies framework is infinitely distributive.

Back to Constant Propagation

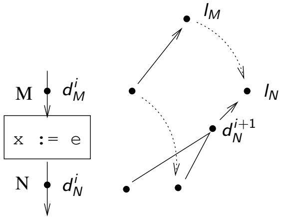
- f_n^{CP} is monotonic
- f_n^{CP} is *not* distributive.
  - Consider node n with statement $y := x * x$. Show two CP values P_1 and P_2 such that $f_n(P_1 \sqcup P_2) \sqsupset f_n(P_1) \sqcup f_n(P_2)$.
- The $nstate'$ functions are all distributive.

3. Kildall's algo computes LFP

- Let $\overline{d}$ be values computed by Kildall's algo upon termination, and $\overline{l}$ be LFP of $\overline{f}$. Let l_N denote $\overline{l}[N]$, l_M denote $\overline{l}[M]$, etc.
- Intermediate vector $\overline{d}^i$ after any step i is bounded above by $\overline{l}$. We prove this using induction on number of steps.
- Let N by any program point whose value gets updated in Step $i+1$.

3. Kildall's algo computes LFP

Case (the node before N is a non-join node):



Explanation:

- $d_M^i \sqsubseteq l_M$ and $d_N^i \sqsubseteq l_N$ by inductive hypothesis.

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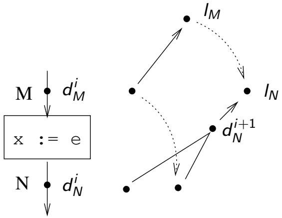

Explanation:

- $d_M^i \sqsubseteq l_M$ and $d_N^i \sqsubseteq l_N$ by inductive hypothesis.
- $l_N = f_{MN}(l_M)$, because $\bar{l}$ is a solution to the equations and because we have the equation $x_N = f_{MN}(x_M)$.

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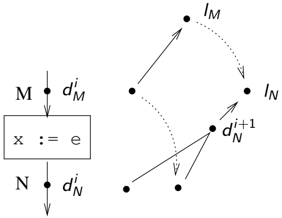

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- $l_N = f_{MN}(l_M)$, because $\bar{l}$ is a solution to the equations and because we have the equation $x_N = f_{MN}(x_M)$.
- Therefore, due to monotonicity of f_{MN}, $f_{MN}(d_M^i) \sqsubseteq l_N$.

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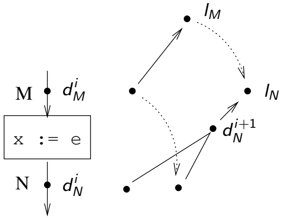

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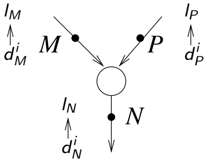
- $l_N = f_{MN}(l_M)$, because $\bar{l}$ is a solution to the equations and because we have the equation $x_N = f_{MN}(x_M)$.
- Therefore, due to monotonicity of f_{MN}, $f_{MN}(d_M^i) \sqsubseteq l_N$.
- Since $d_N^{i+1} = d_N^i \sqcup f_{MN}(d_M^i)$, we derive $d_N^{i+1} \sqsubseteq l_N$.

3. Kildall's algo computes LFP



Case (the node before N is a join node):

- Let M and P be the points that precede the join node. Let d^i_M, d^i_P, d^i_N be the data values at the respective program points after Step i.

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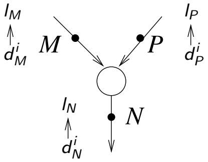

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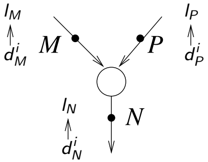
- Say propagation happens from M to N in Step i (argument is similar if propagation happened from P to N).

3. Kildall's algo computes LFP



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- Since $\bar{l}$ is a solution to the equations, and since we have the equation $x_N = x_M \sqcup x_P$, it follows that $l_N = l_M \sqcup l_P$. In other words, $l_M \sqsubseteq l_N$. In conjunction with $d^i_M \sqsubseteq l_M$ (inductive hypothesis), we get $d^i_M \sqsubseteq l_N$.

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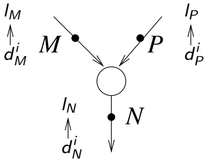

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- By inductive hypothesis, $d_N^i \sqsubseteq l_N$. Therefore, $(d_N^{i+1} = (d_M^i \sqcup d_N^i)) \sqsubseteq l_N$.

Thus it follows that $\bar{d} \leq \bar{l}$.

3. Kildall's algo computes LFP

Let $\overline{d}$ be the vector computed by the algorithm upon termination.
We now show that $\overline{d} \geq \overline{f}(\overline{d})$ (i.e. $\overline{d}$ is a postfixpoint of $\overline{f}$)
Let N be any program point.
Case (the node before N is a non-join node):

- Let M be the point that precedes this node. By definition of $\overline{f}$, $(\overline{f}(\overline{d}))[N]$ is equal to $f_{MN}(d_M)$.
- Since all points are unmarked, value d_M must have been propagated to N. That is, d_N must dominate $f_{MN}(d_M)$. That is, d_N dominates $(\overline{f}(\overline{d}))[N]$.

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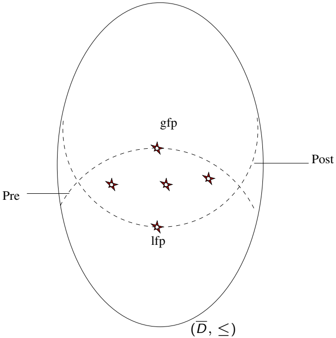
Case (the node before N is a join node):

- Let M and P be the points that precede the join node. By definition of $\overline{f}$, $(\overline{f}(\overline{d}))[N]$ is equal to $d_M \sqcup d_P$.
- Since all points are unmarked, value d_M and d_P must have been propagated to N. That is, d_N must dominate both d_M and d_P. That is, d_N dominates $d_M \sqcup d_P$. Hence, d_N dominates $(\overline{f}(\overline{d}))[N]$.

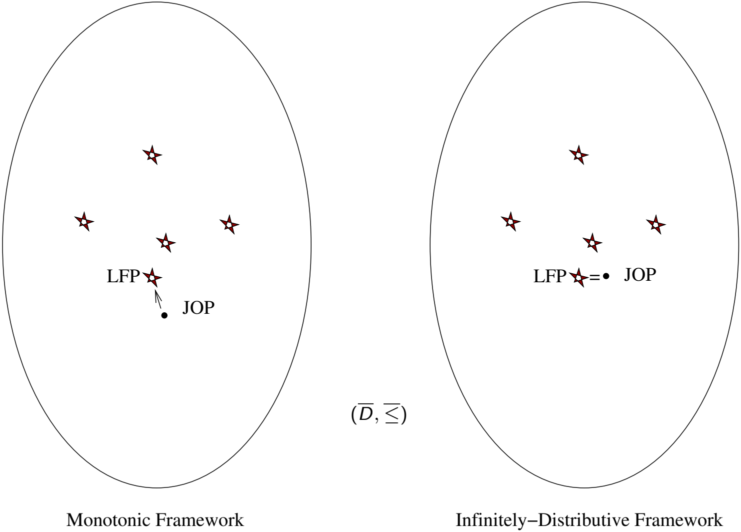
3. Kildall's algo computes LFP

- Therefore, by Knaster-Tarski theorem, $\bar{l} = glb(Post)$, and hence $\overline{d} \geq \bar{l}$.



- We have earlier proved that $\overline{d} \leq \bar{l}$. Therefore, it follows that $\overline{d} = \bar{l}$.

Correctness



Kildall's algo always computes LFP.

Overview of correctness

- Every program induces a set of equations on variables whose domain is lattice D. The equations, in turn, induce a function $\overline{f} : \overline{D} \to \overline{D}$.
- If each f_i is monotone and D is a complete lattice then $\overline{f}$ has a least fix-point $\mathrm{LFP}(\overline{f})$.
  - If each f_i is infinitely distributive, then $\mathrm{JOP} = \mathrm{LFP}(\overline{f})$.
  - Otherwise, if each f_i is only monotonic, $\mathrm{JOP} \leq \mathrm{LFP}(\overline{f})$.

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- If each f_i is monotone and D is a complete lattice then $\overline{f}$ has a least fix-point LFP($\overline{f}$).
  - If each f_i is infinitely distributive, then JOP $=$ LFP($\overline{f}$).
  - Otherwise, if each f_i is only monotonic, JOP $\leq$ LFP($\overline{f}$).
- Kildall's algorithm, for monotone frameworks:
  - Solution *at any point* during its execution is $\leq$ LFP($\overline{f}$)
  - If and when it terminates, solution is equal to LFP($\overline{f}$)
  - Note this is a stronger claim than "Kildall's algo computes JOP for distributive frameworks" [Killdall, 'POPL 73].
  - Kildall's algorithm is not only for program analysis. It can be used to find least solution to *any* set of simultaneous equations, as long as (a) domain of variables' values is a complete lattice, (b) each variable occurs in the lhs of a unique equation, and (c) all operators occurring in rhs's are monotone.

Summary of sufficient conditions

	Termination	LFP $\geq$ JOP	LFP $=$ JOP	Kild computes LFP upon termination
f_{MN}'s monotonic	$\surd$	$\surd$		$\surd$
No inf. asc. chains	$\surd$			
Inf. distributive			$\surd$	

- Each column is a property, and each row is a sufficient condition
- For a property to hold, *each* sufficient condition mentioned in its column needs to hold