# Kildall's algorithm for over-approximate JOP

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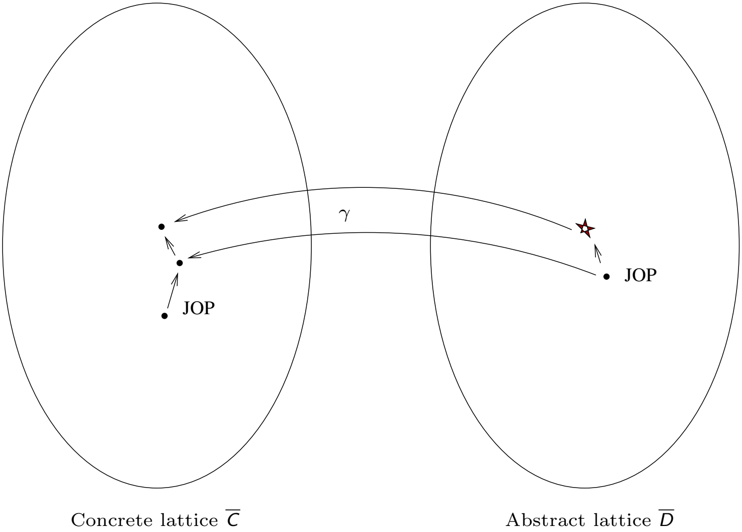
# Why over-approximation of JOP in abstract lattice is useful



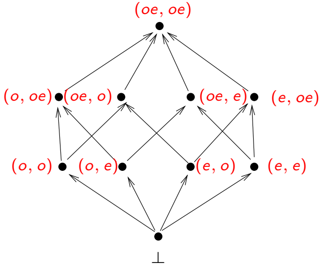
# Kildall's algorithm to compute over-approximation of JOP

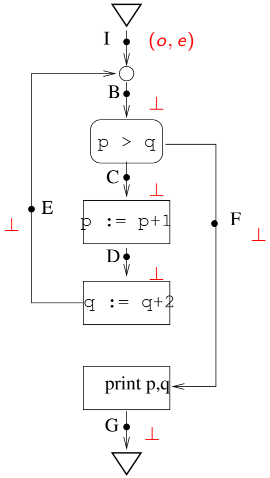
Input: An instance $(P, d_0)$ of a monotone data-flow framework $((D, \leq), F)$.
Output: For each program point $N$ in $P$, a data-value $d_N$ such that $\mathrm{JOP}_N^{d_0} \leq d_N$.

- Initialize data value at each program point to $\perp$, entry point to $d_0$.
- Mark all points.
- Repeat while there is a marked point:
  - Choose a marked point $M$ with value $d_M$, unmark it, and "propagate" it to successor points. That is, for each successor $N$ of $M$: (1) replace old value at $N$ by $f_{MN}(d_M) \sqcup d_N$, and (2) Mark $N$ if it was already marked or if new value strictly dominates than old value.
- Return data values at each point as over-approx of JOP.

# Kildall's algo on parity interpretation example

Underlying lattice



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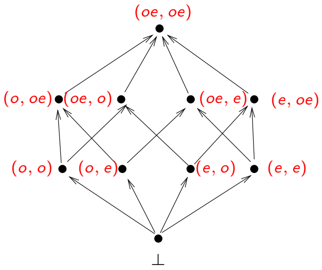


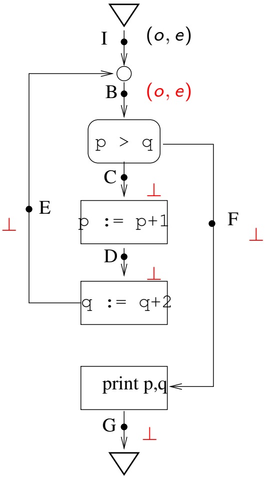

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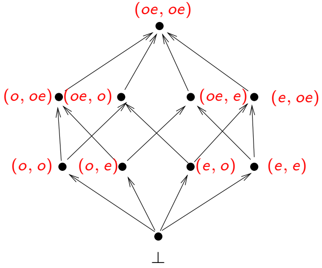


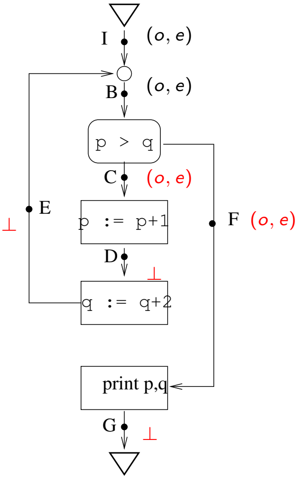

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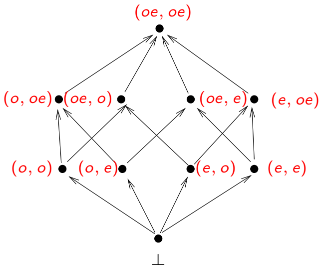


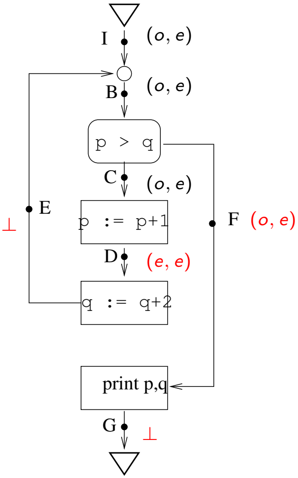

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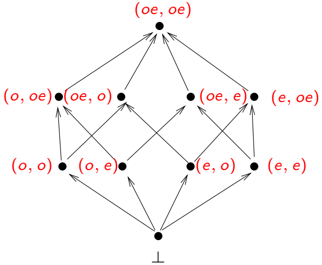
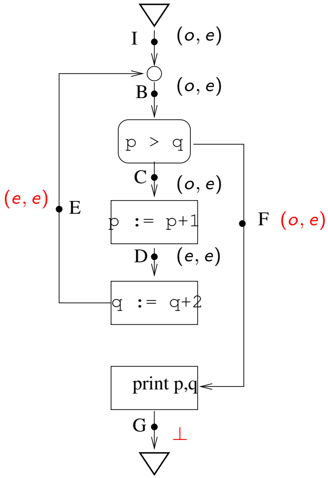
Underlying lattice



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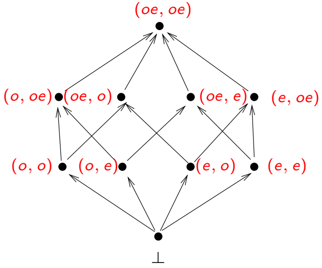


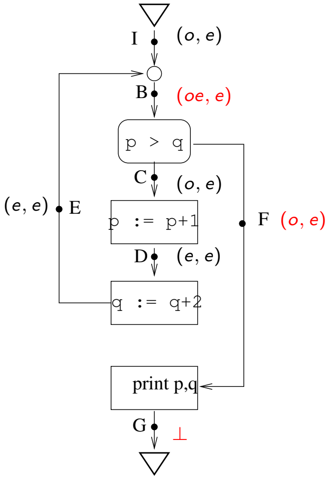

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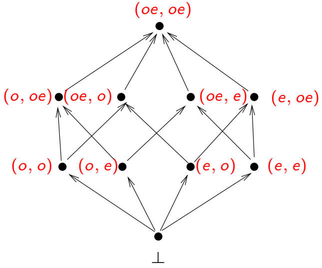
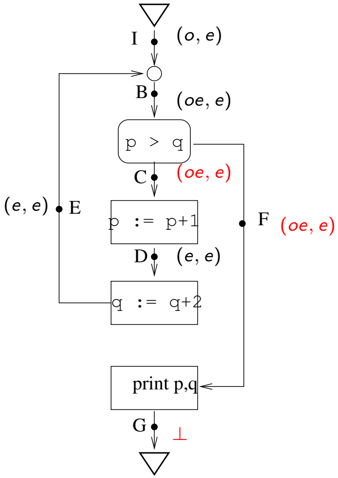
Underlying lattice



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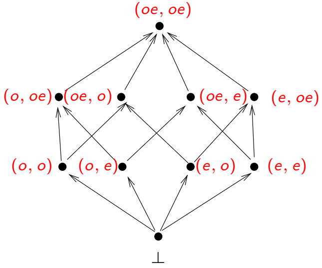
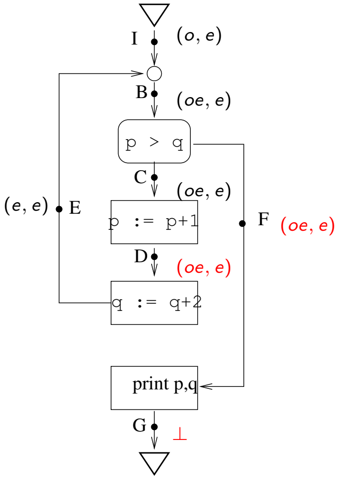
Underlying lattice



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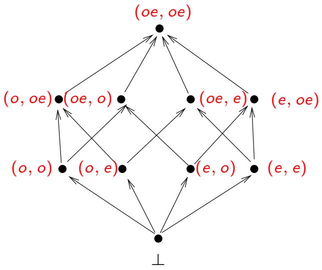
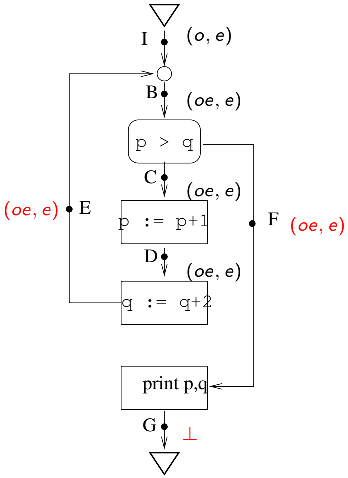
Underlying lattice



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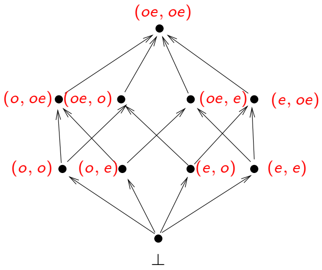


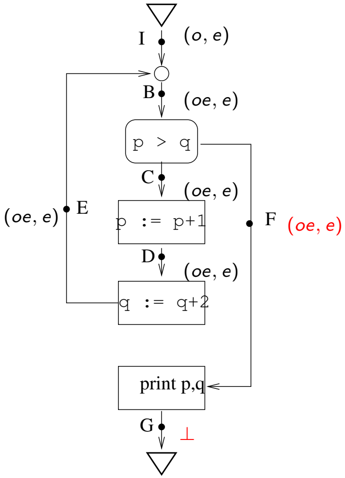

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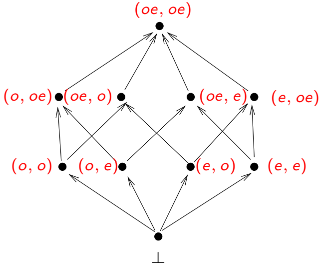
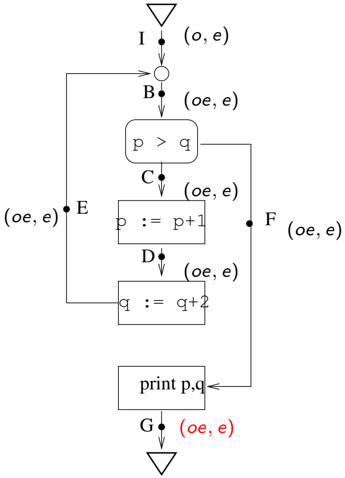
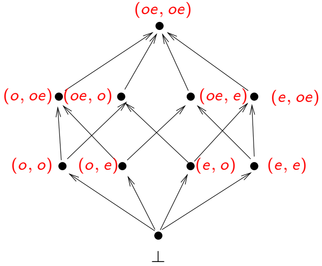
Underlying lattice

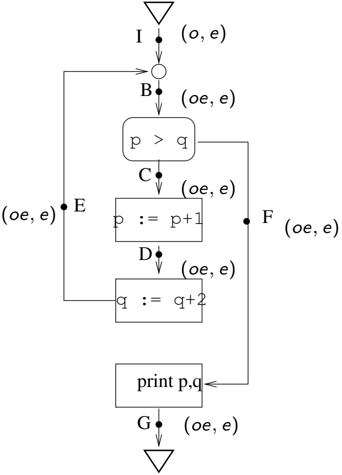


# Kildall’s algo on parity interpretation example

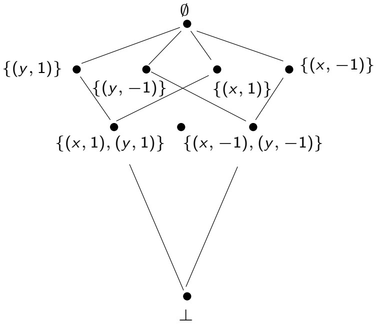
Underlying lattice

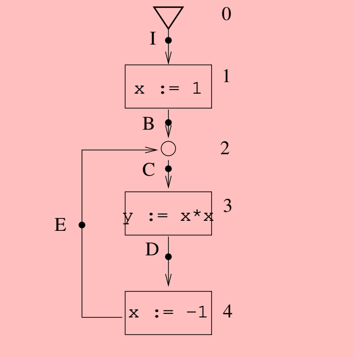



Values computed coincide with JOP values.

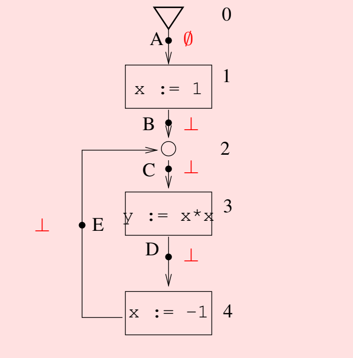
# Constant propagation example



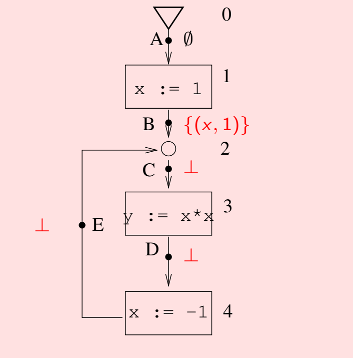




| ProgPt | JOP values |
|---|---|
| A | $\emptyset$ |
| B | $\{(x,1)\}$ |
| C | $\emptyset$ |
| D | $\{(y,1)\}$ |
| E | $\{(x,-1),(y,1)\}$ |

# Kildall's algo on CP example: 1



# Kildall's algo on CP example: 2



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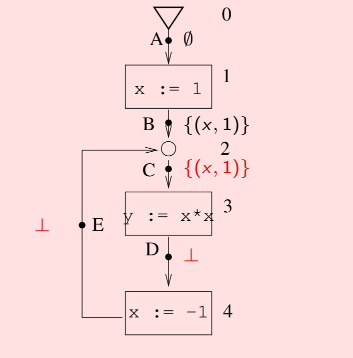
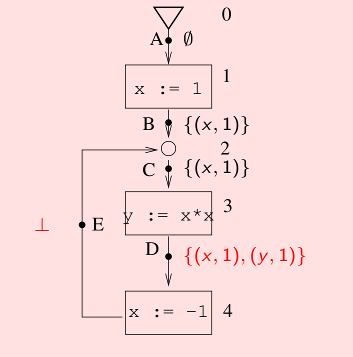
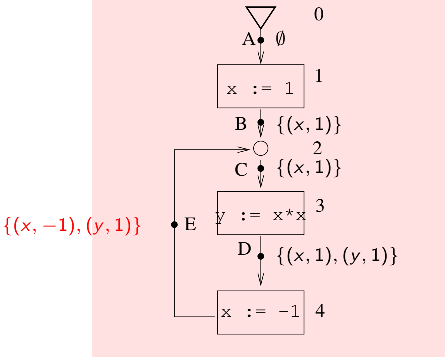


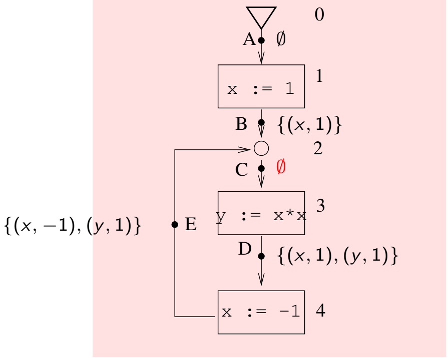
## Kildall's algo on CP example: 4



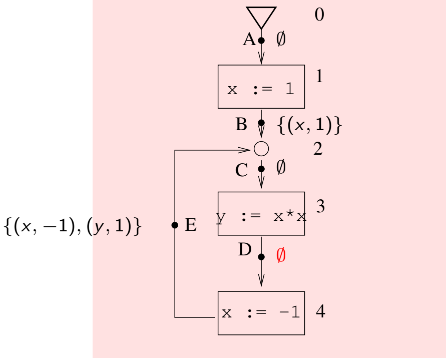
# Kildall's algo on CP example: 5



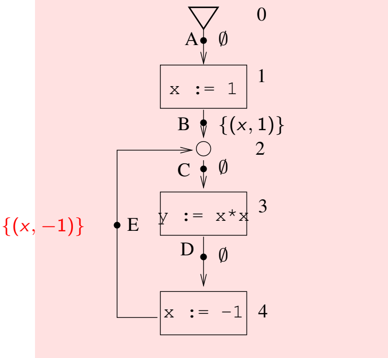
# Kildall's algo on CP example: 6



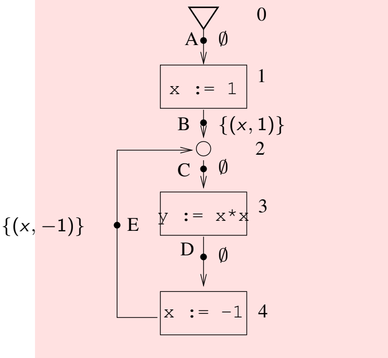
# Kildall's algo on CP example: 7



# Kildall's algo on CP example: 8

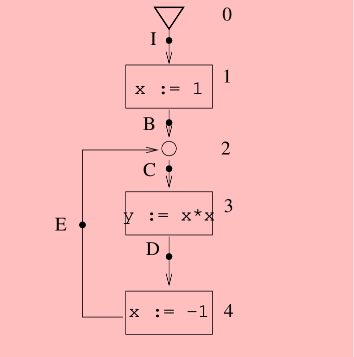


# Kildall's algo on CP example: 9



## Kildall's algo vs Actual Constant data

| ProgPt | Actual JOP values | Kildall's data |
|---|---|---|
| A | $\emptyset$ | $\emptyset$ |
| B | $\{(x,1)\}$ | $\{(x,1)\}$ |
| C | $\emptyset$ | $\emptyset$ |
| D | $\{(y,1)\}$ | $\emptyset$ |
| E | $\{(x,-1),(y,1)\}$ | $\{(x,-1)\}$ |



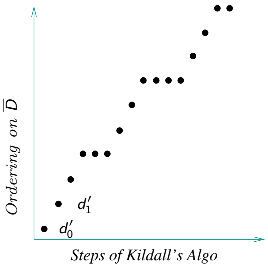

Note that Kildall's values are $\geq$ the actual JOP values at all points.

## What Kildall's algo computes

- Always terminates if lattice has no infinite chains.
- In general, computes the least solution to a system of equations induced by the given instance of the analysis.
- This value is always an over-approximation of the JOP for the given instance.

# Termination of Kildall's algo

- Let $\overline{d}_i$ be the vector of values after the $i$-th step of algo.
- At step $i+1$ either $\overline{d}_{i+1}$ strictly dominates $\overline{d}_i$, or $\overline{d}_{i+1} = \overline{d}_i$. In the latter case number of marks *decreases*.
- The maximum length of any contiguous non-"climbing" sequence is equal to the number of program points.
- Moreover, the maximum number of "climbing" steps in algorithm is at most the length of any chain in the lattice $\overline{D}$.
- Therefore, the algorithm is guaranteed to terminate on finite-height lattices.



## Induced Equations

The program induces a set of data-flow equations:

$$
\begin{array}{lcll}
x_I & = & d_0 & \text{for entry point } I \\
x_N & = & f_{MN}(x_M) & \text{for an assignment or conditional node } n \text{ with} \\
 & & & \text{with incoming point } M \text{ and outgoing point } N \\
x_M & = & x_K \sqcup x_L & \text{for a junction node with incoming points K,L} \\
 & & & \text{ and outgoing M.} \\
 & \cdots & & \text{etc.}
\end{array}
$$

## Induced Equations

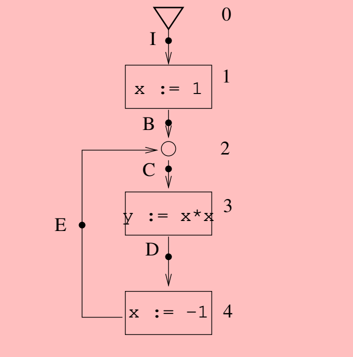
The program induces a set of data-flow equations:

| | | | |
|---|---|---|---|
| $x_I$ | $=$ | $d_0$ | for entry point $I$ |
| $x_N$ | $=$ | $f_{MN}(x_M)$ | for an assignment or conditional node $n$ with with incoming point $M$ and outgoing point $N$ |
| $x_M$ | $=$ | $x_K \sqcup x_L$ | for a junction node with incoming points K,L and outgoing M. |
| | $\cdots$ | | etc. |

Note: The collecting semantics is a solution to the above equations.

## Example equations

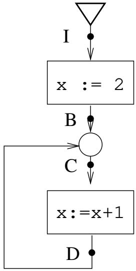
$$
\begin{aligned}
x_I &= d_0 \\
x_B &= f_{IB}(x_I) \\
x_C &= x_B \sqcup x_E \\
x_D &= f_{CD}(x_C) \\
x_E &= f_{DE}(x_D)
\end{aligned}
$$



## Equations can have multiple solutions

Exercise: Give two solutions to equations induced for this program

- Use lattice of subsets of concrete stores, with integer values for `x`.
- Write down induced equations.
- Give two different solutions to the equations. Let $d_0 = State$.



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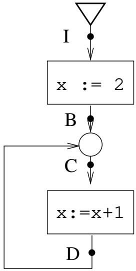


Note: collecting semantics of any program is the least solution to its data-flow equations using the concrete lattice (to be shown).

# Function $\overline{f}$ induced by equations

Equations:

$$
\begin{aligned}
x_I &= d_0 \\
x_B &= f_{IB}(x_I) \\
x_C &= x_B \sqcup x_E \\
x_D &= f_{CD}(x_C) \\
x_E &= f_{DE}(x_D)
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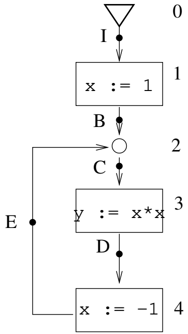


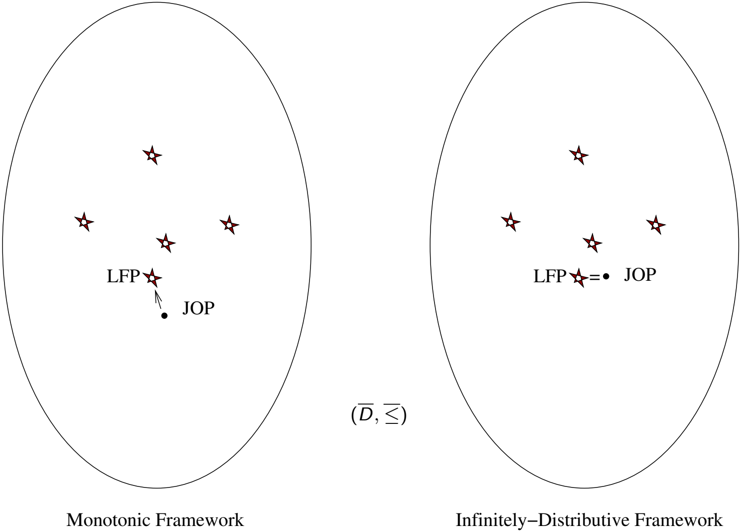

Corresponding $\overline{f}$ function:

$$\overline{f}(d_I, d_B, d_C, d_D, d_E) = (d_0, f_1(d_I), d_B \sqcup d_E, f_3(d_C), f_4(d_D))$$

## Natural ordering on solutions to Eq

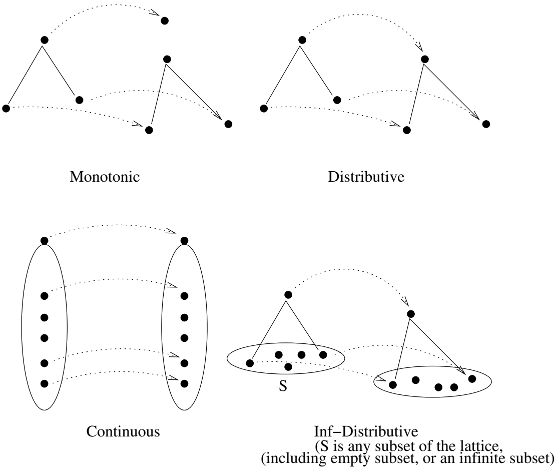
- Consider "vectorised" lattice $\overline{D} = (D^k, \overline{\leq})$, where $D$ is the underlying lattice.
- Each solution to the equations is a point in this vectorised lattice.
- The solutions are precisely the fix-points of the function $\overline{f}$: $\overline{D} \rightarrow \overline{D}$.
- If $D$ is a complete lattice and $f_i$'s are monotone, then $\overline{D}$ is complete and $\overline{f}$ is monotone.
    - Note: Concrete analysis satisfies these properties. So do many abstract interpretations.
- Therefore, Knaster-Tarski theorem applies. Therefore, there exists a least solution to $\overline{f}$.
- Kildall's algorithm computes this lfp (if it terminates).
    - So does the Kleene iteration $\perp_{\overline{D}}, \overline{f}(\perp_{\overline{D}}), \overline{f}^2(\perp_{\overline{D}}), \ldots$ if it reaches a stable value.

## Correctness




Kildall's algo always computes LFP of $\overline{f}$.

# Monotonicity, distributivity, and continuity



# 1. JOP $\leq$ LFP for monotone framework

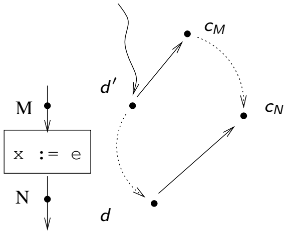
- Let $\overline{c}$ be any FP of $\overline{f}$. Consider any program point $N$. Let $c_N \equiv \overline{c}[N]$.
- **Claim**: For any path $p$, if $N$ is the point at the end of $p$, $c_N$ dominates $d \equiv f_p(d_0)$ reaching $N$.
  The argument is by induction on length of path $p$.
  - Base case $|p| = 0$: Then $N = I$, and $d = c_N = d_0$.
  - Let path $p$ be of length $i + 1$. Let $M$ be the program that $p$ passes through just before reaching $N$. Let $d'$ be $f_p^M(d_0)$, where $f_p^M$ is the path transfer function of the prefix of path $p$ that ends at point $M$. The inductive hypothesis is that $d' \sqsubseteq c_M$.
    The rest of the proof is in two cases.

# 1. JOP ≤ LFP for monotone framework

Case (node between $M$ and $N$ is not a join node):

Since $\overline{c}$ is a solution to the equations, and since the equation for $x_N$ is $x_N = f_{MN}(x_M)$, we have $c_N = f_{MN}(c_M)$.

Now, since $d = f_{MN}(d')$, by monotinicity of $f_{MN}$, and from the hypothesis $d' \sqsubseteq c_M$, it follows that $d \sqsubseteq c_N$.

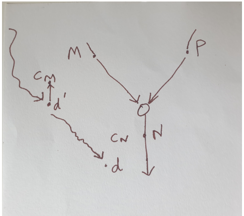


## 1. JOP $\leq$ LFP for monotone framework

Case (node between $M$ and $N$ is a join node):
Let $P$ be the other predecessor of the join node.

1. $d = d'$ (because join nodes have identity transfer function)
2. The dataflow equation for $x_N$ is $x_N = x_M \sqcup x_P$. Since $\overline{c}$ is a solution to the equations, $c_N = c_M \sqcup c_P$. That is, $C_M \sqsubseteq C_N$.
3. By inductive hypothesis, $d' \sqsubseteq c_M$.

The observations above imply that $d \sqsubseteq c_N$.



# 1. JOP $\leq$ LFP for monotone framework

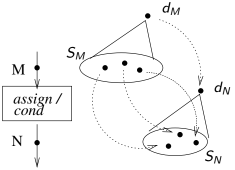
- That is, for every path $p$ that reaches a point $N$, $f_p(d_0) \sqsubseteq c_N$.
- Therefore, JOP $d_N$ at $N$ is $\sqsubseteq c_N$

## 2. JOP = LFP for infinitely-distributive framework

Proof: Enough to show that the JOP $\overline{d}$ is a fixpoint of $\overline{f}$. We denote $\overline{d}[M]$ as $d_M$, $\overline{d}[N]$ as $d_N$, etc.

# 2. JOP = LFP for infinitely-distributive framework

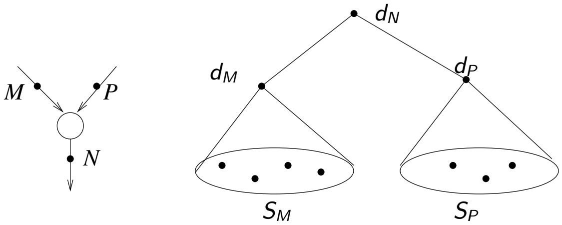
Proof: Enough to show that the JOP $\overline{d}$ is a fixpoint of $\overline{f}$. We denote $\overline{d}[M]$ as $d_M$, $\overline{d}[N]$ as $d_N$, etc.
Let $N$ be any program point.
Case (the node before $N$ is not a join node):




- Let $S_M$ (resp. $S_N$) be the set of all facts that reach $M$ (resp. $N$) along all paths.
- It is clear that $S_N = \{f_{MN}(s) | s \in S_M\}$.
- It is clear that the JOP $d_M$ at $M$ is equal to $\sqcup S_M$, and the JOP $d_N$ at $N$ is equal to $\sqcup S_N$.
- Therefore, by the previous two observations, and due to infinite distributivity, it follows that $d_N = f_{MN}(d_M)$.
- Therefore, $\overline{d}$ satisfies $N$'s equation, which is $x_N = f_{MN}(x_M)$.

# 2. JOP = LFP for infinitely-distributive framework

Case (the node before $N$ is a join node):




- Say $S_M$ (resp. $S_P$ resp. $S_N$) is the set of lattice values reaching $M$ along all paths (resp. reaching $P$ resp. reaching $N$).
- Clearly, $d_M$ (resp. $d_P$ resp. $d_N$) is equal to $\sqcup S_M$ (resp. $\sqcup S_P$ resp. $\sqcup S_N$).
- It is clear that $S_N = S_M \cup S_P$. Therefore, $d_N = d_M \sqcup d_P$.
- Therefore, $\overline{d}$ satisfies $N$'s equation, which is $x_N = x_M \sqcup x_P$.

## 2. JOP = LFP for infinitely-distributive framework

- Since the argument in the previous two slides applies at all points $N$, we have shown that the vector $\overline{d}$ satisfies all the equations, and is hence a fix-point of $\overline{f}$.
- Note: Lattice is finite, and functions are pairwise distributive, and $f_i(\bot) = \bot$ implies framework is infinitely distributive.

## Some examples

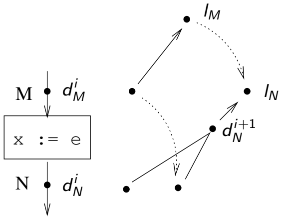
- $f_n^{CP}$ is *not* distributive for the node $n$ with statement $y := x * x$.
  - Show two CP values $P_1$ and $P_2$ such that $f_n(P_1 \sqcup P_2) \sqsupset f_n(P_1) \sqcup f_n(P_2)$.
- The $nstate'$ functions are all infinitely distributive.
  - Therefore, collecting semantics is the LFP to the equations when *nstate'* transfer functions are used.

## 3. Kildall's algo computes LFP

- Let $\overline{d}$ be values computed by Kildall's algo upon termination, and $\overline{l}$ be LFP of $\overline{f}$. Let $l_N$ denote $\overline{l}[N]$, $l_M$ denote $\overline{l}[M]$, etc.
- Intermediate vector $\overline{d}^i$ after any step $i$ is bounded above by $\overline{l}$. We prove this using induction on number of steps.
- Let $N$ by any program point whose value gets updated in Step $i + 1$.

## 3. Kildall's algo computes LFP

Case (the node before $N$ is a non-join node):




Explanation:

- $d_M^i \sqsubseteq l_M$ and $d_N^i \sqsubseteq l_N$ by inductive hypothesis.

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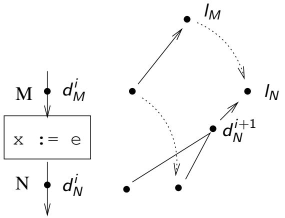


Explanation:

- $d^i_M \sqsubseteq l_M$ and $d^i_N \sqsubseteq l_N$ by inductive hypothesis.
- $l_N = f_{MN}(l_M)$, because $\bar{l}$ is a solution to the equations and because we have the equation $x_N = f_{MN}(x_M)$.

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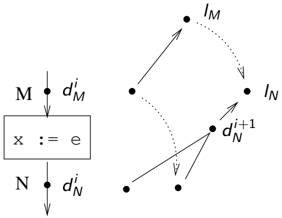


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- Therefore, due to monotonicity of $f_{MN}$, $f_{MN}(d_M^i) \sqsubseteq l_N$.

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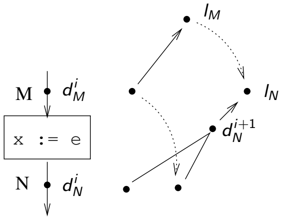


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- Therefore, due to monotonicity of $f_{MN}$, $f_{MN}(d_M^i) \sqsubseteq l_N$.
- Since $d_N^{i+1} = d_N^i \sqcup f_{MN}(d_M^i)$, we derive $d_N^{i+1} \sqsubseteq l_N$.

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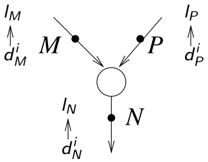
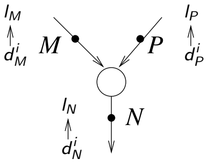



Case (the node before $N$ is a join node):

- Let $M$ and $P$ be the points that precede the join node. Let $d_M^i, d_P^i, d_N^i$ be the data values at the respective program points after Step $i$.

## 3. Kildall's algo computes LFP

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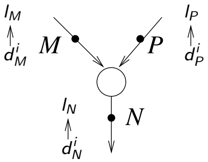


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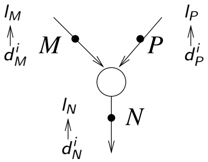


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- Since $\bar{l}$ is a solution to the equations, and since we have the equation $x_N = x_M \sqcup x_P$, it follows that $l_N = l_M \sqcup l_P$. In other words, $l_M \sqsubseteq l_N$. In conjunction with $d^i_M \sqsubseteq l_M$ (inductive hypothesis), we get $d^i_M \sqsubseteq l_N$.
- By inductive hypothesis, $d^i_N \sqsubseteq l_N$. Therefore, $(d^{i+1}_N = (d^i_M \sqcup d^i_N)) \sqsubseteq l_N$.

Thus it follows that $\bar{d} \leq \bar{l}$.

# 3. Kildall's algo computes LFP

Let $\overline{d}$ be the vector computed by the algorithm upon termination.
We now show that $\overline{d} \geq \overline{f}(\overline{d})$ (i.e. $\overline{d}$ is a postfixpoint of $\overline{f}$)
Let $N$ be any program point.
Case (the node before $N$ is a non-join node):

- Let $M$ be the point that precedes this node. By definition of $\overline{f}$, $(\overline{f}(\overline{d}))[N]$ is equal to $f_{MN}(d_M)$.
- Since all points are unmarked, value $d_M$ must have been propagated to $N$. That is, $d_N$ must dominate $f_{MN}(d_M)$. That is, $d_N$ dominates $(\overline{f}(\overline{d}))[N]$.

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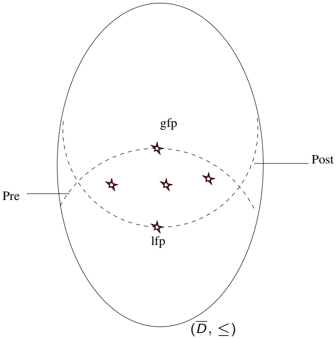
Case (the node before $N$ is a join node):

- Let $M$ and $P$ be the points that precede the join node. By definition of $\overline{f}$, $(\overline{f}(\overline{d}))[N]$ is equal to $d_M \sqcup d_P$.
- Since all points are unmarked, value $d_M$ and $d_P$ must have been propagated to $N$. That is, $d_N$ must dominate both $d_M$ and $d_P$. That is, $d_N$ dominates $d_M \sqcup d_P$. Hence, $d_N$ dominates $(\overline{f}(\overline{d}))[N]$.

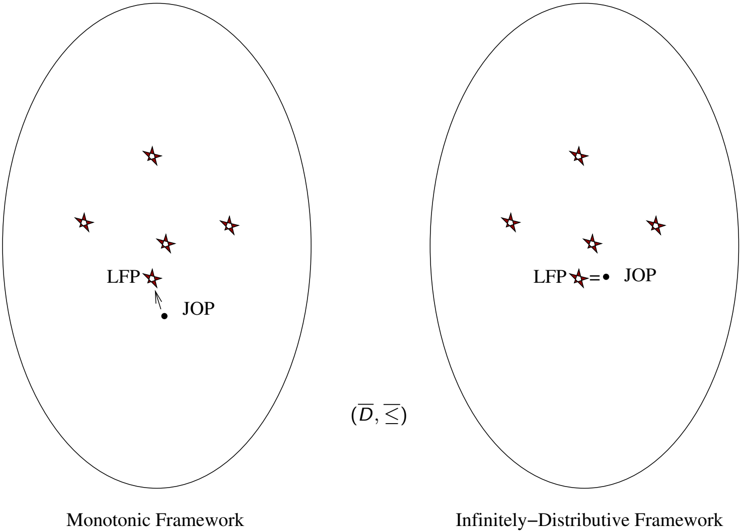
## 3. Kildall's algo computes LFP

- Therefore, by Knaster-Tarski theorem, $\bar{l} = glb(Post)$, and hence $\overline{d} \geq \bar{l}$.




- We have earlier proved that $\overline{d} \leq \bar{l}$. Therefore, it follows that $\overline{d} = \bar{l}$.

## Correctness




Kildall's algo always computes LFP.

## Overview of correctness

- Every program induces a set of equations on variables whose domain is lattice $D$. The equations, in turn, induce a function $\overline{f} : \overline{D} \rightarrow \overline{D}$.
- If each $f_i$ is monotone and $D$ is a complete lattice then $\overline{f}$ has a least fix-point $\text{LFP}(\overline{f})$.
  - If each $f_i$ is infinitely distributive, then $\text{JOP} = \text{LFP}(\overline{f})$.
  - Otherwise, if each $f_i$ is only monotonic, $\text{JOP} \leq \text{LFP}(\overline{f})$.

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  - If each $f_i$ is infinitely distributive, then JOP $=$ LFP($\overline{f}$).
  - Otherwise, if each $f_i$ is only monotonic, JOP $\leq$ LFP($\overline{f}$).
- Kildall's algorithm, for monotone frameworks:
  - Solution *at any point* during its execution is $\leq$ LFP($\overline{f}$)
  - If and when it terminates, solution is equal to LFP($\overline{f}$)
  - Note this is a stronger claim than "Kildall's algo computes JOP for distributive frameworks" [Killdall, 'POPL 73].
  - Kildall's algorithm is not only for program analysis. It can be used to find least solution to *any* set of simultaneous equations, as long as (a) domain of variables' values is a complete lattice, (b) each variable occurs in the lhs of a unique equation, and (c) all operators occurring in rhs's are monotone.

# Summary of sufficient conditions

| | Termination | LFP $\geq$ JOP | LFP $=$ JOP | Kild computes LFP upon termination |
|---|---|---|---|---|
| $f_{MN}$'s monotonic | | $\surd$ | | $\surd$ |
| No infinite chains | $\surd$ | | | |
| Inf. distributive | | | $\surd$ | |

- Each column is a property, and each row is a sufficient condition
- For a property to hold, *each* sufficient condition mentioned in its column needs to hold