Data-flow Analysis / Abstract Interpretation

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IISc

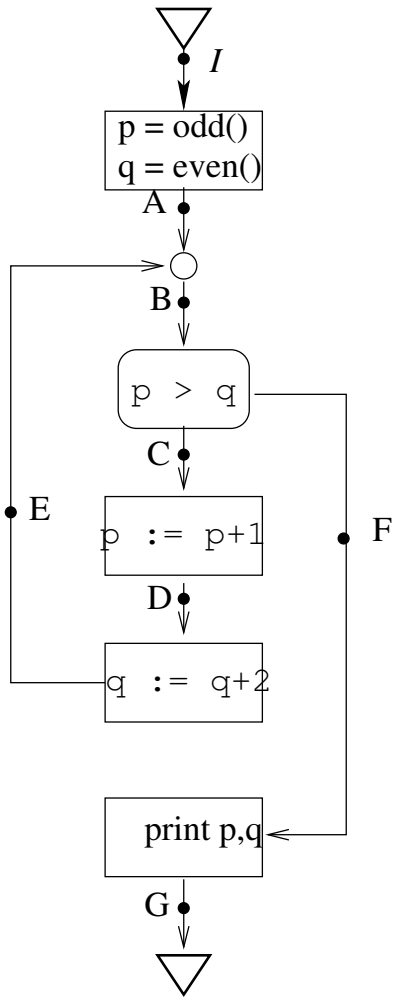
What is data-flow analysis

- "Computing 'safe' approximations to the set of values / behaviours arising dynamically at run time, statically or at compile time."
- Typically used by compiler writers to optimize running time of compiled code.
  - Constant propagation: Is the value of a variable constant at a particular program location.
  - Replace `x := y + z` by `x := 17` during compilation.
- More recently, used for verifying properties of programs.

Collecting semantics – example

Collecting semantics of a program = set of (concrete) states occurring at each program point.

Path	Concrete states
I	$\{(i, j)\}$ (given)



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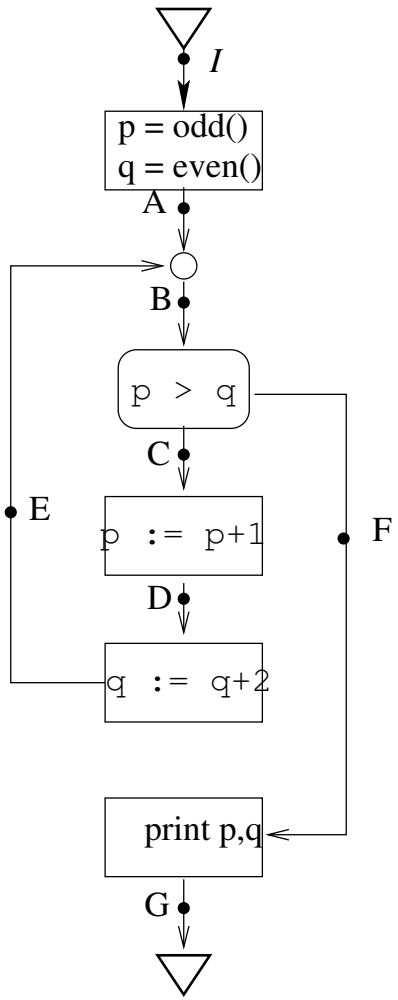

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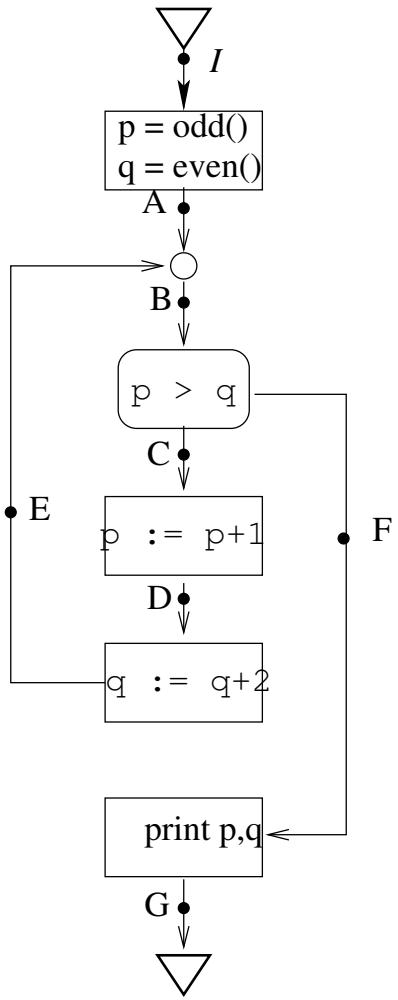

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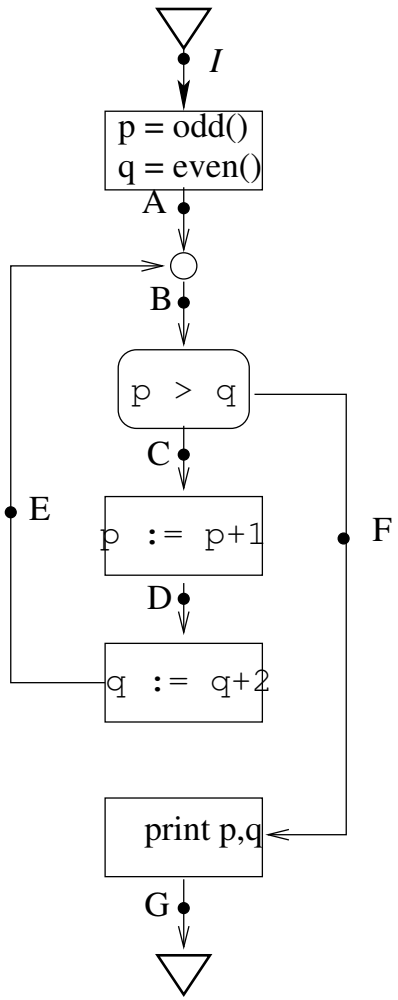

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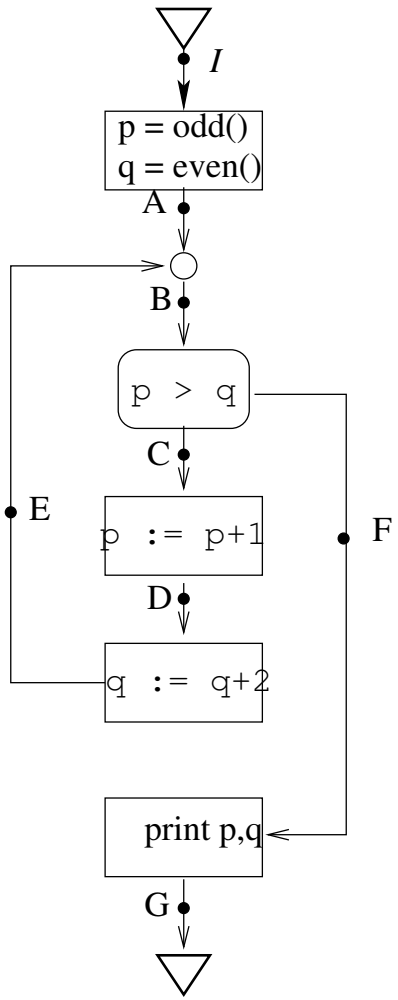

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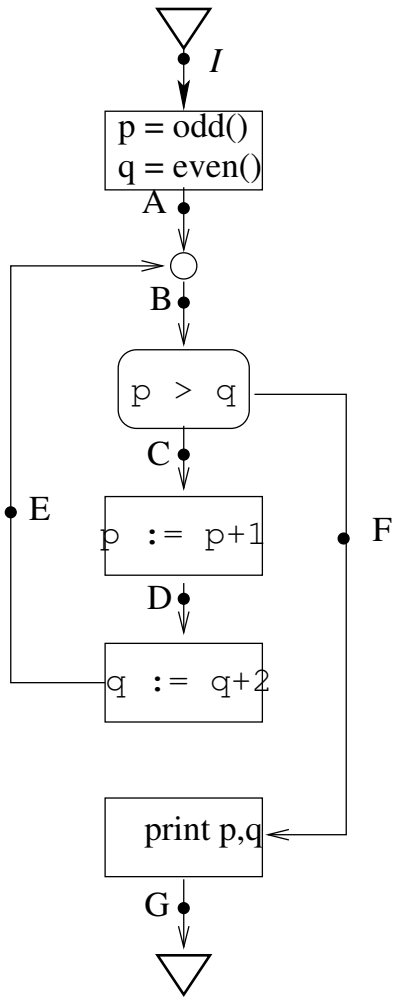

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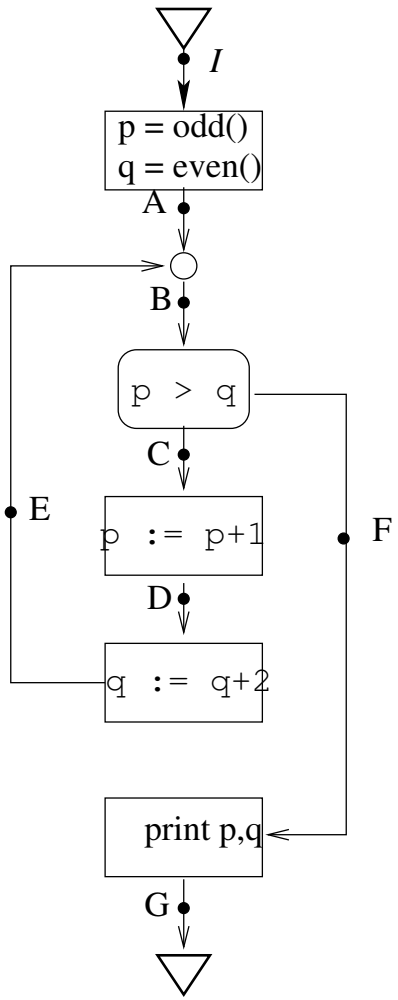

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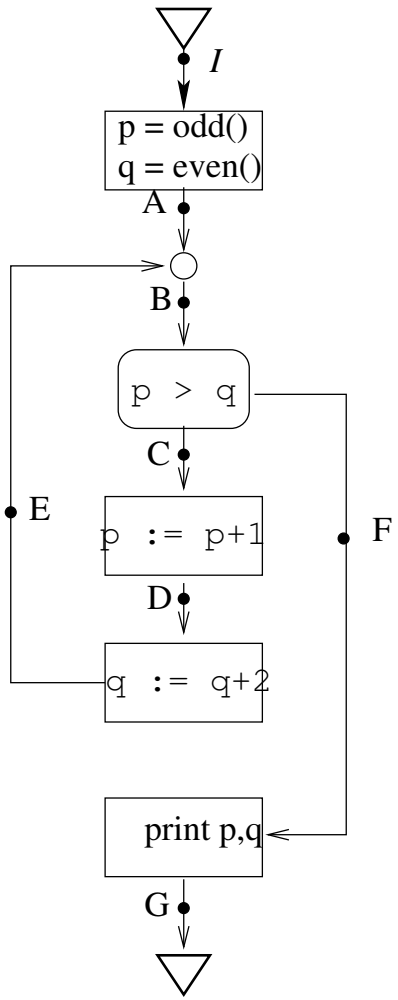

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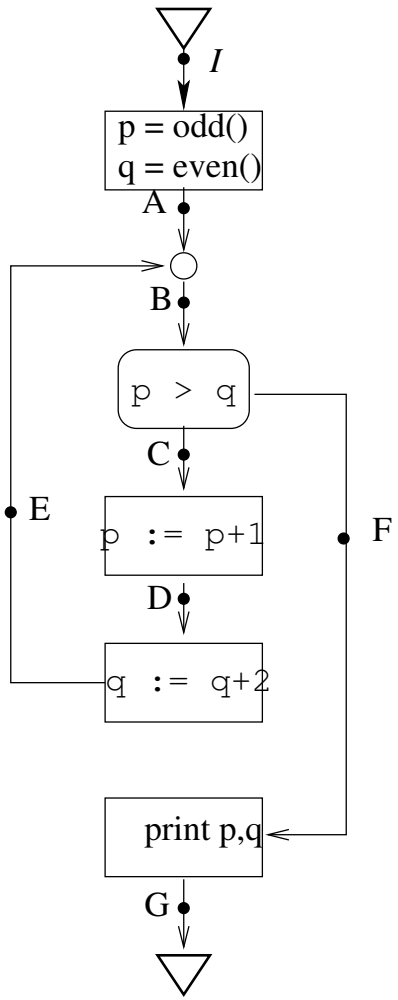

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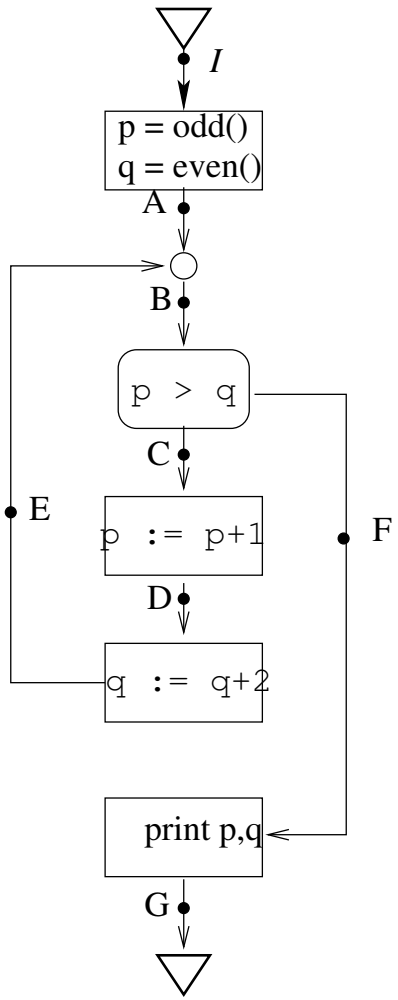

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Therefore, collecting semantics:

I	$\{(i,j)\}$
A	$\{(i,j) \mid i$ odd, j even$\}$
B	$\{(i,j) \mid i$ odd, j even$\} \cup \{(i,j) \mid i$ even, j even, $i \geq j\}$
C	$\{(i,j) \mid j$ even, $i > j\}$
D	$\{(i,j) \mid j$ even, $i > j+1\}$
E	$\{(i,j) \mid j$ even, $i \geq j\}$
F	$\{(i,j) \mid i$ odd, j even, $i < j\} \cup \{(i,j) \mid i$ even, j even, $i = j\}$
G	$\{(i,j) \mid i$ odd, j even, $i < j\} \cup \{(i,j) \mid i$ even, j even, $i = j\}$

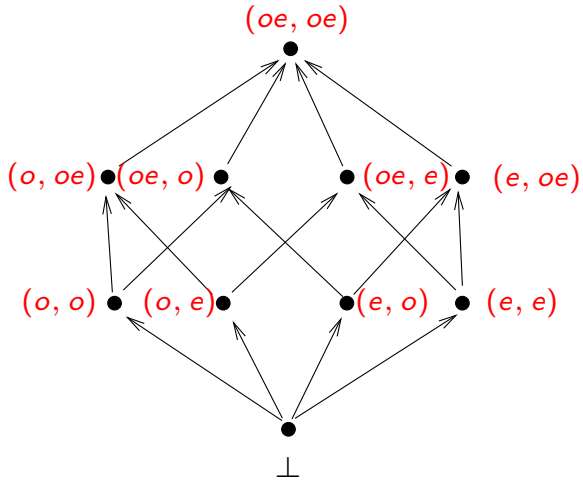
An abstract interpretation

Components of an abstract interpretation:

- Set of abstract states D, forming a complete lattice.
- "Concretization" function $\gamma : D \rightarrow 2^{State}$, which associates a set of concrete states with each abstract state.
- Set of transfer functions, with signature $D \rightarrow D$, which interpret assignments and conditionals using abstract states:
  - a function $f_n : D \rightarrow D$ for each assignment node n
  - a function $f_{n,T} : D \rightarrow D$ and a function $f_{n,F} : D \rightarrow D$ for each conditional node n

Abstract interpretation – example

- Abstract lattice D



- Transfer function for an assignment node n: `p := p+q`

$$f_n(s) = \begin{cases} \bot & \text{if } s \text{ is } \bot \\ (o, s[q]) & \text{if } s[\text{p}] \text{ is o and } s[\text{q}] \text{ is e,} \\ & \quad \text{or } s[\text{p}] \text{ is e and } s[\text{q}] \text{ is o} \\ (e, s[q]) & \text{if both } s[\text{p}] \text{ and } s[\text{q}] \text{ are o} \\ & \quad \text{or both } s[\text{p}] \text{ and } s[\text{q}] \text{ are e} \\ (oe, s[q]) & \text{otherwise} \end{cases}$$

- The concretization function γ
  - $\gamma((oe, oe)) =$

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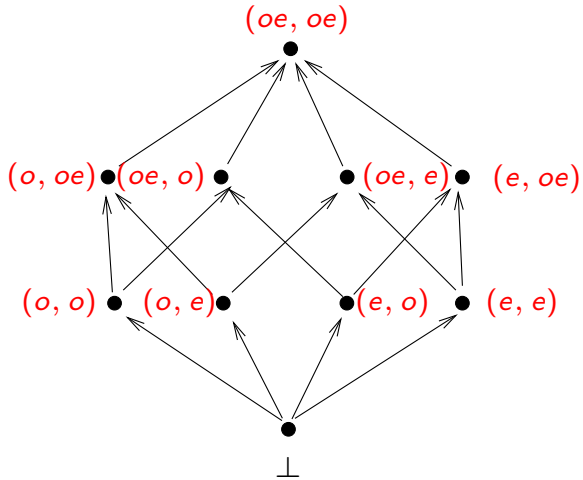

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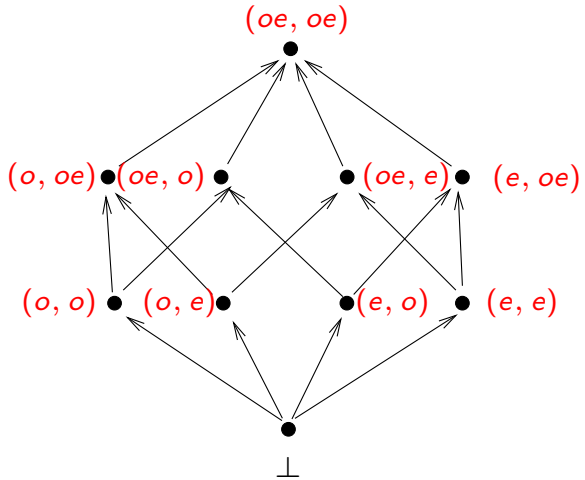

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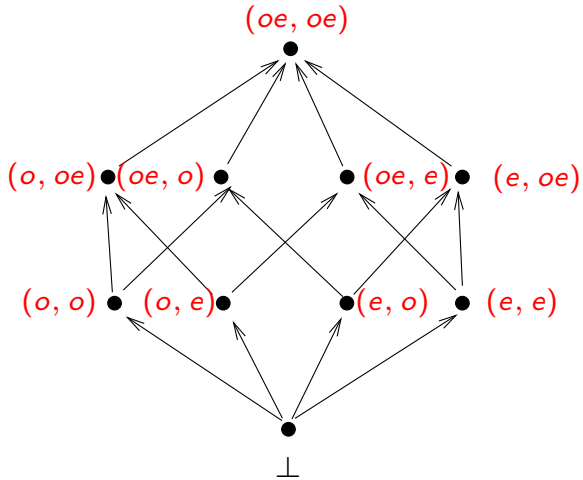

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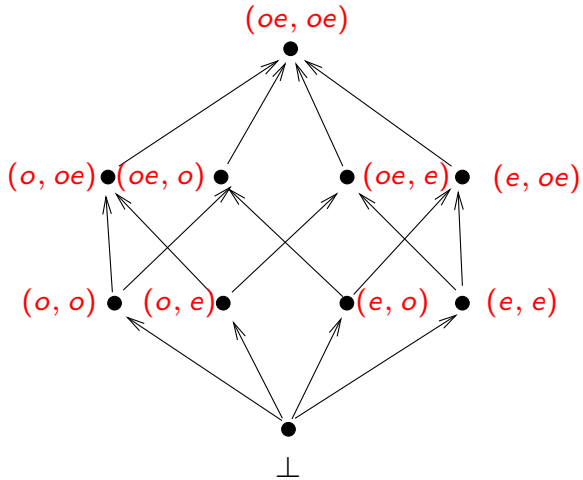

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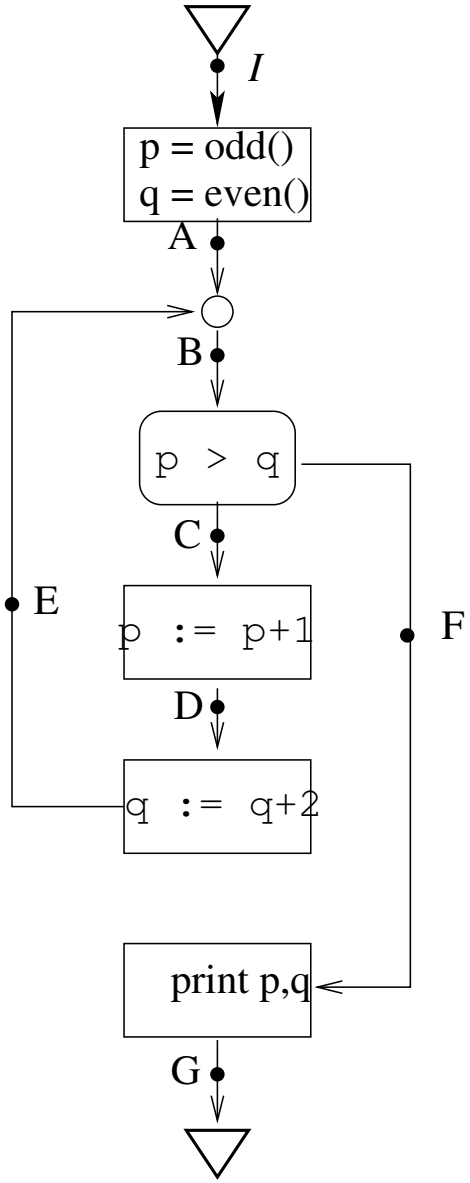

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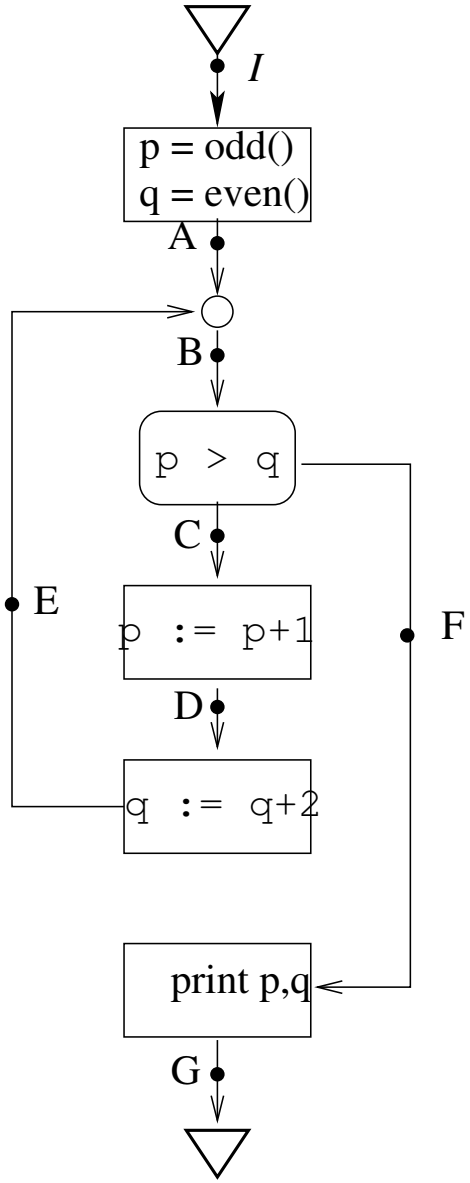

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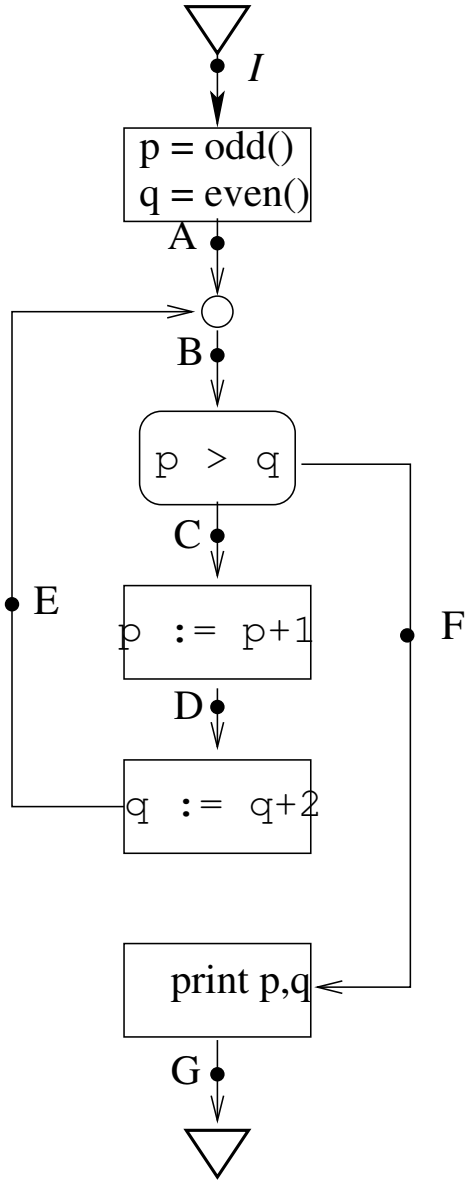

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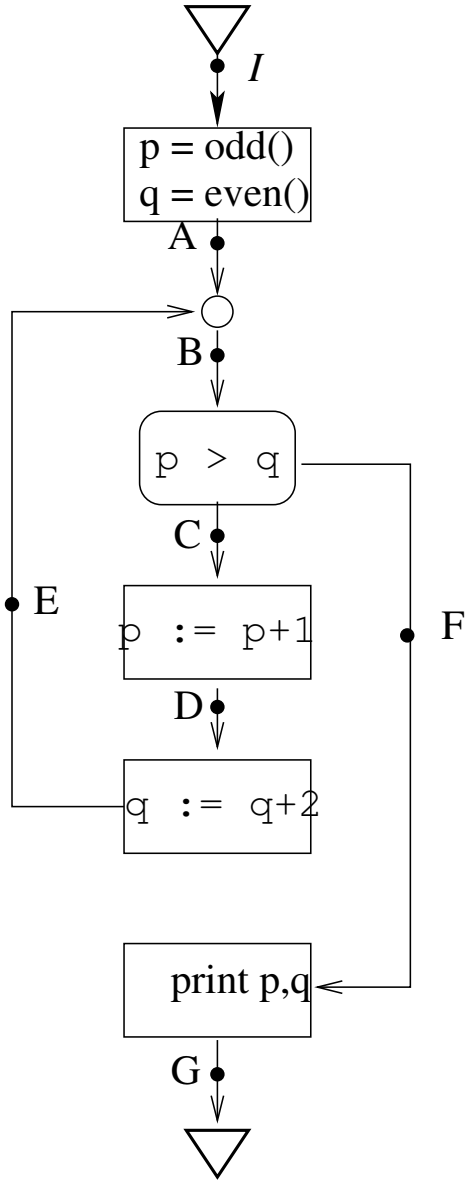

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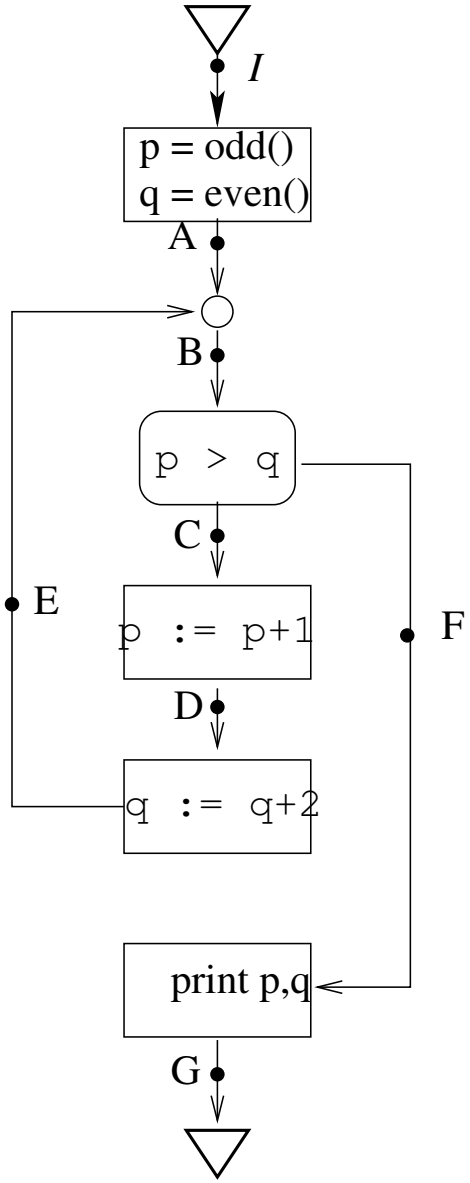

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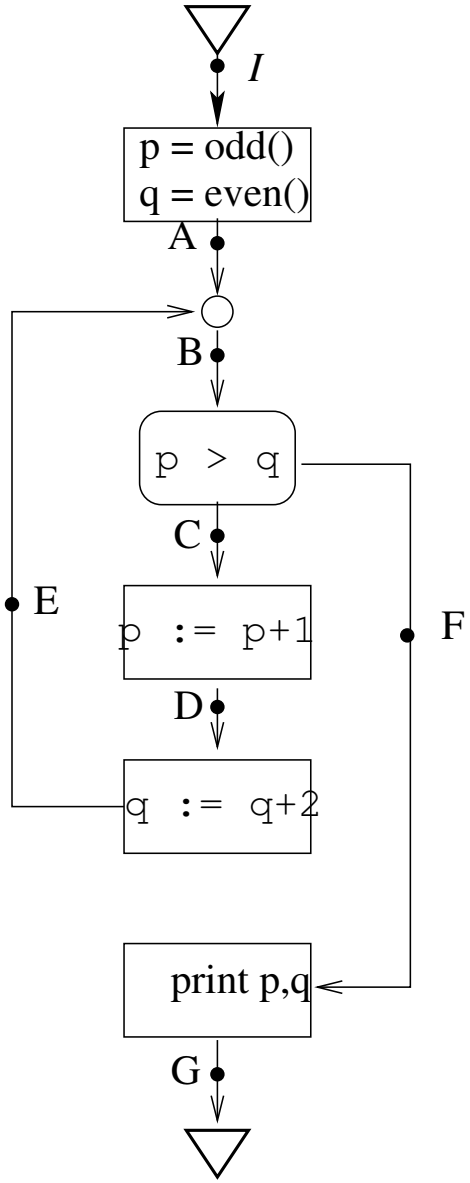

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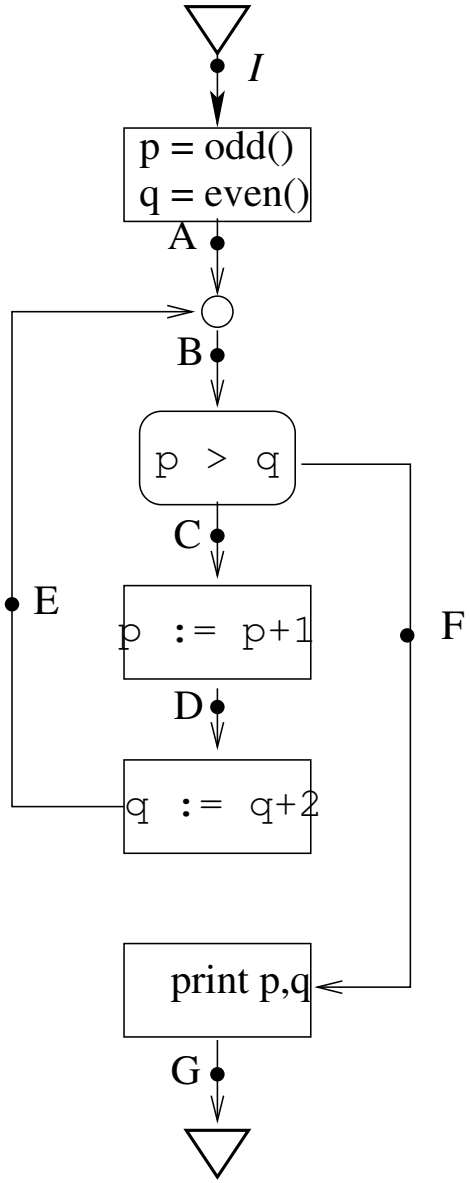

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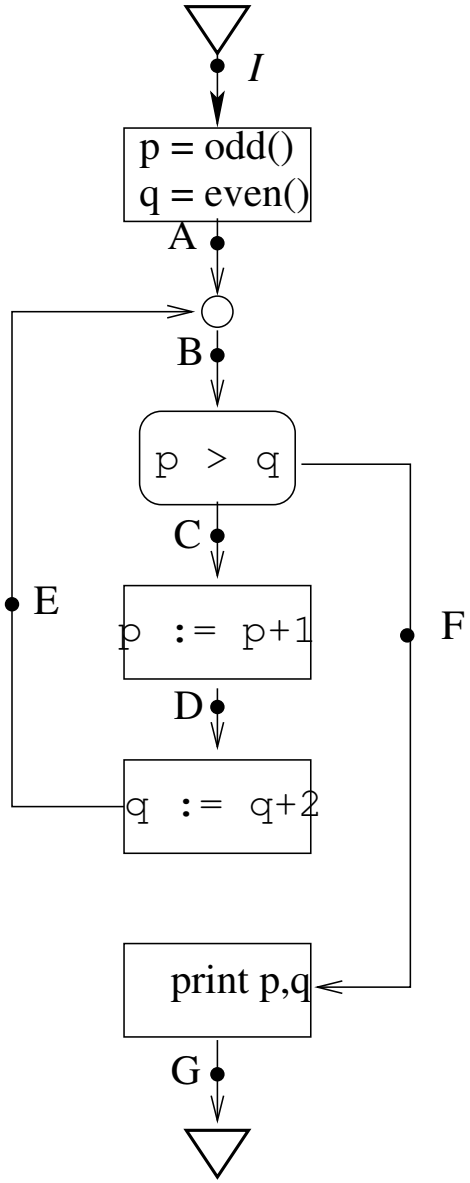

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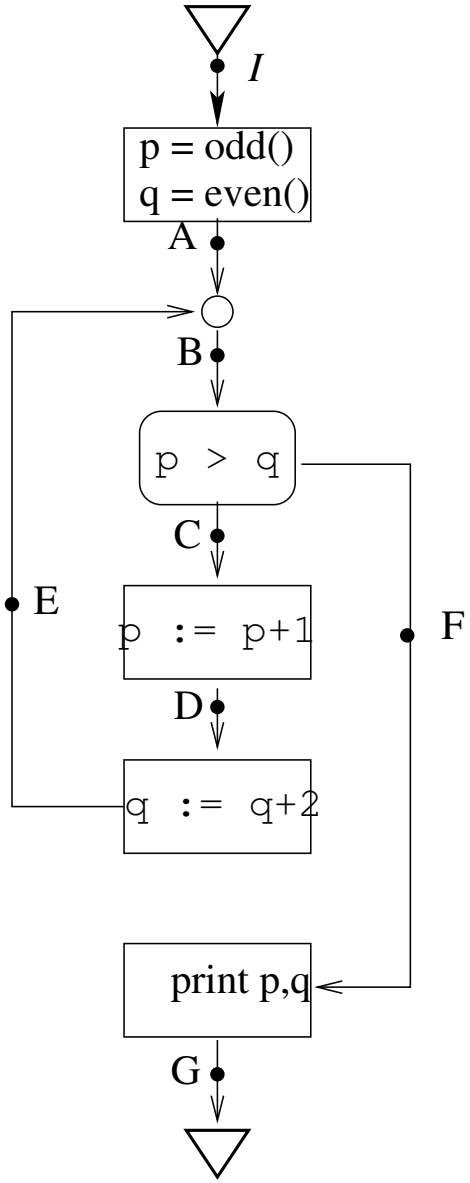

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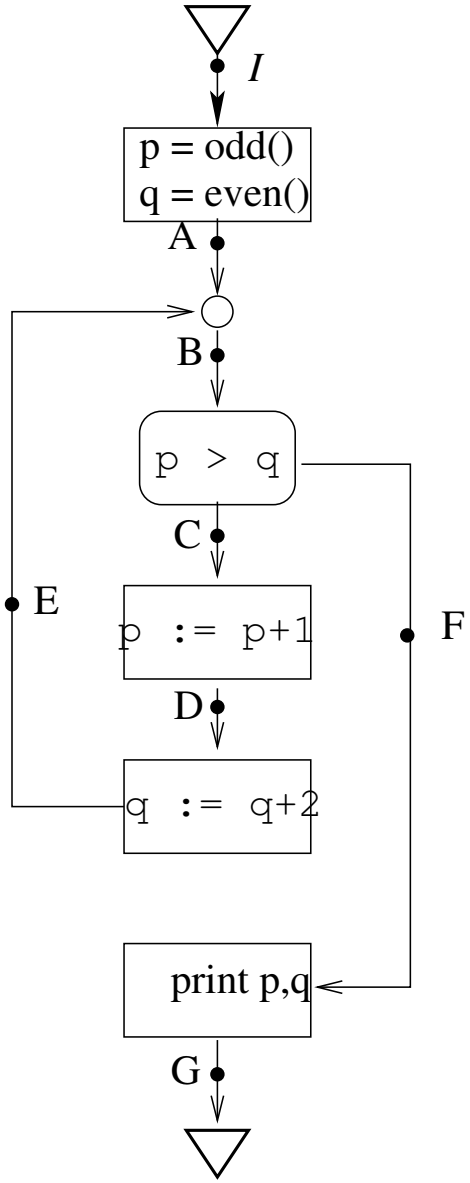

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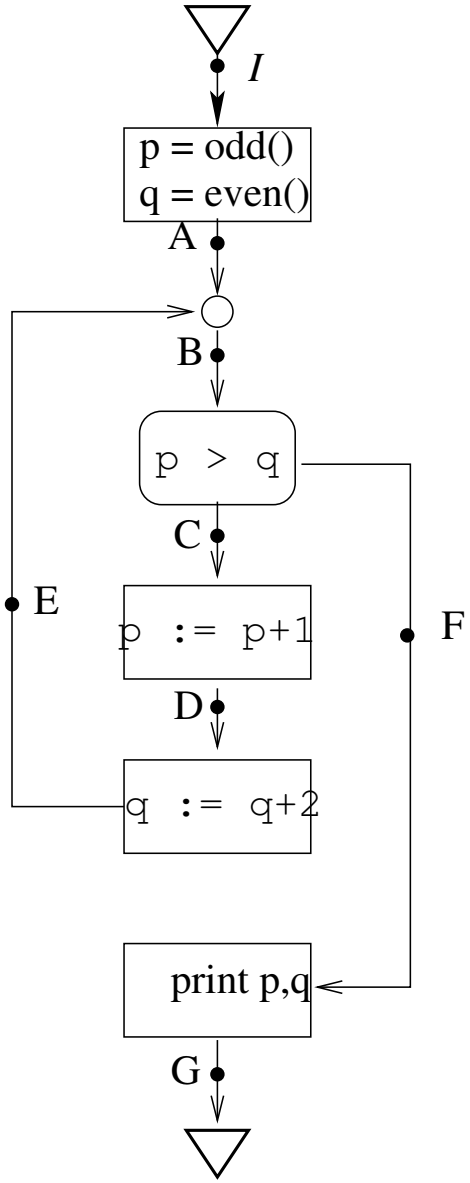

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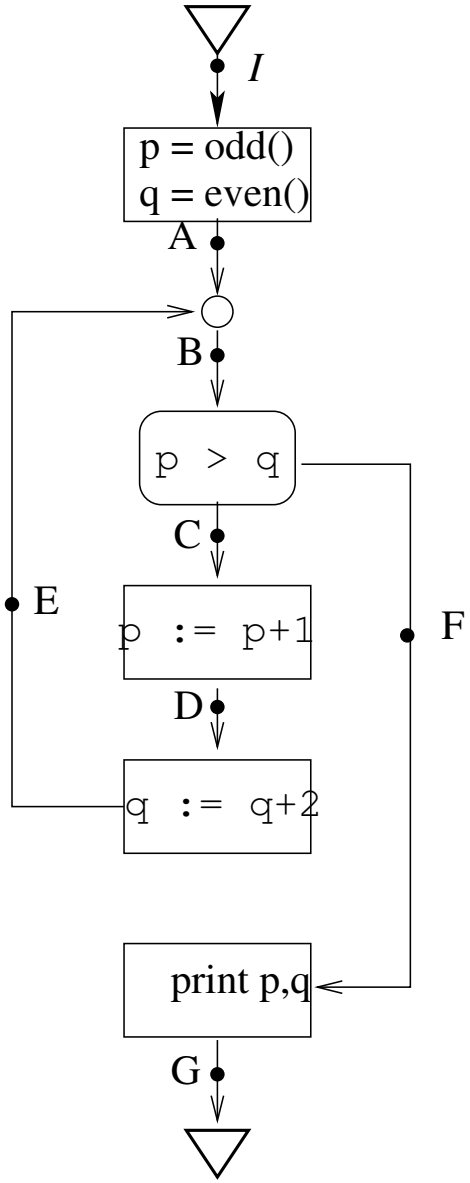

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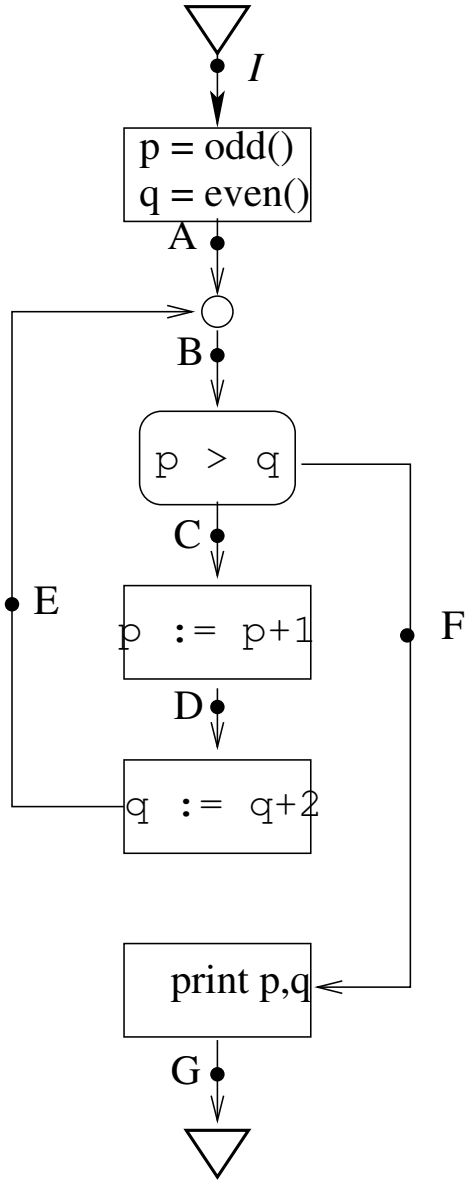
Collecting abstract values – example

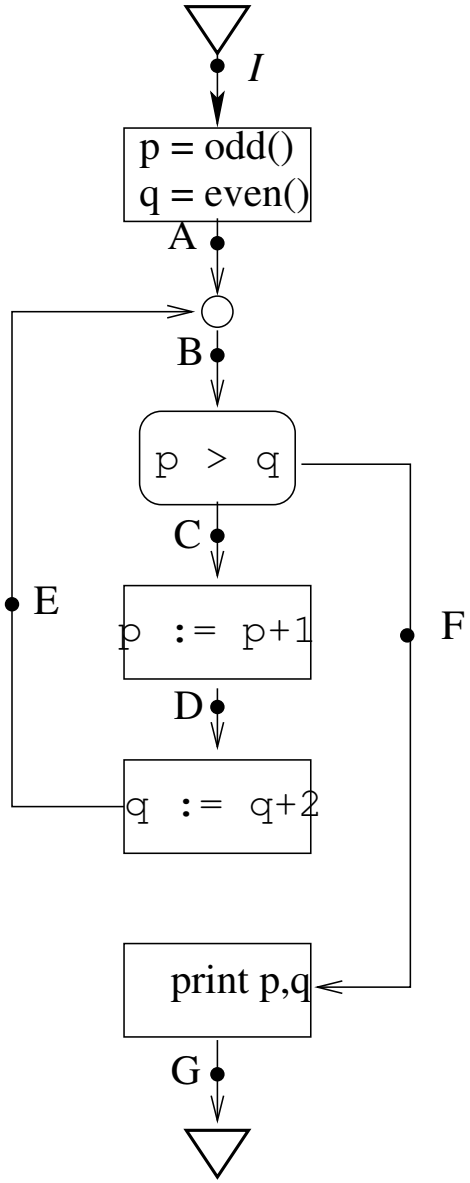


Path	Abstract value
I	(oe, oe) (given)
IA	(o, e)
IAB	(o, e)
IABC	(o, e)
IABCD	(e, e)
IABCDE	(e, e)
IABCDEB	(e, e)
IABCDEBC	(e, e)
IABCDEBCD	(o, e)
IABCDEBCDE	(o, e)
IABF	(o, e)
IABCDEBF	(e, e)

Therefore, joining all abstract values at each point:

I	(oe, oe)
A	(o, e)
B	$(o, e) \sqcup (e, e) = (oe, e)$
C	$(o, e) \sqcup (e, e) = (oe, e)$
D	$(e, e) \sqcup (o, e) = (oe, e)$
E	$(e, e) \sqcup (o, e) = (oe, e)$
F	$(o, e) \sqcup (e, e) = (oe, e)$
G	$(o, e) \sqcup (e, e) = (oe, e)$

Collecting abstract values – example



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G	$(o, e) \sqcup (e, e) = (oe, e)$

This is *abstract join-over-all-paths* (JOP) solution.

Comparison of abstract JOP states and collecting states

Abstract JOP:

A	(o, e)
B	(oe, e)
C	(oe, e)
D	(oe, e)
E	(oe, e)
F	(oe, e)
G	(oe, e)

Collecting states:

A	$\{(i,j) \mid i \text{ odd}, j \text{ even}\}$
B	$\{(i,j) \mid i \text{ odd}, j \text{ even}\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i \geq j\}$
C	$\{(i,j) \mid j \text{ even}, i > j\}$
D	$\{(i,j) \mid j \text{ even}, i > j+1\}$
E	$\{(i,j) \mid j \text{ even}, i \geq j\}$
F	$\{(i,j) \mid i \text{ odd}, j \text{ even}, i < j\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i = j\}$
G	$\{(i,j) \mid i \text{ odd}, j \text{ even}, i < j\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i = j\}$

Comparison of abstract JOP states and collecting states

Abstract JOP:		Collecting states:	
A	(o, e)	A	$\{(i,j) \mid i \text{ odd}, j \text{ even}\}$
B	(oe, e)	B	$\{(i,j) \mid i \text{ odd}, j \text{ even}\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i \geq j\}$
C	(oe, e)	C	$\{(i,j) \mid j \text{ even}, i > j\}$
D	(oe, e)	D	$\{(i,j) \mid j \text{ even}, i > j+1\}$
E	(oe, e)	E	$\{(i,j) \mid j \text{ even}, i \geq j\}$
F	(oe, e)	F	$\{(i,j) \mid i \text{ odd}, j \text{ even}, i < j\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i = j\}$
G	(oe, e)	G	$\{(i,j) \mid i \text{ odd}, j \text{ even}, i < j\} \cup \{(i,j) \mid i \text{ even}, j \text{ even}, i = j\}$

Note that at each point γ image of abstract solution is over-approximation of collecting states.

A given abstract interpretation is said to be *correct* if, for all abstract states $d_0 \in D$, for all programs P and for all program points p in P,

γ image of join of all abstract states arising at p (i.e., abstract JOP solution at p), with d_0 as the initial abstract value at P's entry

$\supseteq$

collecting semantics at p, with $\gamma(d_0)$ as the initial set of concrete states at P's entry

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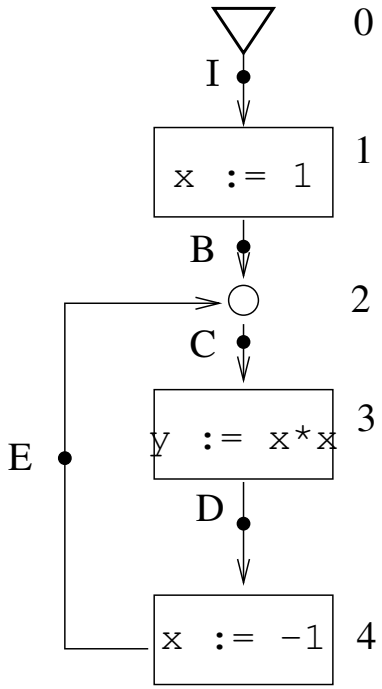
We will study later certain sufficient conditions for a given abstract interpretation to be correct.

Another example program



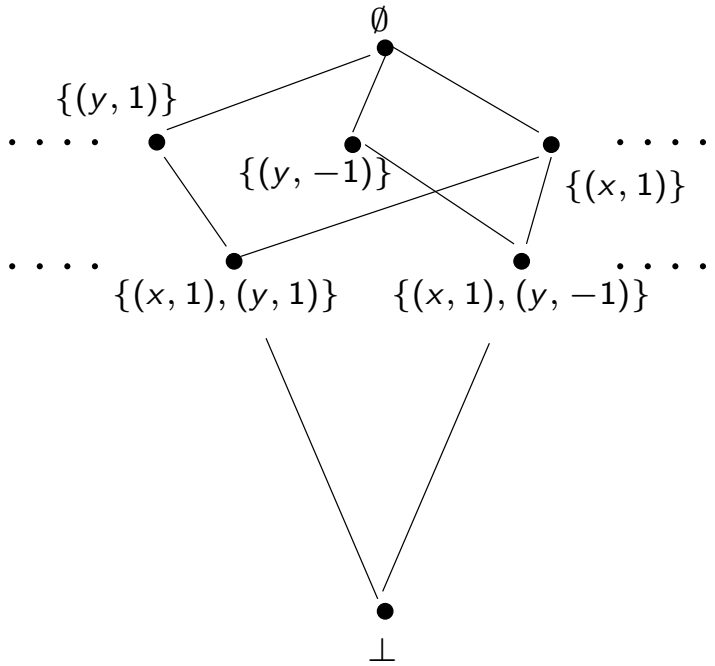
Path	Characterization of concrete states
I	*true* (given)
IB	$x = 1$
IBC	$x = 1$
IBCD	$x = 1 \wedge y = 1$
IBCDE	$x = -1 \wedge y = 1$
IBCDEC	$x = -1 \wedge y = 1$
IBCDECD	$x = -1 \wedge y = 1$
...	$x = -1 \wedge y = 1$

Therefore, collecting semantics:

Point	Characterization of concrete states
I	*true*
B	$x = 1$
C	$(x = 1) \vee (x = -1 \wedge y = 1)$
D	$(y = 1) \wedge (x = -1 \vee x = 1)$
E	$x = -1 \wedge y = 1$

Abstract interpretation for constant propagation

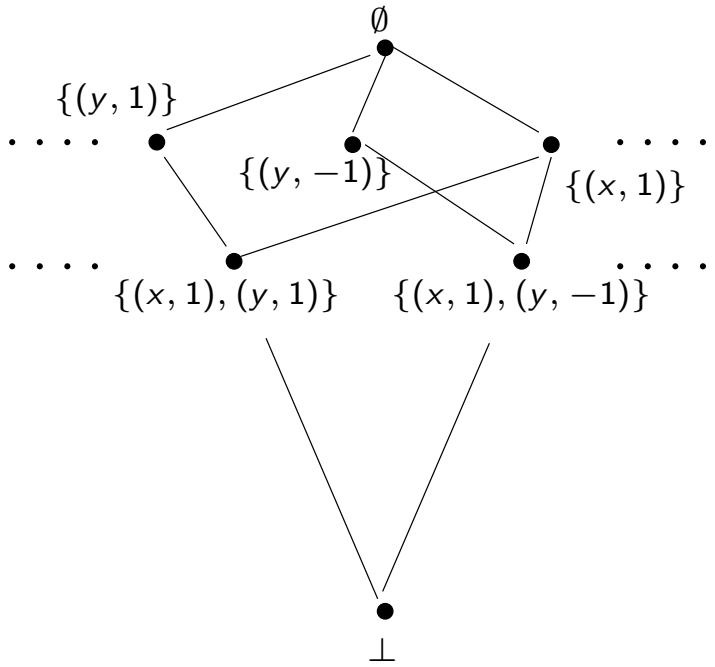
- Abstract lattice D



- Concretization function: What is $\gamma(d)$?

Abstract interpretation for constant propagation

- Abstract lattice D



- Concretization function: What is $\gamma(d)$?

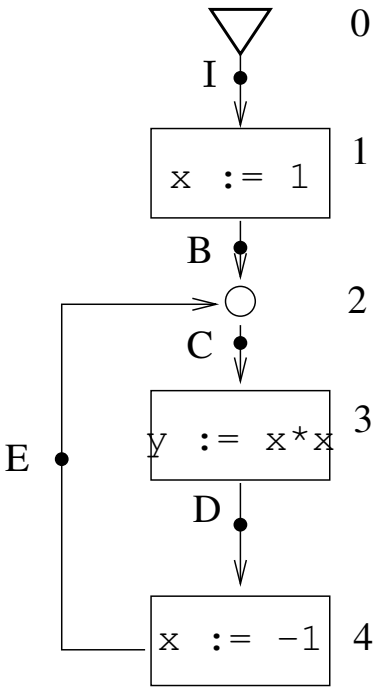
$$
\begin{array}{lcl}
\bot & \mapsto & \{\} \\
\emptyset & \mapsto & \mathit{State} \\
\{(x,c)\} & \mapsto & \{(c,j) |\ j \text{ is any value}\} \\
\{(x,c),(y,d)\} & \mapsto & \{(c,d)\}
\end{array}
$$

Abstract interpretation for constant propagation – contd.

Transfer function for assignment node n of the form $x := exp$.

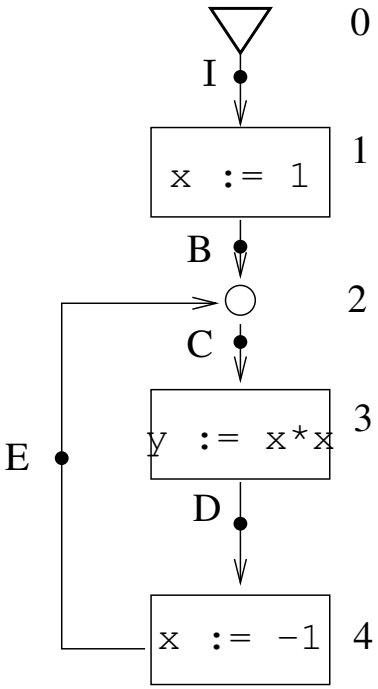
$$
\begin{aligned}
f_n(P) &= \bot, \text{ if } P \text{ is } \bot \\
&= \{(y, c) \in P \mid y \text{ is any variable other than } x\} \cup \{(x, d)\}, \\
&\quad \text{if all variables in } exp \text{ have constant values in } P, \text{ and if} \\
&\quad \text{exp evaluates to } d \text{ with these constant values} \\
&= \{(y, c) \in P \mid y \text{ is any variable other than } x\}, \text{ otherwise}
\end{aligned}
$$

JOP using abstract lattice



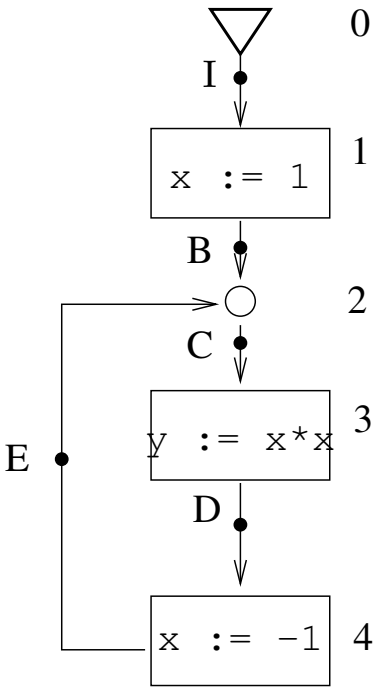
Path	Abstract value at end of path
I	$\emptyset$

JOP using abstract lattice



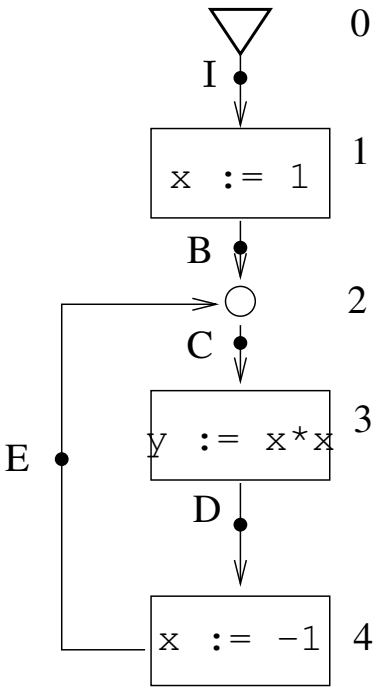
Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$

JOP using abstract lattice



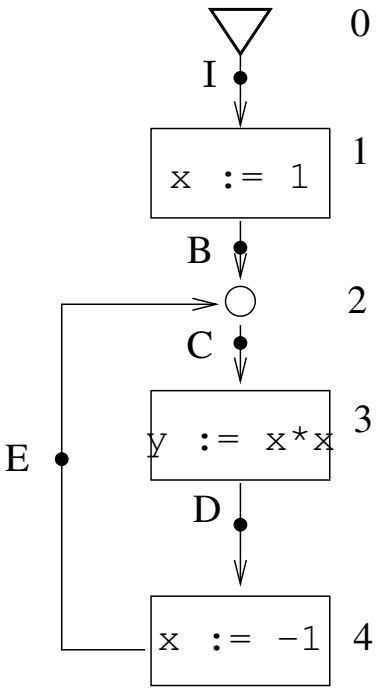
Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$

JOP using abstract lattice



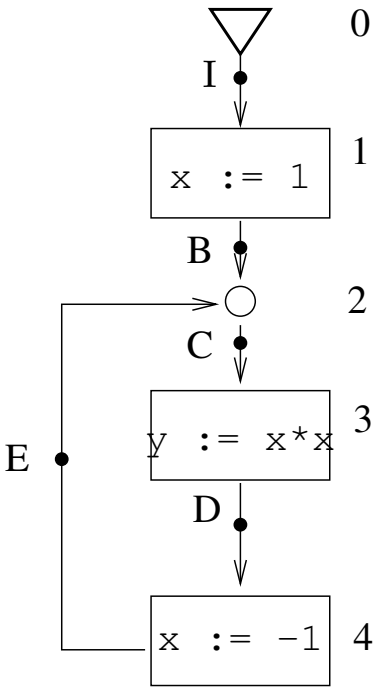
Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$

JOP using abstract lattice



Path	Abstract value at end of path
I	$\emptyset$
IB	$\{(x, 1)\}$
IBC	$\{(x, 1)\}$
IBCD	$\{(x, 1), (y, 1)\}$
IBCDE	$\{(x, -1), (y, 1)\}$

JOP using abstract lattice



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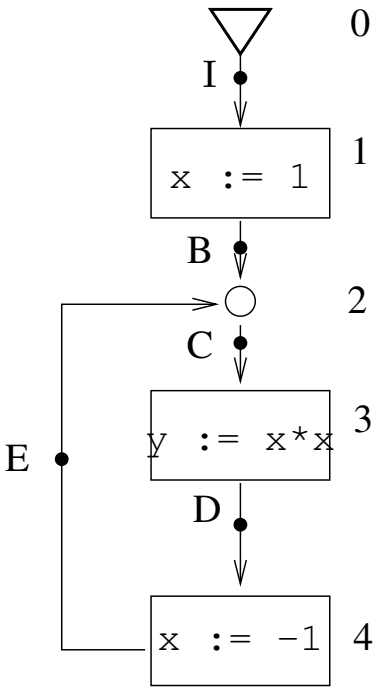
JOP using abstract lattice



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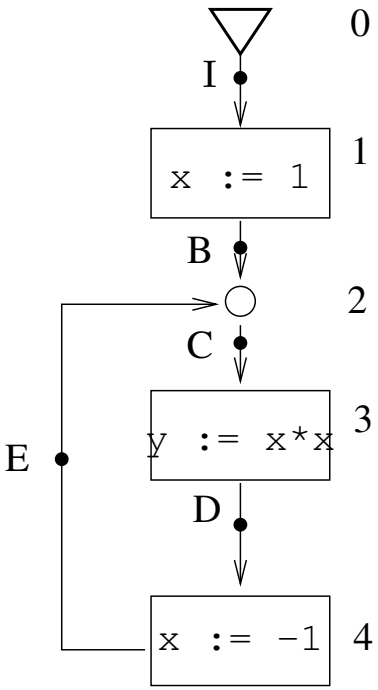
JOP using abstract lattice



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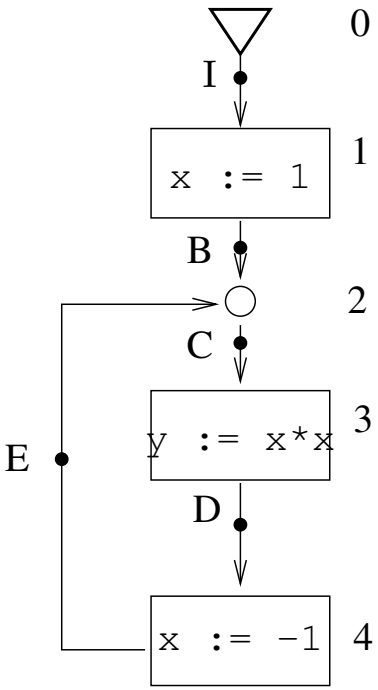
JOP using abstract lattice



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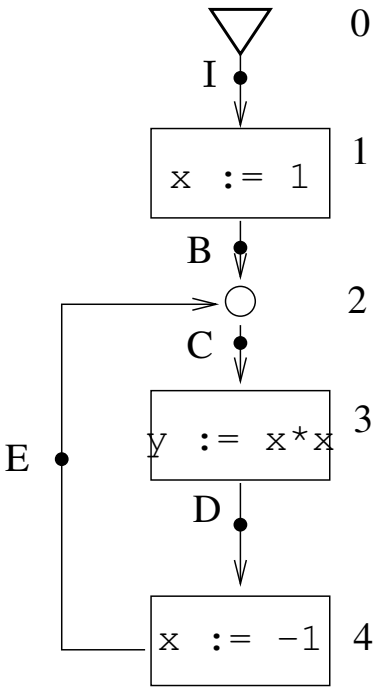
Point	Abstract JOP value
I	$\emptyset$

JOP using abstract lattice



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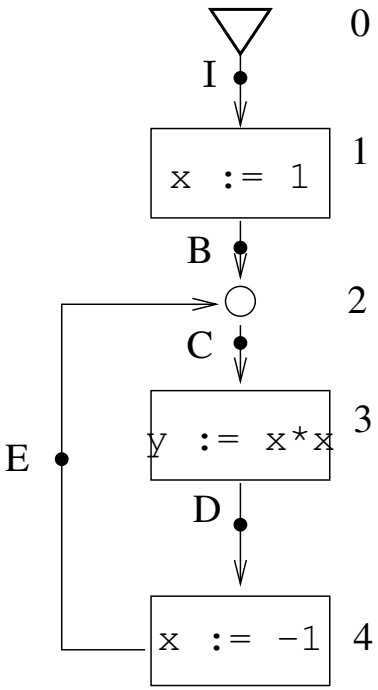
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$

JOP using abstract lattice



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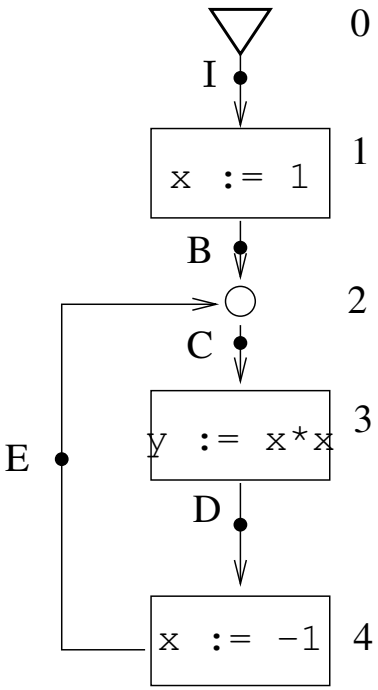
IBCDEC	$\{(x, -1), (y, 1)\}$
IBCDECD	$\{(x, -1), (y, 1)\}$
...	$\{(x, -1), (y, 1)\}$

Point	Abstract JOP value
I	$\emptyset$
B	$\{(x, 1)\}$
C	$\emptyset$

JOP using abstract lattice



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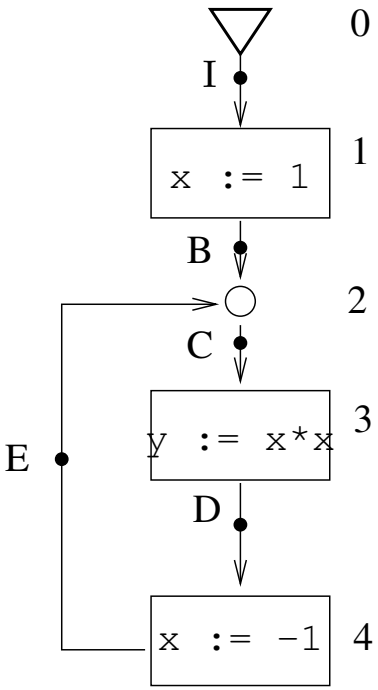
IBCDECD	$\{(x,-1),(y,1)\}$
...	$\{(x,-1),(y,1)\}$

Point	Abstract JOP value
I	$\emptyset$
B	$\{(x,1)\}$
C	$\emptyset$
D	$\{(y,1)\}$

JOP using abstract lattice



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...	$\{(x,-1),(y,1)\}$

Point	Abstract JOP value
I	$\emptyset$
B	$\{(x,1)\}$
C	$\emptyset$
D	$\{(y,1)\}$
E	$\{(x,-1),(y,1)\}$

Correctness in previous example

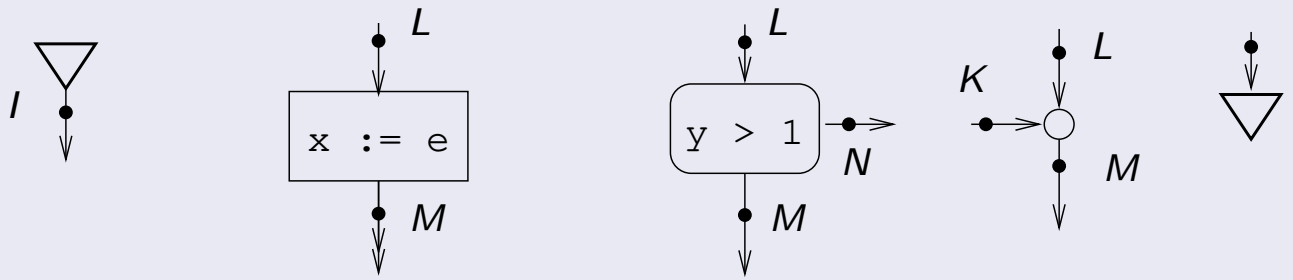
Verify that

- at points I, B and E
  γ(abstract JOP value) $=$ collecting semantics.
- at points C and D
  γ(abstract JOP value) $\supset$ collecting semantics.
- the abstract transfer functions given are the *best* possible for the given lattice L. That is, imprecision is due to the lattice, not the transfer functions.

Formal definition of control-flow graphs

Programs are finite directed graphs with following nodes (statements):

Nodes or statements in a program



- Expressions:

$$e ::= c \mid x \mid e + e \mid e - e \mid e * e.$$

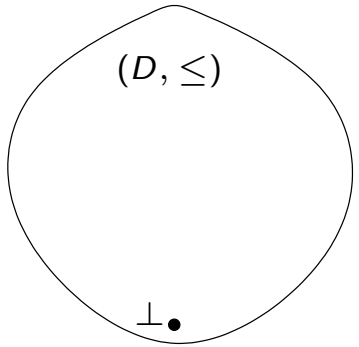
- Boolean expressions:

$$be ::= tt \mid ff \mid e \leq e \mid e = e \mid \neg be \mid be \vee be \mid be \wedge be.$$

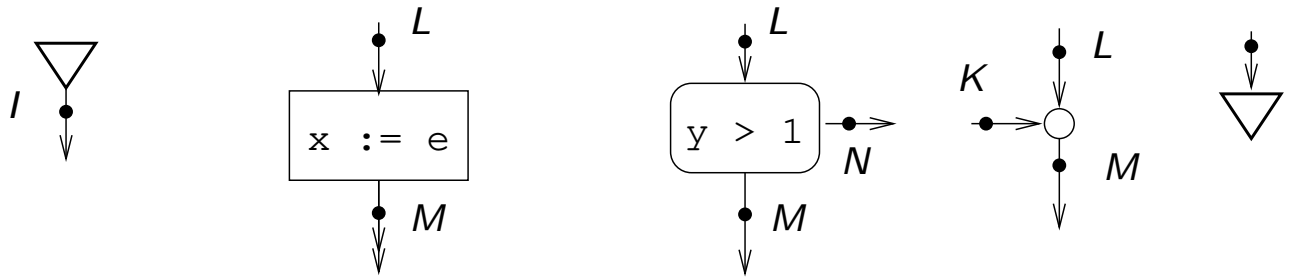
- Assume unique initial program point I.

Formal definition of an abstract interpretation

- Complete join semi-lattice $(D, \leq)$, with a least element $\bot$.
- Concretization function $\gamma : D \to 2^{State}$
- $\bot \in D$ represents unreachability of the program point (i.e., $\gamma(\bot)$ should be equal to $\emptyset$). Also, $\gamma(\top)$ should be *State*.



- We require transfer functions f_{LM}, f_{LN}, f_{KM} for all scenarios below:



- We assume transfer functions are monotonic, and satisfy $f(\bot) = \bot$.
- For junction nodes, both transfer functions should be *identity*

What we want to compute for a given program

- Path in a program: Sequence of connected edges or program points, beginning at initial point I
- Transfer functions extend to paths in program:

$$f_{IBCD} = f_{CD} \circ f_{BC} \circ f_{IB}.$$

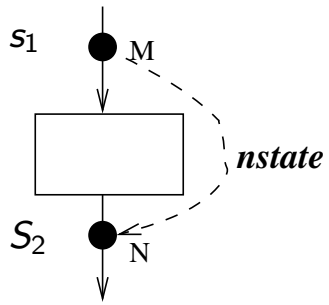
  where $(f_a \circ f_b)(x)$ is defined as $f_a(f_b(x))$.
- f_p is $\lambda d.\bot \Rightarrow$ path p is *infeasible*.
- Join over all paths (JOP) definition: For each program point N

$$d_N = \bigsqcup_{\text{paths } p \text{ from } I \text{ to } N} f_p(d_0).$$

  where d_0 is a given initial abstract value at entry node.

Formalization of collecting semantics

- Let *Val* be the set of all *concrete* values; e.g., $Integer \cup Boolean$.
- *State* is normally the domain $Var \rightarrow Val$. *However, in general, it can be any semantic domain.*
- Program semantics is given by the functions $nstate_{MN} : State \rightarrow 2^{State}$



- These induce the functions $nstate' : 2^{State} \rightarrow 2^{State}$

$$nstate'_{MN}(S_1 \in 2^{State}) = \bigcup_{s_1 \in S_1} nstate_{MN}(s_1)$$

Formalization of collecting semantics – contd.

- Collecting semantics SS is a map $\mathit{ProgramPoints} \rightarrow 2^{\mathit{State}}$
- It can be seen that at each program point N, the collecting semantics $\mathrm{SS}(N)$ is equal to:

$$\bigcup_{p \text{ path from } I \text{ to } N} \mathit{nstate'}_p(S_0).$$

where I is entry point of CFG, S_0 is the given initial set of states, and $\mathit{nstate'}_p$ is composition of $\mathit{nstate'}$ functions of edges that constitute p.

Instantiating a generic lattice and transfer functions implementation to a given program

- Implement a generic lattice element class, such that when one constructs a lattice element, one can provide the variable names along with other internal details
  - For e.g., $\{(p, o), (q, e)\}$, or $\{(x, 1), (y, 2)\}$
- Implement just three generic transfer functions: one for assignment statements, one for *true* branch of conditionals, and one for *false* branch of conditionals
  - Each transfer function takes, in addition to the incoming data flow fact (i.e., lattice element), the actual assignment/condition node as an argument